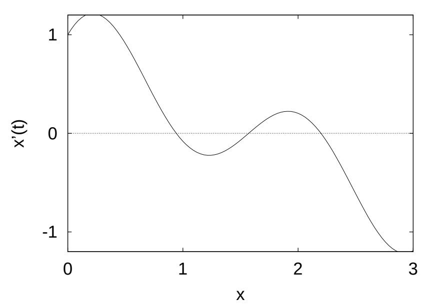
5.3 Stable and unstable equilibria

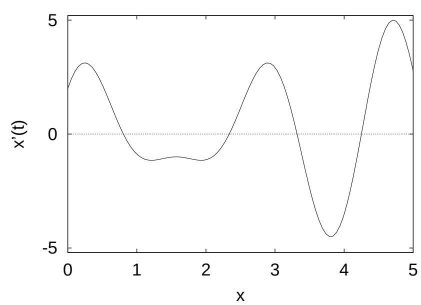
MATHEMATICAL TECHNIQUES

♠ From the following graphs of the rate of change as a function of the state variable, identify stable and unstable equilibria by checking whether the rate of change is an increasing or decreasing function of the state variable.

- EXERCISE **5.3.1**



- EXERCISE **5.3.2**



♠ Use the stability theorem to evaluate the stability of the equilibria of the following autonomous differential equations.

- EXERCISE **5.3.3**
$\frac{dx}{dt} = 1 - x^2$ (as in exercise 5.2.1). Compare your results with the phase-line in exercise 5.2.15.
- EXERCISE **5.3.4**
$\frac{dx}{dt} = 1 - e^x$ (as in exercise 5.2.2). Compare your results with the phase-line in exercise 5.2.16.
- EXERCISE **5.3.5**
$\frac{dy}{dt} = y\cos(y)$ (as in exercise 5.2.3). Compare your results with the phase-line in exercise 5.2.17.
- EXERCISE **5.3.6**
$\frac{dz}{dt} = \frac{1}{z} - 3$ (as in exercise 5.2.4). Compare your results with the phase-line in exercise 5.2.18.

♠ Find the the stability of the equilibria of the following autonomous differential equations that include parameters.

- EXERCISE **5.3.7**
$\frac{dx}{dt} = 1 - ax$ (as in exercise 5.2.5). Suppose that $a > 0$.
- EXERCISE **5.3.8**
$\frac{dx}{dt} = cx + x^2$ (as in exercise 5.2.6). Suppose that $c > 0$.
- EXERCISE **5.3.9**
$\frac{dW}{dt} = \alpha e^{\beta W} - 1$ (as in exercise 5.2.7). Suppose that $\alpha > 0$ and $\beta < 0$.
- EXERCISE **5.3.10**
$\frac{dy}{dt} = ye^{-\beta y} - ay$ (as in exercise 5.2.8). Suppose that $\beta < 0$ and $a > 1$.

♠ As with discrete-time dynamical systems, equilibria can act strange when the slope of the rate of change function is exactly equal to the critical value of zero.

• EXERCISE **5.3.11**
Consider the differential equation $\frac{dx}{dt} = x^2$. Find the equilibrium, graph the rate of change as a function of x, and draw the phase-line diagram. Would you consider the equilibrium to be stable or unstable?

• EXERCISE **5.3.12**
Consider the differential equation $\frac{dx}{dt} = -(1-x)^4$. Find the equilibrium, graph the rate of change as a function of x, and draw the phase-line diagram. Would you consider the equilibrium to be stable or unstable?

• EXERCISE **5.3.13**
Graph a rate of change function which has a slope of 0 at the equilibrium but the equilibrium is stable. What is the sign of the second derivative at the equilibrium? What is the sign of the third derivative at the equilibrium?

• EXERCISE **5.3.14**
Graph a rate of change function which has a slope of 0 at the equilibrium but the equilibrium is unstable. What is the sign of the second derivative at the equilibrium? What is the sign of the third derivative at the equilibrium?

♠ The fact that the rate of change function is continuous means that many behaviors are impossible for an autonomous differential equation.

• EXERCISE **5.3.15**
Try to draw a phase line diagram with two stable equilibria in a row. Use the Intermediate Value Theorem to sketch a proof of why this is impossible.

• EXERCISE **5.3.16**
Why is it impossible for a solution of an autonomous differential equation to oscillate?

