5.6 The Phase-Plane

MATHEMATICAL TECHNIQUES

♠ Finding equilibria coupled differential equations requires solving **simultaneous equations**. The following are linear equations, where the only possibilities are 0 solutions, 1 solution, or a whole line of solutions. For each of the following pairs, solve each equation for y in terms of x, set the two equations for x equal, and solve for x. Check that both equations give the same value for y. Sketch a graph with y on the vertical axis.

• EXERCISE **5.6.1**

$$\begin{aligned} -3 - y + 3x &= 0 \\ -2 + 2y - 4x &= 0 \end{aligned}$$

• EXERCISE **5.6.2**

$$\begin{aligned} -6 + 3y + 3x &= 0 \\ -2 - 2y - 6x &= 0 \end{aligned}$$

• EXERCISE **5.6.3**

$$\begin{aligned} 3 - 3y + 3x &= 0 \\ -2 + 2y - 2x &= 0 \end{aligned}$$

What goes wrong? Use your graph with y on the vertical axis to explain the problem.

• EXERCISE **5.6.4**

$$\begin{aligned} 8 - 2y + 4x &= 0 \\ -2 + y - 2x &= 0 \end{aligned}$$

What goes wrong? Use your graph with y on the vertical axis to explain the problem.

♠ Finding equilibria nonlinear coupled differential equations requires solving nonlinear simultaneous equations which can have any number of solutions. For each of the following pairs, solve each equation for y in terms of x, set the two equations for x equal, and solve for x. Check that both equations give the same value for y. Sketch a graph with y on the vertical axis.

• EXERCISE **5.6.5**

$$\begin{aligned} -5 - y + 3x^2 + 2x &= 0 \\ -2 + 2y - 4x &= 0 \end{aligned}$$

• EXERCISE **5.6.6**

$$\begin{aligned} -3 - y + 3x^2 + 2x &= 0 \\ -2 + 2y - 4x + 2x^2 &= 0 \end{aligned}$$

• EXERCISE **5.6.7**

$$\begin{aligned} -y^2 + x^2 &= 0 \\ -2 + 2y - 4x &= 0 \end{aligned}$$

Solving the first equation for y in terms of x does not give a function. Graph the relation and find the solutions.

• EXERCISE **5.6.8**

$$\begin{aligned} y(y - x^2) &= 0 \\ -6 + 2y - 4x &= 0 \end{aligned}$$

Solving the first equation for y in terms of x does not give a function. Graph the relation and find the solutions.

• EXERCISE **5.6.9**

$$\begin{aligned} x(y - x) &= 0 \\ -6 + 2y - 4x &= 0 \end{aligned}$$

Solving the first equation for y in terms of x does not give a function (and includes a vertical section). Graph the relation and find the solutions.

• EXERCISE **5.6.10**

$$\begin{aligned} (x - 1)(y^2 - x^2) &= 0 \\ -2 + 2y - 6x &= 0 \end{aligned}$$

Solving the first equation for y in terms of x does not give a function and includes a vertical section. Graph the relation and find the solutions.

♠ Graph the nullclines in the phase-plane and find the equilibria of the following.

• EXERCISE **5.6.11**

Predator-prey model with $\lambda = 1.0$, $\delta = 3.0$, $\epsilon = 0.002$, $\eta = 0.005$.

• EXERCISE **5.6.12**

Predator-prey model with $\lambda = 1.0$, $\delta = 3.0$, $\epsilon = 0.005$, $\eta = 0.002$.

• EXERCISE **5.6.13**

Newton's law of cooling with $\alpha = 0.01$ and $\alpha_2 = 0.1$.

• EXERCISE **5.6.14**

Newton's law of cooling with $\alpha = 0.1$ and $\alpha_2 = 0.5$.

• EXERCISE **5.6.15**

Competition model with $\lambda = 2.0$, $\mu = 1.0$, $K_a = 10^6$, $K_b = 10^7$.

• EXERCISE **5.6.16**

Competition model with $\lambda = 1.0$, $\mu = 2.0$, $K_a = 10^6$, $K_b = 10^7$. How do the results compare with those in exercise 5.6.15? Why?

♠ Redraw the phase-planes for the following problems but make the other choice for the vertical variable. Check that you get the same equilibrium.

