5.7 Solutions in the phase-plane

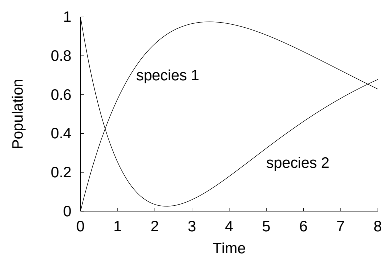
MATHEMATICAL TECHNIQUES

♠ Suppose the following functions are solutions of some differential equation. Graph these as functions of time and as phase-plane trajectories for $0 \le t \le 2$. Mark the position at $t = 0$, $t = 1$ and $t = 2$.

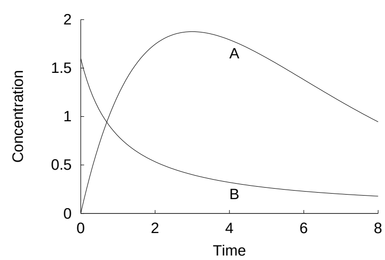
- EXERCISE **5.7.1**
  $x(t) = t$, $y(t) = 3t$.
- EXERCISE **5.7.2**
  $a(t) = 2e^{-t}$, $b(t) = e^{-2t}$.
- EXERCISE **5.7.3**
  $f(t) = 1 + t$, $g(t) = e^{-t}$.
- EXERCISE **5.7.4**
  $x(t) = t - 2t^2 + t^3$, $y(t) = 4 - t^2$.

♠ From the following graphs of solutions of differential equations as functions of time, graph the matching phase-plane trajectory.

- EXERCISE **5.7.5**

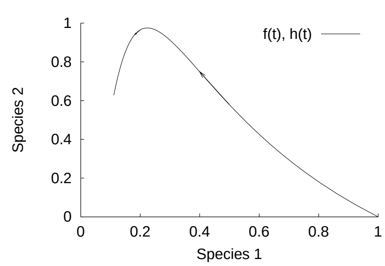


- EXERCISE **5.7.6**

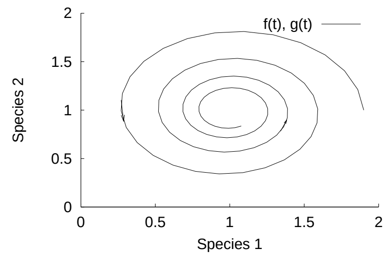


♠ From the following graphs of phase-plane trajectories graph the matching solutions of differential equations as functions of time.

- EXERCISE **5.7.7**

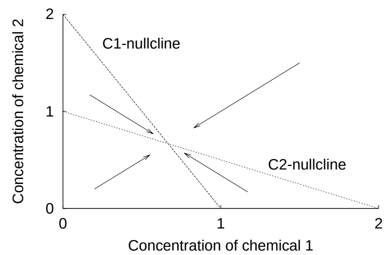


- EXERCISE **5.7.8**

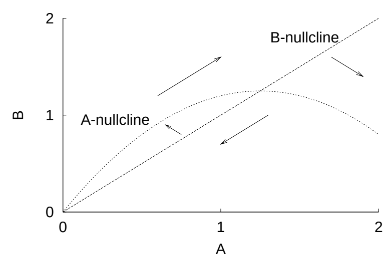


♠ On the following phase-plane diagrams, use the direction arrows to sketch phase-plane trajectories starting from two different initial conditions.

• EXERCISE **5.7.9**



• EXERCISE **5.7.10**



♠ Use the information in the phase-plane diagram to draw direction arrows on the nullclines.

• EXERCISE **5.7.11**

The diagram in exercise 5.7.9.

• EXERCISE **5.7.12**

The diagram in exercise 5.7.10.

♠ Compare solutions estimated with Euler's method with the phase-plane diagram and direction arrows found in the text for the competition equations

$$\frac{da}{dt} = \mu(1 - \frac{a+b}{K_a})a$$
$$\frac{db}{dt} = \lambda(1 - \frac{a+b}{K_b})b$$

starting from the given initial conditions. Assume that $\mu = 2.0$, $\lambda = 2.0$, $K_a = 1000$ and $K_b = 500$.

• EXERCISE **5.7.13**

Start from $a = 750$ and $b = 500$. Take two steps, with a step length of $\Delta t = 0.1$, as in exercise 5.5.9.

• EXERCISE **5.7.14**

Start from $a = 250$ and $b = 500$. Take two steps, with a step length of $\Delta t = 0.1$, as in exercise 5.5.10.

♠ Compare solutions estimated with Euler's method with the phase-plane diagram and direction arrows found in the text for Newton's law of cooling

$$\begin{aligned}\frac{dH}{dt} &= \alpha(A-H)\\ \frac{dA}{dt} &= \alpha_2(H-A)\end{aligned}$$

with the given parameter values and starting from the given initial conditions.

