6.5 Conditional Probability

MATHEMATICAL TECHNIQUES

♠ For the given sample spaces, find a set of mutually exclusive and collective exhaustive events with the given number of elements.

- EXERCISE **6.5.1**

S=$\{0,1,2,3,4\}$. Find a set of two mutually exclusive and collective exhaustive events.

- EXERCISE **6.5.2**

S=$\{0,1,2,3,4\}$. Find a set of three mutually exclusive and collective exhaustive events.

- EXERCISE **6.5.3**

S={1, 2, 3, 4, ...}, the set of all positive integers. Find a set of two mutually exclusive and collective exhaustive events.

- EXERCISE **6.5.4**

S={1, 2, 3, 4, ...}, the set of all positive integers. Find a set of three mutually exclusive and collective exhaustive events.

♠ In each of the following cases, find $\Pr(A \cap B)$, $\Pr(A|B)$ and $\Pr(B|A)$.

- EXERCISE **6.5.5**

As in exercise 1.3.5, the sample space is S=$\{0,1,2,3,4\}$, $\Pr(\{0\}) = 0.2$, $\Pr(\{1\}) = 0.3$, $\Pr(\{2\}) = 0.4$, $\Pr(\{3\}) = 0.1$ and $\Pr(\{4\}) = 0.0$, A=$\{0,1,2\}$ and B=$\{0,2,4\}$.

- EXERCISE **6.5.6**

As in exercise 1.3.5, the sample space is S=$\{0,1,2,3,4\}$, $\Pr(\{0\}) = 0.2$, $\Pr(\{1\}) = 0.3$, $\Pr(\{2\}) = 0.4$, $\Pr(\{3\}) = 0.1$ and $\Pr(\{4\}) = 0.0$. Suppose now that A=$\{1,2,3\}$ and B=$\{2,3,4\}$.

- EXERCISE **6.5.7**

As in exercise 1.3.6, the sample space is S=$\{0,1,2,3,4\}$, $\Pr(\{0\}) = 0.1$, $\Pr(\{1\}) = 0.3$, $\Pr(\{2\}) = 0.4$, $\Pr(\{3\}) = 0.1$, $\Pr(\{4\}) = 0.1$, A=$\{0,2\}$ and B=$\{3,4\}$.

- EXERCISE **6.5.8**

As in exercise 1.3.6, the sample space is S=$\{0,1,2,3,4\}$, $\Pr(\{0\}) = 0.1$, $\Pr(\{1\}) = 0.3$, $\Pr(\{2\}) = 0.4$, $\Pr(\{3\}) = 0.1$, $\Pr(\{4\}) = 0.1$. Suppose now that A=$\{1,2\}$ and B=$\{1,2,3,4\}$.

♠ Somebody invents a three-sided die that gives scores of 1,2 or 3, each with probability 1/3. Two such die are rolled. Use the law of total probability to find the following probabilities, and then find them directly by counting.

- EXERCISE **6.5.9**

The probability that the total on the two die is 4 or more. (To use the law of total probability, find the probability that the score is 4 or more if the first die gives a 1, if the first die gives a 2, and if the first die gives a 3.)

- EXERCISE **6.5.10**

The probability that the total on the two die is 5 or more.

- EXERCISE **6.5.11**

The probability that the total on the two die is odd.

- EXERCISE **6.5.12**

The probability that the second roll was larger than the first.

♠ Consider again the three-sided die that gives a score of 1 with probability 1/3, a score of 2 with probability 1/3, and a score of 3 with probability 1/3. Two such die are rolled. Use Bayes' Theorem to find the following conditional probabilities. Check your result by direct counting.

- EXERCISE **6.5.13**

Find the probability that the first roll is a 3 if the total of the two rolls is greater than 4 (based on exercise 6.5.9).

- EXERCISE **6.5.14**

Find the probability that the first roll is a 3 if the total of the two rolls is greater than 5 (based on exercise 6.5.10).

- EXERCISE **6.5.15**

Find the probability that the first roll is a 3 if the total of the two rolls is odd (based on exercise 6.5.11).

• EXERCISE **6.5.16**
Find the probability that the first roll is a 1 if the second roll is greater than the first (based on exercise 6.5.12).

♠ Four balls are placed in a jar, two red, one blue and one yellow. Two are removed at random.