• EXERCISE **5.6.17**

The equations in exercise 5.6.11.

• EXERCISE **5.6.18**

The equations in exercise 5.6.12.

• EXERCISE **5.6.19**
The equations in exercise 5.6.15.
• EXERCISE **5.6.20**
The equations in exercise 5.6.16.

♠ If each state variable in a system of autonomous differential equations does not respond to changes in the value of the other, but depends only on a constant value, the two equations can be considered separately. In this case, the phase-plane is particularly simple. Find the nullclines and equilibria in the following cases.

• EXERCISE **5.6.21**
The situation in exercise 5.5.7.
• EXERCISE **5.6.22**
The situation in exercise 5.5.8.

APPLICATIONS

♠ Find the nullclines and equilibria for the following predator-prey models.

• EXERCISE **5.6.23**
The model in exercise 5.5.21.
• EXERCISE **5.6.24**
The model in exercise 5.5.22.
• EXERCISE **5.6.25**
The model in exercise 5.5.23.
• EXERCISE **5.6.26**
The model in exercise 5.5.24.

♠ Find and graph the nullclines, and find the equilibria for the following models.

• EXERCISE **5.6.27**
The model found in exercise 5.5.25.
• EXERCISE **5.6.28**
The model found in exercise 5.5.26.
• EXERCISE **5.6.29**
The model found in exercise 5.5.27.
• EXERCISE **5.6.30**
The model found in exercise 5.5.28.

♠ The models of diffusion derived in exercises 5.5.5 and 5.5.6 assume that the membrane between the vessels is equally permeable in both directions. Suppose instead that the constant of proportionality governing the rate at which chemical moves differs in the two directions. In each of the following cases,

a. Find the rate at which chemical moves from the smaller to the larger vessel.

b. Find the rate at which chemical moves from the larger to the smaller vessel.

c. Find the rate of change of the amount of chemical in each vessel.

d. Divide by the volumes V_1 and V_2 to find the rate of change of concentration.

e. Find and graph the nullclines.

f. What are the equilibria? Do they make sense?

• EXERCISE **5.6.31**
The constant of proportionality governing the rate at which chemical enters the cell is three times as large as the constant governing the rate at which it leaves (as in exercise 5.1.33).
• EXERCISE **5.6.32**
The constant of proportionality governing the rate at which chemical enters the cell is half as large as the constant governing the rate at which it leaves (as in exercise 5.1.34).

♠ In our model of competition, the per capita growth rate of types a and b are functions only of the total population size. This means that reproduction is reduced just as much by an individual of type a as by an individual of type b. In many systems, each type interferes differently with type a than with type b. Check that the given set of equation matches the assumptions in each of the following cases, and find and graph the equilibria and nullclines.

• EXERCISE **5.6.33**

Suppose that individuals of type b reduce the per capita growth rate of type a by half as much as individuals of type a, and that individuals of type a reduce the per capita type b growth rate by twice as much as individuals of type b. The equations are

$$\begin{aligned}\frac{da}{dt} &= (1 - \frac{a + b/2}{1000})a \\ \frac{db}{dt} &= (1 - \frac{2a + b}{1000})b.\end{aligned}$$

• EXERCISE **5.6.34**

Suppose that individuals of type b reduce the per capita type a growth rate by half as much as individuals of type a, and that individuals of type a reduce the per capita type b growth rate by half as much as individuals of type b. The equations are

$$\begin{aligned}\frac{da}{dt} &= (1 - \frac{a + b/2}{1000})a \\ \frac{db}{dt} &= (1 - \frac{a/2 + b}{1000})b.\end{aligned}$$

(There should be four equilibria).

♠ Draw the nullclines and find equilibria of the following extensions of the basic disease model.

• EXERCISE **5.6.35**

The model in exercise 5.5.31. Find the nullclines and equilibria of this model when $\alpha = 2.0$ and $\mu = 1.0$.

• EXERCISE **5.6.36**

The model in exercise 5.5.32. Find the nullclines and equilibria of this model when $\alpha = 2.0$ and $\mu = 1.0$.

• EXERCISE **5.6.37**

The model in exercise 5.5.33. Find the nullclines and equilibria of this model when $\alpha = 2.0$, $\mu = 1.0$, and $k = 0.5$.