- EXERCISE **5.7.15**

Suppose $\alpha = 0.3$ and $\alpha_2 = 0.1$. Start from $H = 60$ and $A = 20$. Take two steps, with a step length of $\Delta t = 0.1$, as in exercise 5.5.13.

- EXERCISE **5.7.16**

Suppose $\alpha = 0.3$ and $\alpha_2 = 0.1$. Start from $H = 0$ and $A = 20$. Take two steps, with a step length of $\Delta t = 0.1$, as in exercise 5.5.14.

- EXERCISE **5.7.17**

Suppose $\alpha = 3.0$ and $\alpha_2 = 1.0$. Start from $H = 60$ and $A = 20$. Take two steps, with a step length of $\Delta t = 0.25$, as in exercise 5.5.15. Does this diagram help explain what went wrong?

- EXERCISE **5.7.18**

Suppose $\alpha = 3.0$ and $\alpha_2 = 1.0$. Start from $H = 0$ and $A = 20$. Take two steps, with a step length of $\Delta t = 0.5$, as in exercise 5.5.16. Does this diagram help explain what went wrong?

♠ Draw the nullclines and direction arrows for the following models of springs. Make sure to include positive and negative values for the position x and the velocity v.

- EXERCISE **5.7.19**

The model in exercise 5.5.17.

- EXERCISE **5.7.20**

The model in exercise 5.5.18.

♠ Sketch the given solution of the following models of springs first as a pair of functions of time, and then in the phase-plane. Check that the solution follows the arrows.

- EXERCISE **5.7.21**

The solution $x(t) = \cos(t)$ (exercise 5.5.19) of the spring equation in exercise 5.5.17.

- EXERCISE **5.7.22**

The solution $x(t) = e^{-t}\cos(t)$ (exercise 5.5.20) of the spring equation with friction in exercise 5.5.18.

APPLICATIONS

♠

For the following problems, add direction arrows to the phase-plane.

- EXERCISE **5.7.23**

The model in exercise 5.5.21.

- EXERCISE **5.7.24**

The model in exercise 5.5.22.

- EXERCISE **5.7.25**

The model in exercise 5.5.25.

- EXERCISE **5.7.26**

The model in exercise 5.5.26.

- EXERCISE **5.7.27**

The model in exercise 5.5.27.

- EXERCISE **5.7.28**

The model in exercise 5.5.28.

- EXERCISE **5.7.29**

The model in exercise 5.6.33.

- EXERCISE **5.7.30**
  The model in exercise 5.6.34.
- EXERCISE **5.7.31**
  The model in exercise 5.6.35.
- EXERCISE **5.7.32**
  The model in exercise 5.6.36.
- EXERCISE **5.7.33**
  The model in exercise 5.6.37.
- EXERCISE **5.7.34**
  The model in exercise 5.6.38.
- EXERCISE **5.7.35**
  The model in exercise 5.6.39.
- EXERCISE **5.7.36**
  The model in exercise 5.6.40.

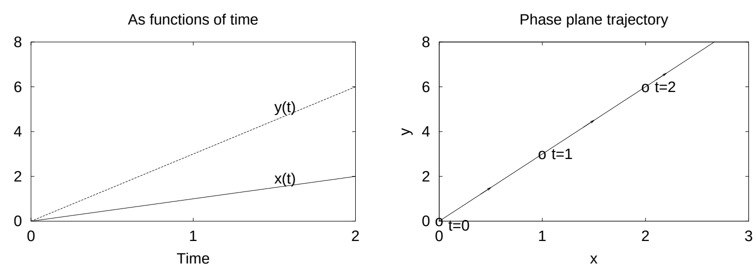
♠ For the following problems, use the direction arrows on your phase-plane to sketch a solution starting from the given initial condition.

- EXERCISE **5.7.37**
  The model in exercise 5.7.25 starting from (1500,200). Is there another path for the solution that is consistent with the direction arrows?.
- EXERCISE **5.7.38**
  The model in exercise 5.7.26 starting from (1500,200).
- EXERCISE **5.7.39**
  The model in exercise 5.7.27 starting from (200,300).
- EXERCISE **5.7.40**
  The model in exercise 5.7.28 starting from (200,300).
- EXERCISE **5.7.41**
  The model in exercise 5.7.31 starting from (0.5,1).
- EXERCISE **5.7.42**
  The model in exercise 5.7.32 starting from (0.5,1).
- EXERCISE **5.7.43**
  The model in exercise 5.7.35 starting from (0.5,1).
- EXERCISE **5.7.44**
  The model in exercise 5.7.36 starting from (0.5,0,5). Can you be sure that the solution behaves exactly like your picture?

Chapter 6

Answers

5.7.1.



