## 6.4 Probability Theory

### MATHEMATICAL TECHNIQUES

♠ For the given sets A and B, find $A \cap B$, $A \cup B$, and $A^c$ (the complement of A).

• EXERCISE **6.4.1**

A and B are subsets of the set S=$\{0,1,2,3,4\}$. A=$\{0,1,2\}$ and B=$\{0,2,4\}$.

• EXERCISE **6.4.2**

A and B are subsets of the set S=$\{0,1,2,3,4,5\}$. A=$\{0,1,2\}$ and B=$\{0,2,4,5\}$.

• EXERCISE **6.4.3**

A and B are subsets of the set of all positive integers, $\{1,2,3,\ldots\}$, with A=$\{1,2,6,10\}$ and B=$\{2,4,5\}$.

• EXERCISE **6.4.4**

A and B are subsets of the set of all negative integers, $\{-1,-2,-3,\ldots\}$, with the set A=$\{-1,-4,-6,-10\}$ and the set B=$\{-2,-4,-10,-11\}$.

♠ For the given sets and sample spaces, show that the assignment of probabilities is mathematically consistent and use them to compute the requested probability.

• EXERCISE **6.4.5**

The sample space is S=$\{0,1,2,3,4\}$. Suppose that $\Pr(\{0\}) = 0.2$, $\Pr(\{1\}) = 0.3$, $\Pr(\{2\}) = 0.4$, $\Pr(\{3\}) = 0.1$ and $\Pr(\{4\}) = 0.0$. Find Pr(A) and Pr($A^c$) if A=$\{0,1,2\}$ and Pr(B) if B=$\{0,2,4\}$. Is $\Pr(A \cup B) = \Pr(A)+\Pr(B)$? Why or why not?

• EXERCISE **6.4.6**

The sample space is S=$\{0,1,2,3,4\}$. Suppose that $\Pr(\{0\}) = 0.1$, $\Pr(\{1\}) = 0.3$, $\Pr(\{2\}) = 0.4$, $\Pr(\{3\}) = 0.1$ and $\Pr(\{4\}) = 0.1$. Find Pr(A) and Pr($A^c$) if A=$\{0,2\}$ and Pr(B) if B=$\{3,4\}$. Is $\Pr(A \cup B) = \Pr(A) + \Pr(B)$? Why or why not?

• EXERCISE **6.4.7**

The sample space is S=$\{0,1,2,3,4\}$. Suppose that $\Pr(\{0\}) = 0.2$, $\Pr(\{1\}) = 0.1$, $\Pr(\{2\}) = 0.4$ and $\Pr(\{3\}) = 0.1$. Find Pr(A) and Pr($A^c$) if A=$\{4\}$ and Pr(B) if B=$\{3,4\}$.

• EXERCISE **6.4.8**

The sample space is S=$\{0,1,2,3,4\}$. Suppose that $\Pr(\{0\}) = 0.2$, $\Pr(\{0,1\}) = 0.3$, $\Pr(\{0,1,2\}) = 0.5$ and $\Pr(\{0,1,2,3\}) = 0.8$. Find $\Pr(\{1\})$, $\Pr(\{2\})$, $\Pr(\{3\})$ and $\Pr(\{4\})$.

♠ Draw Venn diagrams with sets A, B and C satisfying the following requirements.

• EXERCISE **6.4.9**

A and B disjoint, B and C disjoint, A and C not disjoint.

• EXERCISE **6.4.10**

A and B disjoint, B and C not disjoint, A and C not disjoint.

• EXERCISE **6.4.11**

No two sets disjoint, but $A \cap B \cap C$ empty.

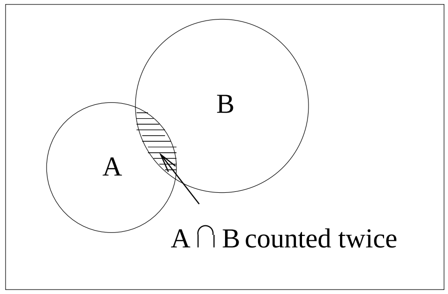
• EXERCISE **6.4.12**

No two sets disjoint, and $A \cap B \cap C$ non-empty.

♠ The following formula gives the probability of the union of two events,

$$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B).$$

As indicated in the figure, adding the area in A and B counts the area in the intersection $A \cap B$ twice. Subtracting the area of the intersection once corrects for this double counting.




Test this formula on the following examples.

• EXERCISE **6.4.13**

The sets A and B in exercise 6.4.5.

• EXERCISE **6.4.14**

The sets A and B in exercise 6.4.6.

• EXERCISE **6.4.15**

Using the probabilities in exercise 6.4.5, check the formula on the sets $C = \{1, 2, 3\}$ and $D = \{0, 1, 2\}$.

• EXERCISE **6.4.16**

Using the probabilities in exercise 6.4.6, check the formula on the sets $C = \{2, 3, 4\}$ and $D = \{0, 1, 2\}$.