• EXERCISE **6.5.17**
You are told that the first ball removed was red. What is the probability that the second is red?

• EXERCISE **6.5.18**
You are told that at least one of the two removed is red. What is the probability that both are?

• EXERCISE **6.5.19**
As in exercise 6.5.17, but the first ball is replaced (but remembered) before the second ball is drawn. What is the probability that the second is red?

• EXERCISE **6.5.20**
As in exercise 6.5.18, but the first ball is replaced (but remembered) before the second ball is drawn. What is the probability that both are red?

APPLICATIONS

♠ Give a set of three mutually exclusive and collectively exhaustive sets for each of the following sample spaces.

• EXERCISE **6.5.21**
The situation in exercise 1.3.17.

• EXERCISE **6.5.22**
The situation in exercise 1.3.18.

• EXERCISE **6.5.23**
The situation in exercise 1.3.19.

• EXERCISE **6.5.24**
The situation in exercise 1.3.20.

• EXERCISE **6.5.25**
The situation in exercise 1.3.23.

• EXERCISE **6.5.26**
The situation in exercise 1.3.24.

♠ An ecologist is looking for the effects of eagle predation on the behavior of jack rabbits. In each of the following cases,

a. Draw a Venn diagram to illustrate the situation.

b. Find the probability that she saw a jack rabbit conditional on her seeing an eagle. How might you interpret this result? Compare with the overall probability of seeing a jack rabbit.

c. Find the probability that she saw an eagle conditional on her seeing a jack rabbit. How might you interpret this result?

• EXERCISE **6.5.27**
She sees an eagle with probability 0.2 during an hour of observation, a jack rabbit with probability 0.5, and both with probability 0.05.

• EXERCISE **6.5.28**
She sees an eagle with probability 0.2 during an hour of observation, a jack rabbit with probability 0.5, and both with probability 0.15.

♠ A lab is attempting to stain many cells. Young cells stain properly 90% of the time and old cells stain properly 70% of the time.

• EXERCISE **6.5.29**
If 30% of the cells are young, what is the probability that a cell stains properly?

• EXERCISE **6.5.30**
If 70% of the cells are young, what is the probability that a cell stains properly?

♠ Further study of the cell staining problem (exercises 6.5.29 and 6.5.30) reveals that new cells stain properly with probability 0.95, one day old cells stain properly with probability 0.9, two day old cells stain properly with probability 0.8, and three day old cells stain properly with probability 0.5. Suppose

$$\begin{aligned} \Pr(\text{cell is 0 day old}) &= 0.4 \\ \Pr(\text{cell is 1 day old}) &= 0.3 \\ \Pr(\text{cell is 2 days old}) &= 0.2 \\ \Pr(\text{cell is 3 days old}) &= 0.1. \end{aligned}$$

• EXERCISE **6.5.31**
Find the probability that a cell stains properly.

• EXERCISE **6.5.32**
The lab finds a way to eliminate the oldest cells (more than 3 days old) from its stock. What is the probability of proper staining? Write this as a conditional probability.

♠ Use Bayes' Theorem to compute the following. Say whether the stain is a good indicator of the age of the cell.

• EXERCISE **6.5.33**
For the cells in exercise 6.5.29, what is the probability that a cell which stains properly is young? How does this compare with the unconditional probability of 0.3?

• EXERCISE **6.5.34**
For the cells in exercise 6.5.30, what is the probability that a cell which stains properly is young? How does this compare with the unconditional probability of 0.7?

• EXERCISE **6.5.35**
For the cells in exercise 6.5.31, what is the probability that a cell which stains properly is less than one day old? How does this compare with the unconditional probability of 0.4?

• EXERCISE **6.5.36**
For the cells in exercise 6.5.32, what is the probability that a cell which stains properly is less than one day old? How does this compare with the unconditional probability?

♠ Consider a disease with a imperfect test. Let D denote the event of an individual having the disease, N the event of not having the disease, and P the event of a positive result on the test. In each of the following cases, find $\Pr(D|P)$.

• EXERCISE **6.5.37**
$\Pr(D) = 0.2$, $\Pr(N) = 0.8$, $\Pr(P|D) = 1.00$ and $\Pr(P|N) = 0.05$. Compare with the results in the text, when the disease was much less common.