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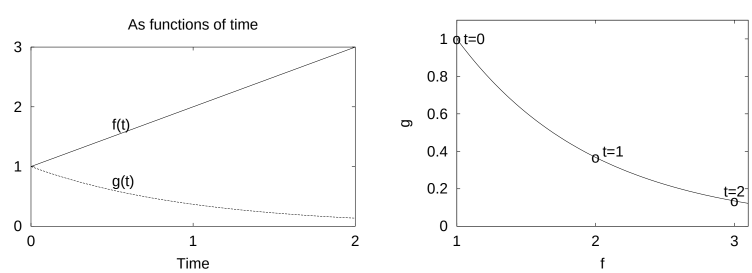

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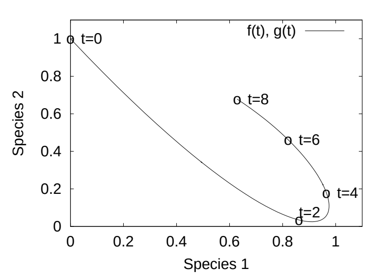

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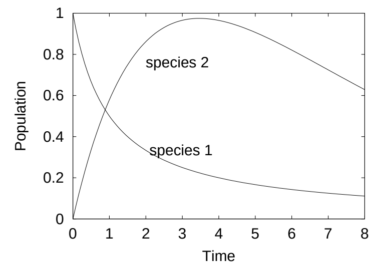

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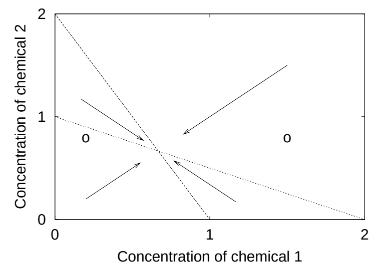

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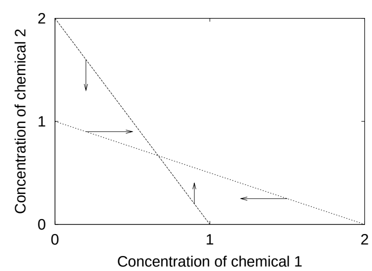
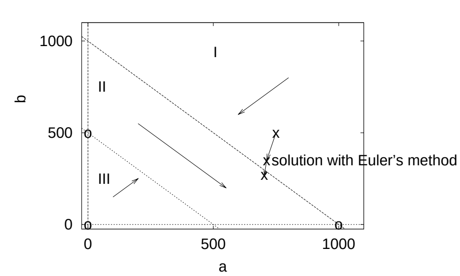
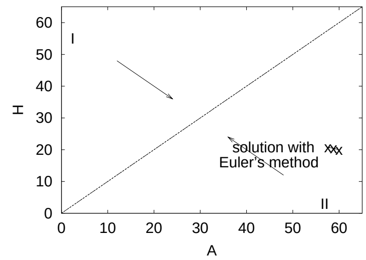


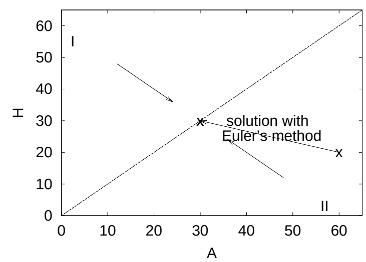
5.7.13. We found that $(\hat{a}(0.1), \hat{b}(0.1)) = (712.5, 350)$, with both values having decreased. This is consistent with the fact that our initial conditions lie in Region I (because $750+500 > 1000$). Similarly, $(\hat{a}(0.2), \hat{b}(0.2)) = (703.6, 271.3)$, with both values having decreased. This is consistent with the fact that our initial conditions lie in Region I (because $712.5 + 350 > 1000$). However, the solution has now moved into region II and a will begin to increase.



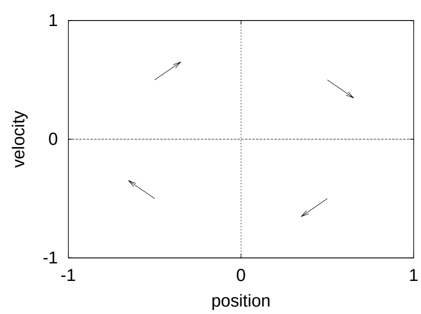
5.7.15. We found that $(\hat{H}(0.1), \hat{A}(0.1)) = (58.8, 20.4)$, with H having decreased and A increased. This is consistent with the fact that our initial conditions lie in Region II (because $60 > 20$). Similarly, $(\hat{H}(0.2), \hat{A}(0.2)) = (57.6, 20.8)$ because our first step landed in region II.



5.7.17. We found that $(\hat{H}(0.1), \hat{A}(0.1)) = (30.0, 30.0)$, with H having decreased and A increased. This is consistent with the fact that our initial conditions lie in Region II (because $60 > 20$). However, $(\hat{H}(0.2), \hat{A}(0.2)) = (30.0, 30.0)$ because our first step landed right on the equilibrium. This overly long jump is a consequence of taking too big a step.