## APPLICATIONS

♠ Give the sample spaces associated with the following experiments. Say how many simple events there are and list them if there are fewer than 10. If there are more than 10, list three simple events.

• EXERCISE **6.4.17**

We cross two plants with genotype **bB** and check the genotype of one offspring.

• EXERCISE **6.4.18**

We cross two plants with genotype **bB** and check the genotype of two offspring.

• EXERCISE **6.4.19**

A molecule jumps in and out of a cell. We record whether the molecule is inside or outside the cell at times 1 and 5.

• EXERCISE **6.4.20**

A molecule jumps in and out of a cell. We record whether the molecule is inside or outside the cell at times 2, 5 and 10.

• EXERCISE **6.4.21**

Two molecules jump in and out of a cell. We record how many molecules are inside at times 1 and 5.

• EXERCISE **6.4.22**

Three molecules jump in and out of a cell. We record how many molecules are inside at times 3 and 5.

• EXERCISE **6.4.23**

We count how many out of 16 plants are taller than 50 cm.

• EXERCISE **6.4.24**

We measure the heights of 2 plants.

♠ We start 100 molecules in a cell and count the number, $N$, that remain after 10 minutes. Give 5 simple events which are included in the following events.

• EXERCISE **6.4.25**

$N < 10$.

• EXERCISE **6.4.26**

$N > 90$.

• EXERCISE **6.4.27**

$N$ is odd.

• EXERCISE **6.4.28**

$30 \leq N \leq 32$ or $68 \leq N \leq 70$.

♠ We start 100 molecules in a cell and count the number, $N$, that remain after 10 minutes. Find the union and intersection of the following events.

- EXERCISE **6.4.29**
  Event A is $N < 10$ and event B is $N > 5$.
- EXERCISE **6.4.30**
  Event A is $N > 10$ and event B is $N < 5$.
- EXERCISE **6.4.31**
  Event A is $N > 10$ and event B is $N > 5$.
- EXERCISE **6.4.32**
  Event A is $20 > N > 10$ and event B is $15 > N > 5$.

♠ We follow 4 individually labeled molecules and record the minute $t_i$ when molecule $i$ leaves the cell. For example, if $t_1 = 1$ $t_2 = 3$, $t_3 = 6$ and $t_4 = 2$, the first molecule left during minute 1, the second left during minute 3, the third left during minute 6, and the fourth left during minute 2. Give 3 simple events which are included in the following events.

- EXERCISE **6.4.33**
  All molecules left before minute 5.
- EXERCISE **6.4.34**
  Molecules 1, 2 and 4 left before minute 5 and molecule 3 left after minute 7.
- EXERCISE **6.4.35**
  All odd numbered molecules left at odd times.
- EXERCISE **6.4.36**
  All odd numbered molecules left at odd times and all even numbered molecules left at even times.

♠ Give two mathematically consistent ways of assigning probabilities to the results of the following experiments. Try to make one of your assignments biologically reasonable.

- EXERCISE **6.4.37**
  The situation in exercise 6.4.17.
- EXERCISE **6.4.38**
  The situation in exercise 6.4.18.

♠ Give two assignments, different from those in the text, of probabilities when counting the number of molecules inside a cell starting from an initial number of 3. Compute $\Pr(N \text{ is odd})$ and $\Pr(N \neq 1)$ in each case.

- EXERCISE **6.4.39**
  Create an assignment where $\Pr(N = 1)$ is larger than the probability of any other simple event.
- EXERCISE **6.4.40**
  Create an assignment where $\Pr(N = 1)$ is equal to the probability of each other simple event.
- EXERCISE **6.4.41**
  Create an assignment where $\Pr(N = 1)$ is smaller than the probability of any other simple event, but not equal to 0.
- EXERCISE **6.4.42**
  Create an assignment where the probabilities get larger as the number of molecules gets larger.

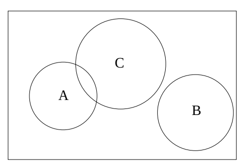
# Chapter 7

# Answers

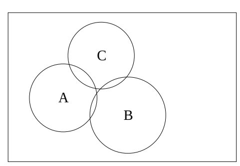
**6.4.1.** A$\cap$ B $= \{0, 2\}$, A$\cup$B $= \{0, 1, 2, 4\}$, $A^c = \{3, 4\}$.
**6.4.3.** A$\cap$ B $= \{2\}$, A$\cup$B $= \{1, 2, 4, 5, 6, 10\}$, $A^c$ = all positive integers except 1, 2, 6 and 10.
**6.4.5.** The probabilities are all positive, and add up to 1. Because we assigned probabilities to the simple events, requirement 3 will take care of itself. $\Pr(A) = 0.2 + 0.2 + 0.4 = 0.9$, $\Pr(A^c) = 1 - \Pr(A) = 1 - 0.9 = 0.1$ and $\Pr(B) = 0.2+0.4+0.0 = 0.6$. Also, $A \cup B = \{0, 1, 2, 4\}$ so $\Pr(A \cup B) = 0.2+0.3+0.4+0.0 = 0.9 \neq \Pr(A)+\Pr(B)$ This is because A and B are not disjoint.
**6.4.7.** The probabilities are all positive. We are not given the probability of the simple event A=$\{4\}$, so we use the requirement that the probabilities add up to 1.