• EXERCISE **6.5.38**
$\Pr(D) = 0.8$, $\Pr(P|D) = 1.00$ and $\Pr(P|N) = 0.1$. Compare with the results in the text and exercise 6.5.37.

♠ In the following cases, the test does not catch every sick peson. Let D denote the event of an individual having the disease, N the event of not having the disease, and P the event of a positive result on the test. Find $\Pr(D|P)$ and $\Pr(D|P^c)$ (the probability that a person who did not test positive has the disease).

• EXERCISE **6.5.39**
$\Pr(D) = 0.2$, $\Pr(N) = 0.8$, $\Pr(P|D) = 0.95$ and $\Pr(P|N) = 0.05$. Compare your results with exercise 6.5.37.

• EXERCISE **6.5.40**
$\Pr(D) = 0.8$, $\Pr(P|D) = 0.95$ and $\Pr(P|N) = 0.1$. Compare your results with exercise 6.5.38.

♠ Consider a dominant gene where plants with genotype **BB** or **Bb** are tall, while plants with genotype **bb** are short. Find the probability that a tall plant has genotype **Bb** when it results from the following crosses.

• EXERCISE **6.5.41**
A plant with genotype **Bb** is crossed with the offspring from a cross between a **BB** plant and a **Bb** plant.

• EXERCISE **6.5.42**
A plant with genotype **Bb** is crossed with the offspring from a cross between a **Bb** plant and a **Bb** plant.

• EXERCISE **6.5.43**
Two offspring from the cross between a **BB** plant and a **Bb** plant are crossed with equal other.

• EXERCISE **6.5.44**
Two tall offspring from the cross between a **Bb** plant and a **Bb** plant are crossed with equal other.

♠ A popular probability problem refers to a once popular game show called "Let's make a deal." In this game, the host (named Monte Hall) hands out large prizes to contestants for no reason at all. In one situation, Monte would show the contestant three doors, named door 1, door 2 and door 3. One would hide a new car, one $500 worth of false eyelashes, and the other a goat (deemed worthless by the purveyors of the show). The contestant picks door 1. But instead of showing her the prize, Monte opens door 3 to reveal the goat.

• EXERCISE **6.5.45**

If the car is really behind door 1, what happens if she switches?

• EXERCISE **6.5.46**

If the car is really behind door 2, what happens if she switches?

• EXERCISE **6.5.47**

Should the contestant switch her guess to door 2?

• EXERCISE **6.5.48**

If she uses the right strategy, what is her probability of getting the new car?

• EXERCISE **6.5.49**

It is later revealed that Monte does not always show what is behind one of the other doors, but does so only when the contestant guessed right in the first place (the so-called "Machiavellian Monte"). How often would a contestant who used the strategy in exercise 6.5.47 get the new car?

• EXERCISE **6.5.50**

What is the right strategy to use for dealing with the Machiavellian Monte? How well would the contestant do?

COMPUTER EXERCISES

• EXERCISE **6.5.51**

Use the command q_p (exercise 1.1.37) that returns 1 with probability p and 0 with probability $1-p$ to simulate the rare disease example.

a. Simulate 100 people who have the disease with probability 0.05 and count up the number with the disease.

b. For each of the remaining people, assume that the probability of a false positive is 0.1. Simulate them and count up the number of positives.

c. What fraction of positive tests identify people who are sick? How does this compare with the mathematical expectation?

d. Try the same experiment where the probability that each person has the disease is 0.4.

• EXERCISE **6.5.52**

Suppose cells fall into three categories: those that are dead, those that are alive but do not stain properly, and those that are alive and do stain properly. Let D_t denotes the number of cells that are dead, N_t the number that are alive but do not stain properly, and S_t those that are alive and do stain properly. Each day, there are two possible transitions.

- Cells that are alive die with probability 0.9.
- Cells that stain properly cease to stain properly with probability 0.8.

a. Start with $S_0 = 100$, $D_0 = 0$, and $N_0 = 0$. Use your computer to simulate the numbers in the next generation.

b. Follow these cells until there are no more cells that stain properly. How long did it take?

c. At each time, what is the fraction of cells that stain properly? What is the fraction of *living* cells that stain properly? Estimate the probability that a cell stains properly and the probability it stains properly conditional on being alive.