• EXERCISE **5.6.38**

The model in exercise 5.5.33. Find the nullclines and equilibria of this model when $\alpha = 2.0$, $\mu = 1.0$, and $k = 4.0$.

• EXERCISE **5.6.39**

The model in exercise 5.5.35. Find the nullclines and equilibria of this model when $\alpha = 2.0$, $\mu = 1.0$, and $b = 2.0$.

• EXERCISE **5.6.40**

The model in exercise 5.5.36. Find the nullclines and equilibria of this model when $\alpha = 2.0$, $\mu = 1.0$, and $b = 1.0$.

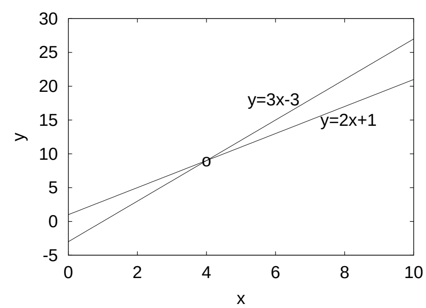
Chapter 6

Answers

5.6.1. Solving each equation for y gives

$$
\begin{aligned}
y &= 3x - 3 \\
y &= 2x + 1.
\end{aligned}
$$

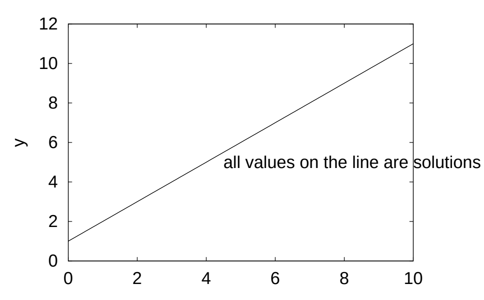
Setting equal, we find $3x - 3 = 2x = 1$, so $x = 4$. Plugging in gives $y = 9$ in each equation.



5.6.3. Solving each equation for y gives

$$
\begin{aligned}
y &= x + 1 \\
y &= x + 1.
\end{aligned}
$$

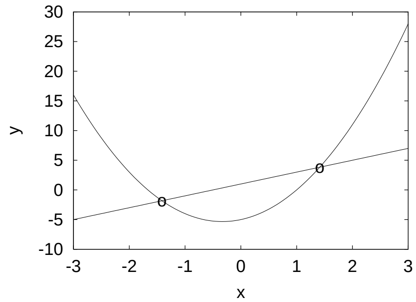
Setting equal, we find $x + 1 = x + 1$. Any value of x is a solution. This happens because the two lines are identical, and any value where $y = x + 1$ is a solution.



5.6.5. Solving each equation for y gives

$$
\begin{aligned}
y &= 3x^2 + 2x - 5 \\
y &= 2x + 1.
\end{aligned}
$$

Setting equal, we find $3x^2 + 2x - 5 = 2x = 1$, so $3x^2 = 6$, and $x = \pm\sqrt{2}$. With $x = \sqrt{2}$, the first equation gives $y = 1 + 2\sqrt{2}$, as does the second. With $x = -\sqrt{2}$, the first equation gives $y = 1 - 2\sqrt{2}$, as does the second.



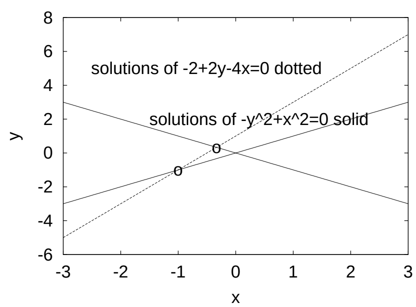
5.6.7. Solving the first equation for y gives $y = x$ **and** $y = -x$. The second gives $y = 2x + 1$. This gives two pairs of equations:

$$\begin{aligned} y &= x \\ y &= 2x + 1 \end{aligned}$$

and

$$\begin{aligned} y &= -x \\ y &= 2x + 1 \end{aligned}$$

Setting the first pair equal, we find $x = 2x + 1$, so $x = -1$ and $y = -1$. Setting the second pair equal, we find $-x = 2x + 1$, so $x = -1/3$ and $y = 1/3$.