♠ When parameter values change, the number and stability of equilibria sometimes changes. Such changes are called bifurcations, and play a central role the study of differential equations. The following illustrate several of the more important bifurcations. In each case, graph the value of equilibria as functions of the parameter value, using a solid line when an equilibrium is stable and a dashed line when an equilibrium is unstable. This picture is called a **bifurcation diagram**.

• EXERCISE **5.3.17**
Consider the equation

$$\frac{dx}{dt} = ax - x^2$$

for both positive and negative values of x. Find the equilibria as functions of a for values of a between -1 and 1. Draw a bifurcation diagram and describe in words what happens at $a = 0$. The change that occurs at $a = 0$ is called a **transcritical bifurcation**.

• EXERCISE **5.3.18**
Consider the equation

$$\frac{dx}{dt} = a - x^2$$

for both positive and negative values of x. Find the equilibria as functions of a for values of a between -1 and 1. Draw a bifurcation diagram and describe in words what happens at $a = 0$. The change that occurs at $a = 0$ is called a **saddle-node bifurcation**.

• EXERCISE **5.3.19**
Consider the equation

$$\frac{dx}{dt} = ax - x^3$$

for both positive and negative values of x. Find the equilibria as functions of a for values of a between -1 and 1. Draw a bifurcation diagram and describe in words what happens at $a = 0$. The change that occurs at $a = 0$ is called a **pitchfork bifurcation**.

• EXERCISE **5.3.20**
Consider the equation

$$\frac{dx}{dt} = ax + x^3$$

for both positive and negative values of x. Find the equilibria as functions of a for values of a between -1 and 1. Draw a bifurcation diagram and describe in words what happens at $a = 0$. The change that occurs at $a = 0$ is a slightly different type of **pitchfork bifurcation** (exercise 5.3.19) called a **sub-critical** (exercise 5.3.19 is **super-critical**). How does your picture differ from a simple mirror image of that in exercise 5.3.19?

APPLICATIONS

♠ Use the stability theorem to check the phase-lines for the following models of bacterial population growth.

- EXERCISE **5.3.21**
  The model in exercises 5.1.27 and 5.2.25.
- EXERCISE **5.3.22**
  The model in exercises 5.1.28 and 5.2.26.
- EXERCISE **5.3.23**
  The model in exercises 5.1.29 and 5.2.27.
- EXERCISE **5.3.24**
  The model in exercises 5.1.30 and 5.2.28.

♠ Use the stability theorem to check the phase-lines for the following models of bacterial population growth.

- EXERCISE **5.3.25**
  The model in exercises 5.1.37 and 5.2.31.
- EXERCISE **5.3.26**
  The model in exercises 5.1.38 and 5.2.32.

♠ A **reaction-diffusion equation** describes how chemical concentration changes due to two factors simultaneously, reaction and movement. A simple model has the form

$$\frac{dC}{dt} = \beta(\Gamma - C) + R(C)$$

The first term describes diffusion, and the second term $R(C)$ is the reaction, which could have a positive or negative sign (depending on whether chemical is being created or destroyed). Suppose that $\beta = 1.0$/min, and $\Gamma = 5.0$ moles/liter. For each of the following forms of of $R(C)$,

a. Describe how the reaction rate depends on the concentration.

b. Find the equilibria and their stability,

c. Describe how absorption changes the results.

- EXERCISE **5.3.27**
  Suppose that $R(C) = -C$.
- EXERCISE **5.3.28**
  Suppose that $R(C) = 0.5C$.
- EXERCISE **5.3.29**
  Suppose that $R(C) = \frac{C}{2+C}$.
- EXERCISE **5.3.30**
  Suppose that $R(C) = -\frac{C}{2+C}$.

♠ Apply the stability theorem for autonomous differential equations to the following equations. Show that your results match what you found in your phase-line diagrams, and give a biological interpretation.

- EXERCISE **5.3.31**
  The equation in exercise 5.2.33.
- EXERCISE **5.3.32**
  The equation in exercise 5.2.34.
- EXERCISE **5.3.33**
  The equation in exercise 5.2.35.

- EXERCISE **5.3.34**
  The equation in exercise 5.2.36.
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  The equation in exercise 5.2.37.
- EXERCISE **5.3.36**
  The equation in exercise 5.2.38.
- EXERCISE **5.3.37**
  The equation in exercise 5.2.39.
- EXERCISE **5.3.38**
  The equation in exercise 5.2.40 with $R = 0.5$.
- EXERCISE **5.3.39**
  The equation in exercise 5.2.41.
- EXERCISE **5.3.40**
  The equation in exercise 5.2.42.