Chapter 2

Answers

6.5.1. One possibility is A= $\{0,1\}$, B= $\{2,3,4\}$.
6.5.3. One possibility is A= $\{1,3,5,\ldots\}$, B= $\{2,4,6,\ldots\}$.
6.5.5. $\Pr(\mathrm{A}\cap\mathrm{B}) = \Pr(\{0,2\}) = 0.2+0.4 = 0.6$. Therefore, $\Pr(\mathrm{A}|\mathrm{B}) = \frac{\Pr(\mathrm{A}\cap\mathrm{B})}{\Pr(\mathrm{B})} = \frac{0.6}{0.9} = 2/3$. $\Pr(\mathrm{B}|\mathrm{A}) = \frac{\Pr(\mathrm{A}\cap\mathrm{B})}{\Pr(\mathrm{A})} = \frac{0.6}{0.6} = 1$.
6.5.7. $\Pr(\mathrm{A}\cap\mathrm{B}) = 0$ because the sets are disjoint. Therefore, $\Pr(\mathrm{A}|\mathrm{B}) = 0$, and $\Pr(\mathrm{B}|\mathrm{A}) = 0$.
6.5.9. Let S be the event four or more, and let F_1 be the event of a 1 on the first roll, F_2 a 2 on the first role, F_3 a 3 on the first. Then $\Pr(\mathrm{S}|F_1) = 1/3$ (because the second roll must be a 3), $\Pr(\mathrm{S}|F_2) = 2/3$ (because the second roll can be a 2 or a 3), $\Pr(\mathrm{S}|F_3) = 1$ (because the second roll can be anything). Then, by the law of total probability,

$$\Pr(\mathrm{S}) = \Pr(\mathrm{S}|F_1)\Pr(F_1) + \Pr(\mathrm{S}|F_2)\Pr(F_2) + \Pr(\mathrm{S}|F_3)\Pr(F_3) = 1/3\cdot 1/3 + 2/3\cdot 1/3 + 1\cdot 1/3 = 2/3.$$

By direct counting, there are 9 possible outcomes, each with probability 1/9. Of these 6

$$(1,3),(2,2),(2,3),(3,1),(3,2),(3,3)$$

give a result greater than 4, for a probability of 6/9=2/3.
6.5.11. Let O be the event odd, and let F_1 be the event of a 1 on the first roll, F_2 a 2 on the first role, F_3 a 3 on the first. Then $\Pr(\mathrm{O}|F_1) = 1/3$ (because the second roll must be a 2), $\Pr(\mathrm{O}|F_2) = 2/3$ (because the second roll can be a 1 or a 3), $\Pr(\mathrm{O}|F_3) = 1$ (because the second roll must be a 2). Then, by the law of total probability,

$$\Pr(\mathrm{S}) = \Pr(\mathrm{S}|F_1)\Pr(F_1) + \Pr(\mathrm{S}|F_2)\Pr(F_2) + \Pr(\mathrm{S}|F_3)\Pr(F_3) = 1/3\cdot 1/3 + 2/3\cdot 1/3 + 1/3\cdot 1/3 = 4/9.$$

By direct counting, there are 9 possible outcomes, each with probability 1/9. Of these 4

$$(1,2),(2,1),(2,3),(3,2)$$

give an odd result, for a probability of 4/9.
6.5.13. Let S be the event four or more, and F_3 a 3 on the first role. By Bayes' Theorem,

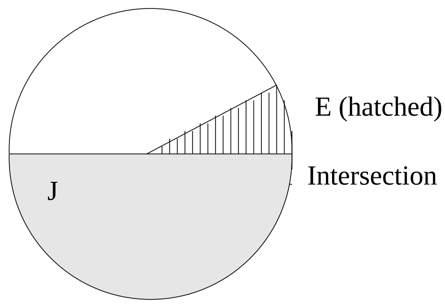
$$\Pr(F_3|\mathrm{S}) = \frac{\Pr(\mathrm{S}|F_3)\Pr(F_3)}{\Pr(\mathrm{S})}.$$