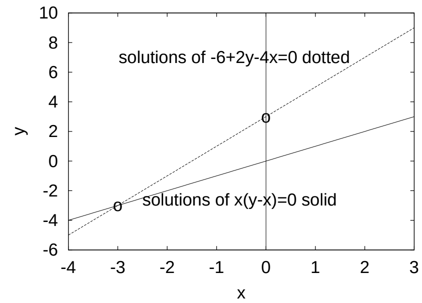
5.6.9. Solving the first equation for y gives $y = x$ and $x = 0$. The second gives $y = 2x + 3$. This gives two pairs of equations:

$$\begin{aligned} y &= x \\ y &= 2x + 3 \end{aligned}$$

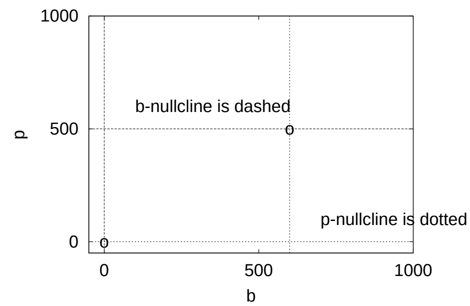
and

$$\begin{aligned} x &= 0 \\ y &= 2x + 3 \end{aligned}$$

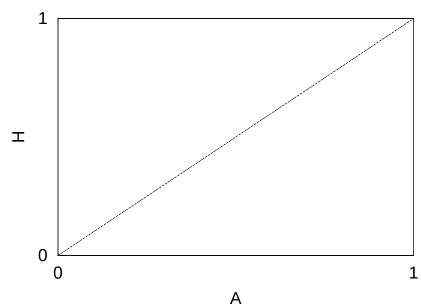
Setting the first pair equal, we find $x = 2x + 3$, so $x = -3$ and $y = -3$. We can substitute $x = 0$ into the second equation of the second pair to find $y = 3$.



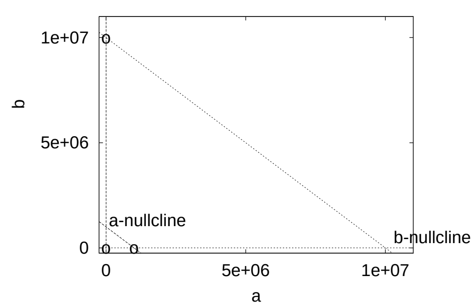
5.6.11. The p-nullcline consists of the two pieces $p = 0$ and $b = \delta/\eta = 600$. The b-nullcline consists of the two pieces $b = 0$ and $p = \lambda/\epsilon = 500$. The equilibria are $(0, 0)$ and $(600, 500)$.



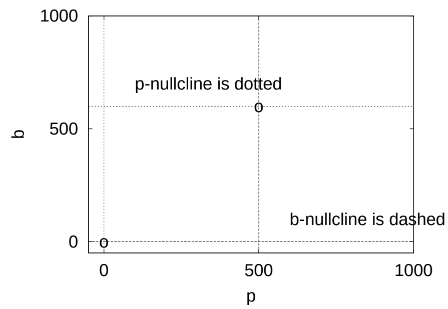
5.6.13. Both nullclines are the line $H = A$, which consists entirely of equilibria.



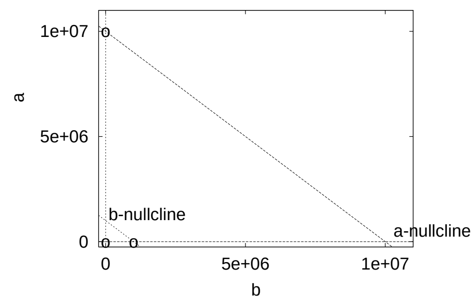
5.6.15. Place b on the vertical axis. The a-nullcline is the two pieces $a = 0$ and $b = 10^6 - a$. The b-nullcline is the two pieces $b = 0$ and $b = 10^7 - a$. The equilibria are $(0, 0)$, $(10^6, 0)$ and $(0, 10^7)$.



5.6.17. Place b on the vertical axis. The p-nullcline consists of the two pieces $p = 0$ and $b = 600$ and the b-nullcline consists of the two pieces $b = 0$ and $p = 500$ as before. The equilibria are $(0, 0)$ and $(500, 600)$.