♠ Exercises 5.3.17–5.3.20 show how the number and stability of equilibria can change when a parameter changes. Often, bifurcations have important biological applications, and bifurcation diagrams help in explaining how the dynamics of a system can suddenly change when a parameter changes only slightly. In each case, graph the equilibria against the parameter value, using a solid line when an equilibrium is stable and a dashed line when an equilibrium is unstable to draw the **bifurcation diagram**.

- EXERCISE **5.3.41**
  Consider the logistic differential equation (exercise 5.1.27) with harvesting proportional to population size, or $\frac{db}{dt} = b(1 - b - h)$ where h represents the fraction harvested. Graph the equilibria as functions of h for values of h between 0 and 2, using a solid line when an equilibrium is stable and a dashed line when an equilibrium is unstable. Even though they do not make biological sense, include negative values of the equilibria on your graph. You should find a **transcritical bifurcation** (exercise 5.3.17) at $h = 1$.
- EXERCISE **5.3.42**
  Suppose $\mu = 1$ in the basic disease model $\frac{dI}{dt} = \alpha I(1 - I) - \mu I$. Graph the two equilibria as functions of α for values of α between 0 and 2, using a solid line when an equilibrium is stable and a dashed line when an equilibrium is unstable. Even though they do not make biological sense, include negative values of the equilibria on your graph. You should find a **transcritical bifurcation** (exercise 5.3.17) at $\alpha = 1$.
- EXERCISE **5.3.43**
  Consider a version of the equation in exercise 5.2.33 that includes the parameter r,

$$\frac{dN}{dt} = \frac{rN^2}{1 + N^2} - N.$$

  Graph the equilibria as functions of r for values of r between 0 and 3, using a solid line when an equilibrium is stable and a dashed line when an equilibrium is unstable. The algebra for checking stability is messy, so it is only necessary to check stability at $r = 3$. You should find a **saddle-node bifurcation** (exercise 5.3.18) at $r = 2$.
- EXERCISE **5.3.44**
  Consider a variant of the basic disease model given by

$$\frac{dI}{dt} = \alpha I^2(1 - I) - I.$$

  Graph the equilibria as functions of α for values of α between 0 and 5, using a solid line when an equilibrium is stable and a dashed line when an equilibrium is unstable. The algebra for checking stability is messy, so it is only necessary to check stability at $\alpha = 5$. You should find a **saddle-node bifurcation** (exercise 5.3.18) at $\alpha = 4$.

♠ Right at a bifurcation point, the stability theorem fails because the the slope of the rate of change function at the equilibrium is exactly zero. In each of the following cases, check that the stability theorem fails, and then draw a phase-line diagram to find the stability.

• EXERCISE **5.3.45**

Analyze the stability of the positive equilibrium in the model from exercise 5.3.44 when $\alpha = 4$, the point where the bifurcation occurs.

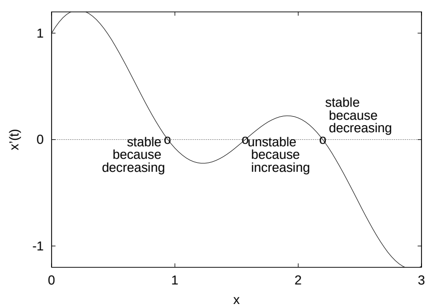
• EXERCISE **5.3.46**

Analyze the stability of the disease model when $\alpha = \mu = 1$, the point where the bifurcation occurs in exercise 5.3.42.

Chapter 6

Answers

5.3.1.



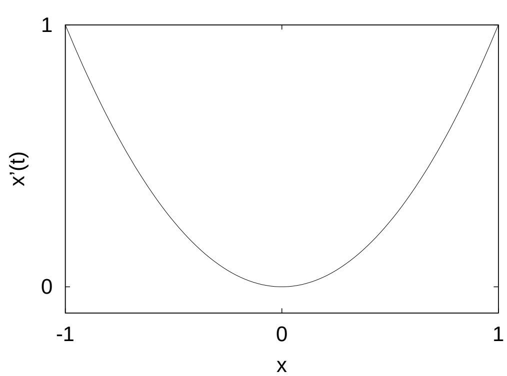
5.3.3. We found equilibria at $x = 1$ and $x = -1$. The derivative of the rate of change is $-2x$, which is negative at $x = 1$ and positive at $x = -1$. Therefore, $x = 1$ is stable, consistent with the inward pointing arrows on the phase-line diagram, and $x = -1$ is unstable, consistent with the outward pointing arrows on the phase-line diagram.