$$\Pr(\{0\}) + \Pr(\{1\}) + \Pr(\{2\}) + \Pr(\{3\}) + \Pr(\{4\}) = 0.2 + 0.1 + 0.4 + 0.1 + \Pr(\{4\}) = 0.8 + \Pr(\{4\}) = 1$$

Therefore, $\Pr(\{4\}) = 1 - 0.8 = 0.2$. $\Pr(A^c) = 1 - \Pr(A) = 1 - 0.2 = 0.8$ and $\Pr(B) = 0.1 + 0.2 = 0.3$.
**6.4.9.**




**6.4.11.**




**6.4.13.** We found that $\Pr(A) = 0.9$, $\Pr(B) = 0.6$ and $\Pr(A \cup B) = 0.9$. Also, $\Pr(A \cap B) = \Pr(\{0{,}2\}) = 0.2{+}0.4 = 0.6$. The formula checks, because $\Pr(A \cup B) = 0.9 = \Pr(A) + \Pr(B) - \Pr(A \cap B) = 0.9 + 0.6 - 0.6$.
**6.4.15.** $\Pr(C) = 0.3 + 0.4 + 0.1 = 0.8$, $\Pr(D) = 0.2 + 0.3 + 0.4 = 0.9$ and $\Pr(C \cup D) = 0.2 + 0.3 + 0.4 + 0.1 = 1.0$, Also, $\Pr(C \cap D) = \Pr(\{1{,}2\}) = 0.3 + 0.4 = 0.7$. The formula checks, because $\Pr(C \cup D) = 1.0 = \Pr(C) + \Pr(D) - \Pr(C \cap D) = 0.8 + 0.9 - 0.7$.
**6.4.17.** Three possible genotypes: **bb**, **Bb** and **BB**.
**6.4.19.** There are 4 simple events: $\{\text{In}, \text{In}\}$, $\{\text{In}, \text{Out}\}$, $\{\text{Out}, \text{In}\}$ and $\{\text{Out}, \text{Out}\}$.
**6.4.21.** There are 9 simple events, described by the number in (which can be 0, 1 or 2) at each time: $\{0, 0\}$, $\{0, 1\}$, $\{0, 2\}$, $\{1, 0\}$, $\{1, 1\}$, $\{1, 2\}$, $\{2, 0\}$, $\{2, 1\}$ and $\{2, 2\}$.
**6.4.23.** There are 17 simple events, the numbers ranging from 0 to 16.
**6.4.25.** $\{N = 1\}$, $\{N = 2\}$, $\{N = 3\}$, $\{N = 4\}$, $\{N = 5\}$.
**6.4.27.** $\{N = 1\}$, $\{N = 3\}$, $\{N = 5\}$, $\{N = 7\}$, $\{N = 9\}$.
**6.4.29.** The union is all numbers between 0 and 100, the intersection is the values 6, 7, 8 and 9.

**6.4.31.** The union is all numbers greater than 5 and less than or equal to 100, and the intersection is all numbers greater than 10 and less than or equal to 100.
**6.4.33.** This requires that $t_1 < 5$, $t_2 < 5$, $t_3 < 5$ and $t_4 < 5$. Three possible ways this could happen are $\{t_1 = 1, t_2 = 1, t_3 = 1, t_4 = 1\}$, $\{t_1 = 1, t_2 = 2, t_3 = 3, t_4 = 4\}$ and $\{t_1 = 4, t_2 = 3, t_3 = 2, t_4 = 1\}$.
**6.4.35.** The odd numbered molecules are 1 and 3. Possibilities are $\{t_1 = 1, t_2 = 1, t_3 = 3, t_4 = 1\}$, $\{t_1 = 1, t_2 = 1, t_3 = 5, t_4 = 2\}$, $\{t_1 = 1, t_2 = 2, t_3 = 5, t_4 = 1\}$.
**6.4.37.** The biologically reasonable assignment is Pr(bb) = 0.25, Pr(Bb) = 0.5, Pr(BB) = 0.25. Another possibility is Pr(bb) = 0.5, Pr(Bb) = 0.25, Pr(BB) = 0.25.
**6.4.39.** We could set $\Pr(N = 1) = 1$ and all the rest to 0. Then $\Pr(N \text{ is odd}) = 1$ and $\Pr(N \neq 1) = 0$.
**6.4.41.** We could set $\Pr(N = 1) = 0.1$ and all the rest to 0.3. Then $\Pr(N \text{ is odd}) = 0.4$ and $\Pr(N \neq 1) = 0.9$.