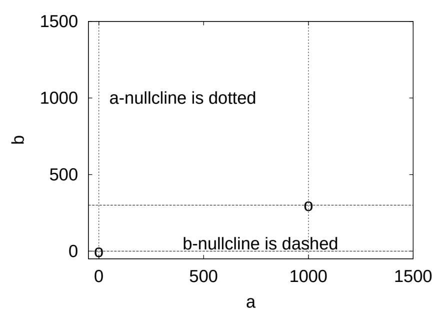
5.6.19. Place a on the vertical axis. The a-nullcline is the two pieces $a = 0$ and $a = 10^6 - b$. The b-nullcline is the two pieces $b = 0$ and $a = 10^7 - b$. The equilibria are $(0, 0)$, $(0, 10^6)$ and $(10^7, 0)$.



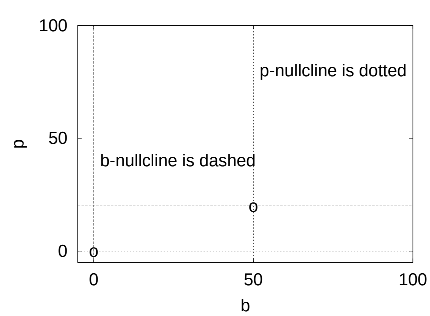
5.6.21. The equations are

$$\frac{da}{dt} = 2(1 - \frac{a}{1000})a$$
$$\frac{db}{dt} = 3(1 - \frac{b}{300})b,$$

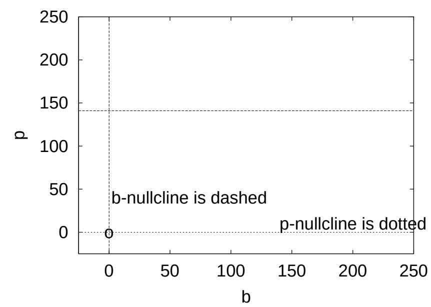
The a-nullcline is $a = 0$ and $a = 1000$. The b-nullcline is $b = 0$ and $b = 300$. The equilibria are $(0, 0)$ and $(1000, 300)$.



5.6.23. With p on the vertical axis, the b-nullcline is $b = 0$ and $p = 20$. The p-nullcline is $p = 0$ or $b = 50$ (which is absurd). The only equilibria are at $(0, 0)$ and $(50, 20)$.



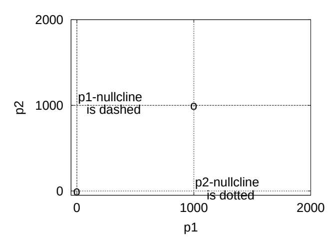
5.6.25. With p on the vertical axis, the b-nullcline is $b = 0$ and $p = \sqrt{20000} = 141$. The p-nullcline is $p = 0$ or $b = -100$ (which is absurd). The only equilibrium is at $(0, 0)$.



5.6.27. We found equations

$$\frac{dp_1}{dt} = (-1.0 + 0.001p_2)p_1$$
$$\frac{dp_2}{dt} = (-1.0 + 0.001p_1)p_2.$$

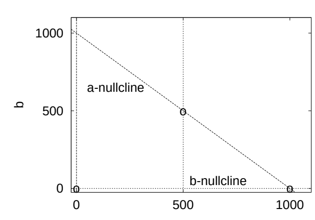
Putting p_2 on the vertical axis, the p_1-nullcline is $p_1 = 0$ and $p_2 = 1000$. The p_2-nullcline is $p_2 = 0$ or $p_1 = 1000$. The equilibria are $(0, 0)$ and $(1000, 1000)$.



5.6.29. We found equations

$$\frac{da}{dt} = \mu(1 - \frac{a+b}{K_a})a$$
$$\frac{db}{dt} = \lambda(1 - \frac{b}{K_b})b.$$

Putting b on the vertical axis, the a-nullcline is $a = 0$ and $b = K_a - a$. The b-nullcline is $b = 0$ or $a = K_b$. The equilibria are $(0, 0)$, $(K_a, 0)$, and $(K_b, K_a - K_b)$ (if $K_a > K_b$).



5.6.31.

a. Let C_1 be the concentration in the first vessel. Then chemical moves from the first to the second at rate βC_1.

b. Let C_2 be the concentration in the second vessel. Then chemical moves from the second to the first at rate $\frac{\beta}{3}C_2$.