5.3.5. We found a equilibria at $y = 0$ and $y = \frac{\pi}{2} + \pi n$. The derivative of the rate of change is $\cos(y) - y\sin(y)$. This is positive at $y = 0$, negative at $y = \pi/2$ and negative at $y = -\pi/2$. Therefore, $y = 0$ is stable (inward pointing arrows), $y = \pi/2$ is unstable (outward pointing arrows), and $y = -\pi/2$ is unstable (outward pointing arrows).

5.3.7. The derivative of the rate of change is $-a$, which is negative for all values of x when $a > 0$. Therefore the equilibrium must be stable.

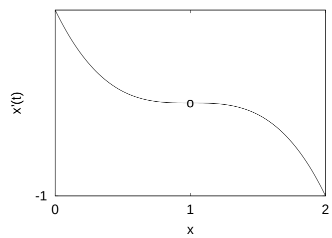
5.3.9. The derivative of the rate of change is $\alpha\beta e^{\beta W}$, which is negative for all values of W when $\alpha > 0$ and $\beta < 0$. Therefore the equilibrium must be stable.

5.3.11. The equilibrium occurs where $x^2 = 0$, or at $x = 0$. The derivative of the rate of change function is $2x$, which is equal to 0 at the equilibrium. The stability theorem does not apply.



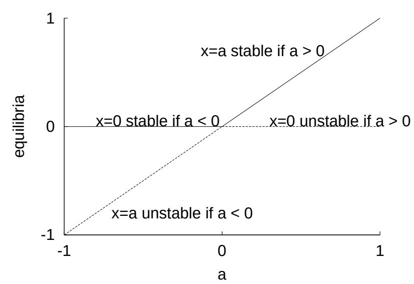
The rate of change is positive except at the equilibrium, so x is increasing except at the equilibrium. This equilibrium is sort of half stable: stable from the left, and unstable to the right.

5.3.13. The second derivative is 0 at the equilibrium. The third derivative must be negative because the function switches from being concave up for values of x less than the equilibrium to concave down for values of x greater than the equilibrium.

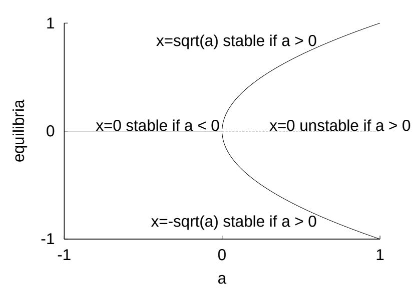


5.3.15. Suppose there are two stable equilibria in a row. Then the rate of change must cross from negative to positive at each. In particular, just above the lower one, the value of the rate of change is positive. Just below the upper one, the value of the rate of change is negative. By the Intermediate Value Theorem, there must be another crossing in between.

5.3.17. The equilibria are at $x = 0$ and $x = a$. The derivative of the rate of change function $f(x) = ax - x^2$ is $f'(x) = a - 2x$. Then $f'(0) = a$, so the equilibrium at $x = 0$ is stable if $a < 0$ and unstable if $a > 0$. $f'(a) = -a$, so the equilibrium at $x = a$ is stable if $a > 0$ and unstable if $a < 0$. At $a = 0$ there is an exchange of stability when the two equilibria cross.



5.3.19. The equilibria are at $x = 0$, $x = \pm\sqrt{a}$. The last two equilibria do not exist if $a < 0$. The derivative of the rate of change function $f(x) = ax - x^3$ is $f'(x) = a - 3x^2$. $f'(0) = a$, so the equilibrium at $x = 0$ is stable if $a < 0$ and unstable if $a > 0$. $f'(\sqrt{a}) = -2a$, so the equilibrium at $x = \sqrt{a}$ is stable when $a > 0$. $f'(-\sqrt{a}) = -2a$, so the equilibrium at $x = -\sqrt{a}$ is also stable when $a > 0$.