5.7.19. The x-nullcline is $v = 0$ and the v-nullcline is $x = 0$. These break the phase-plane into four regions. The position is increasing when the velocity is positive, and the velocity is decreasing when the position is positive.



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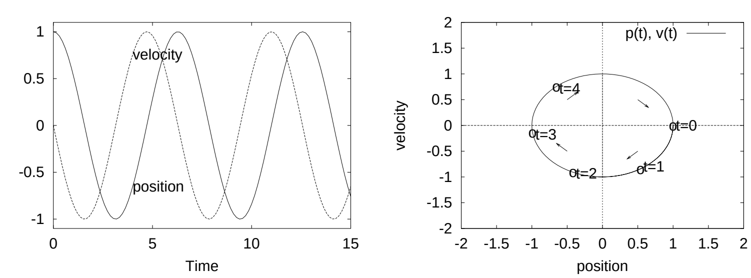

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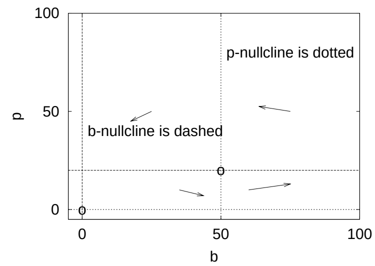

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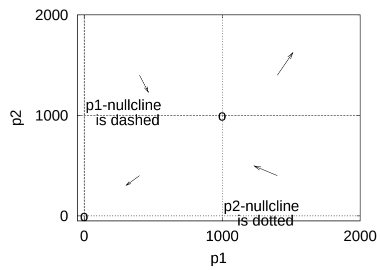
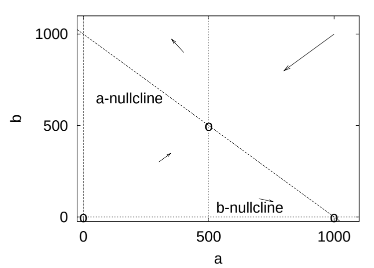


5.7.27.



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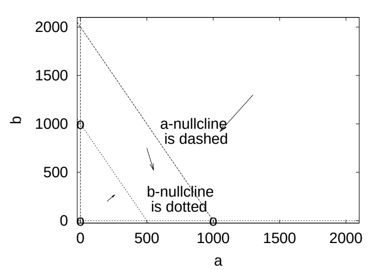

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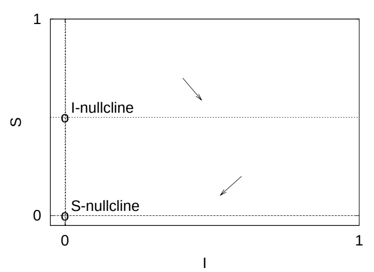
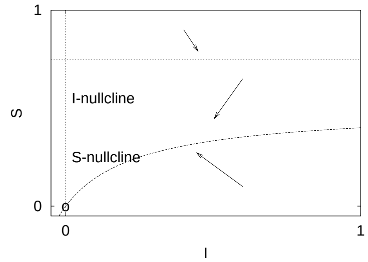


5.7.33.



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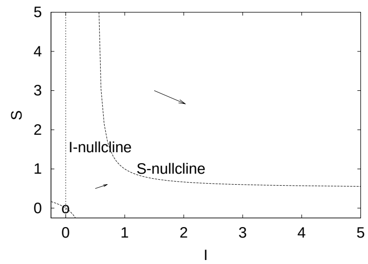

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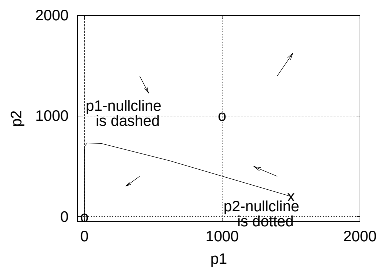
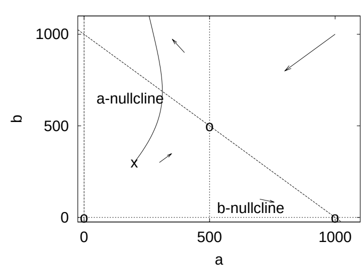


It is also possible that the solution goes to the right of the equilibrium and shoots off the upper right hand part of the phase-plane.

5.7.39.



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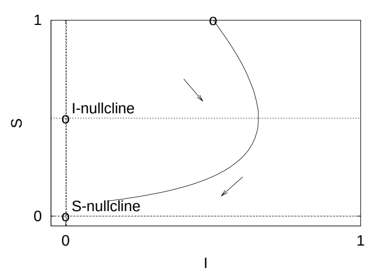

5.7.43.