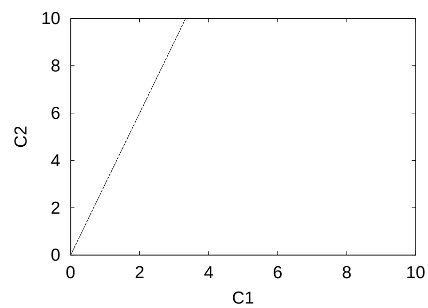
c. Let A_1 and A_2 be the total amounts. Then

$$\frac{dA_1}{dt} = \beta(\frac{C_2}{3} - C_1)$$
$$\frac{dA_2}{dt} = \beta(C_1 - \frac{C_2}{3}).$$

d.

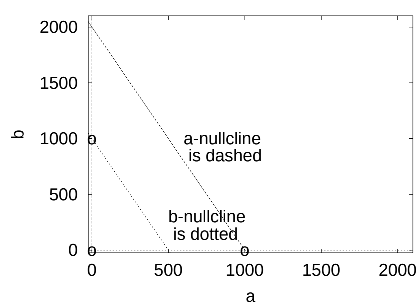
$$\frac{dC_1}{dt} = \frac{\beta}{V_1}(\frac{C_2}{3} - C_1)$$
$$\frac{dC_2}{dt} = \frac{\beta}{V_2}(C_1 - \frac{C_2}{3}).$$

e. Place C_2 on the vertical axis, $C_2 = 3C_1$ is the nullcline for both C_1 and C_2.



f. All points along the line $C_2 = 3C_1$ are equilibria. The equilibrium concentration in the second vessel is three times that in the first.

5.6.33. The equations look right because the per capita growth of a decreases half as quickly as a function of b as it does as a function of a. Therefore, individuals of type b decrease reproduction of individuals of type a only half as much. Similarly, the per capita growth of b decreases twice as quickly as a function of a as it does as a function of b. The a-nullcline is $a = 0$ and $b = 2(1000 - a)$. The b-nullcline is $b = 0$ and $b = 1000 - 2a$.

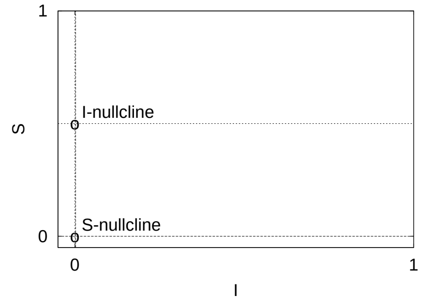


The two nullclines are parallel and do not intersect except at the boundaries.

5.6.35. The equations are

$$\frac{dI}{dt} = \alpha IS - \mu I = 2IS - I$$
$$\frac{dS}{dt} = -\alpha IS = -2IS.$$

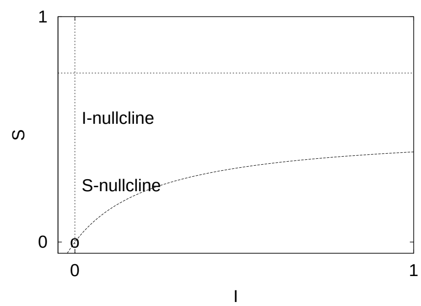
The I-nullcline is the two pieces $I = 0$ and $S = 1/2$. The S-nullcline is the two pieces $I = 0$ and $S = 0$. There are equilibria wherever $I = 0$.



5.6.37. The equations are

$$\frac{dI}{dt} = 2IS - 1.5I$$
$$\frac{dS}{dt} = -2IS + I - 0.5S.$$

The I-nullcline is the two pieces $I = 0$ and $S = 3/4$. The S-nullcline is $S = \frac{I}{2I + 0.5}$. There is only one equilibrium, at $(0, 0)$, because $\frac{3}{4} = \frac{I}{2I + 0.5}$ has no intersection.



5.6.39. The equations are

$$\frac{dI}{dt} = I + 2IS$$
$$\frac{dS}{dt} = S - 2IS + I.$$

The I-nullcline has only one reasonable piece at $I = 0$ (along with $S = -1/2$). The S-nullcline is $S = \frac{I}{2I - 1}$ (which is only defined for $I = 0$ and $I > 1/2$. There is only one equilibrium at $(0, 0)$.