From exercise 6.5.9, $\Pr(\mathrm{S}|F_3) = 1$ and $\Pr(\mathrm{S}) = 2/3$. By assumption, $\Pr(F_3) = 1/3$. Therefore, $\Pr(F_3|\mathrm{S}) = 1/2$. Out of the six possible ways that the score could be 4 or greater, three start with a roll of 3, for a probability of 3/6=1/2.
6.5.15. Let O be the event odd, and F_3 a 3 on the first role. By Bayes' Theorem,

$$\Pr(F_3|\mathrm{O}) = \frac{\Pr(\mathrm{O}|F_3)\Pr(F_3)}{\Pr(\mathrm{O})}.$$

From exercise 6.5.11, $\Pr(\mathrm{O}|F_3) = 1/3$ and $\Pr(\mathrm{O}) = 4/9$. By assumption, $\Pr(F_3) = 1/3$. Therefore, $\Pr(F_3|\mathrm{O}) = 1/4$. Out of the four possible ways that the score could be odd, one starts with a roll of 3, for a probability of 1/4.

6.5.17. Let F be the event "first red," S the event "second red," O the event "one red," and B the event "both red." Then $\Pr(S|F) = \dfrac{\Pr(S \cap F)}{\Pr(F)}$. The probability that both are red is 1/6. The probability that the first is red is 1/2. So the requested probability is 1/3.
6.5.19. Exactly 1/2, because there are 2 out of 4 balls that are red.
6.5.21. There are only three events: **bb**, **Bb**, **BB**. This is the only possible choice.
6.5.23. We could break the four elements in the sample space into those with an "Out" at time 1 (two simple events), and the two simple events $\{\text{In}, \text{In}\}$, $\{\text{In}, \text{Out}\}$.
6.5.25. Let N denote the number of plants taller than 50 cm. We could use $N = 0$, $0 < N < 16$, and $N = 16$.
6.5.27.



a.

b. Let J be the event of seeing a jack rabbit, E the event of seeing an eagle. Then

$$\Pr(J|E) = \frac{\Pr(J \cap E)}{\Pr(E)} = \frac{0.05}{0.2} = 0.25.$$

It is less likely that she will see a rabbit if she sees an eagle. Perhaps rabbits avoid the eagles.

c.

$$\Pr(E|J) = \frac{\Pr(E \cap J)}{\Pr(J)} = \frac{0.05}{0.5} = 0.1.$$

It is less likely that she will see an eagle if she sees a rabbit. Perhaps eagles avoid the rabbits. This seems less likely, but cannot be demonstrated from the data.

6.5.29. Let S be the event that a cell stains properly, Y that a cell is young, and O that it is old. $\Pr(S) = \Pr(S|Y)\Pr(Y) + \Pr(S|O)\Pr(O) = 0.9 \cdot 0.3 + 0.7 \cdot 0.7 = 0.76$.
6.5.31. Let S be the event "stains," A_1 the event "less than 1 day old," A_2 the event "between 1 and 2 days old", A_3 the event "between 2 and 3 days old", and A_4 the event "more than 3 days old." By the law of total probability,

$$\begin{aligned}\Pr(S) &= \Pr(S|A_1)\Pr(A_1) + \Pr(S|A_2)\Pr(A_2) + \Pr(S|A_3)\Pr(A_3) + \Pr(S|A_4)\Pr(A_4) \\ &= 0.95 \cdot 0.4 + 0.9 \cdot 0.3 + 0.8 \cdot 0.2 + 0.5 \cdot 0.1 = 0.76.\end{aligned}$$

6.5.33. We want to find $\Pr(Y|S)$. By Bayes' Theorem,

$$\Pr(Y|S) = \frac{\Pr(S|Y)\Pr(Y)}{\Pr(S)} = \frac{0.9 \cdot 0.3}{0.76} = 0.355.$$

A properly staining cell is only slightly more likely to be young.
6.5.35.

$$\Pr(A_1|S) = \frac{\Pr(S|A_1)\Pr(A_1)}{\Pr(S)} = \frac{0.95 \cdot 0.4}{0.76} = 0.5.$$

A properly staining cell is a bit more likely to be young.
6.5.37. First, find $\Pr(P)$ with the law of total probability,

$$\begin{aligned}\Pr(P) &= \Pr(P|D)\Pr(D) + \Pr(P|N)\Pr(N) \\ &= (1.0)(0.2) + (0.05)(0.8) = 0.24.\end{aligned}$$