5.3.21. This population obeys the autonomous differential equation $\frac{db}{dt} = (1-0.002b)b$ and has equilibria at $b = 500$ and at $b = 0$. The derivative of the rate of change function $f(b) = (1-0.002b)b$ is $f'(b) = 1-0.004b$. Then $f'(0) = 1 > 0$ so $b = 0$ is unstable, and $f'(500) = -1 < 0$ so $b = 500$ is stable.

5.3.23. This population obeys the differential equation $\frac{db}{dt} = (-2 + 0.01b)b$ and has equilibria at $b = 200$ and at $b = 0$. The derivative of the rate of change function $f(b) = (-2+0.01b)b$ is $f'(b) = -2+0.02b$. Then $f'(0) = -2 < 0$ so $b = 0$ is stable, and $f'(200) = 2 > 0$ so $b = 200$ is unstable.

5.3.25. This population obeys the autonomous differential equation $\frac{dp}{dt} = 0.5p(1-p)^2$ and has equilibria at $p = 0$ and at $p = 1$. The derivative of the rate of change function $f(p) = 0.5p(1-p)^2$ is $f'(p) = 0.5(1-p)^2 - p(1-p)$. Then $f'(0) = 0.5 > 0$ so $p = 0$ is unstable, and $f'(1) = 0$ so we can't tell. However, the graph does indicate that this equilibrium is stable.

5.3.27.

a. This is a case where chemical is used up at a rate proportional to its concentration.

b. Solving $f(C) = (5 - C) - C = 0$ gives $C = 2.5$. The derivative of the rate of change function is $f'(C) = -2$, so the equilibrium is stable.

c. Without the reaction, the equilibrium is $C = \Gamma = 5$. Absorption decreases the equilibrium amount.

5.3.29.

a. This is a case where chemical is created at a rate that gets larger as the concentration gets larger, but reaches a maximum of 0.5.

b. Solving $f(C) = (5 - C) + \frac{C}{2+C} = 0$ gives $C = 2 \pm \sqrt{14}$. Only one of these values is positive, at $C = 2 + \sqrt{14} = 5.74$. The derivative of the rate of change function is

$$f'(C) = -1 + \frac{1}{2+C} - \frac{C}{(2+C)^2}$$

and $f'(5.74) = -0.97$, so the equilibrium is stable.

c. Without the reaction, the equilibrium is $C = \Gamma = 5$. Chemical creation increases the equilibrium amount, but only slightly.

5.3.31. The derivative of the rate of change is

$$\frac{d}{dN}\left(\frac{3N^2}{2+N^2} - N\right) = \frac{12N}{(2+N^2)^2} - 1.$$

At $N = 0$, this is -1, so the equilibrium at $N = 0$ is stable. At $N = 1$, this is $1/3$, so the equilibrium at $N = 1$ is unstable. At $N = 2$, this is $-1/3$, so the equilibrium at $N = 2$ is stable. This population of 1 acts as a threshold.

5.3.33. The derivative of the rate of change is

$$\frac{d}{dv}(9.8 - 0.0032v^2) = -0.0064v$$

which is negative for any positive speed, including the equilibrium at $v = 55.3$. The equilibrium is stable, consistent with the inward pointing arrows on the phase-line diagram. The falling object will approach its terminal velocity.

5.3.35. The derivative of the rate of change is

$$\frac{d}{dy}(-2.0\sqrt{y}) = -\frac{2.0}{2\sqrt{y}}.$$

The equilibrium is at $y = 0$, so this derivative is negative infinity, implying that the equilibrium is very stable, as in our phase-line diagram. The cylinder will drain very quickly.

5.3.37. The derivative of the rate of change is

$$\frac{d}{dS}(-\frac{S}{1+S}) = -\frac{1}{(1+S)^2},$$

which is always negative. Any equilibrium must be stable. In particular, the equilibrium at $S = 0$ is stable, consistent with using up a substance.

5.3.39. The derivative of the rate of change is

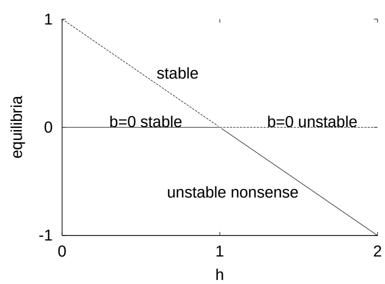
$$\frac{d}{dV}(a_1 V^{2/3} - a_2 V) = \frac{2}{3} a_1 V^{-1/3} - a_2.$$

At the equilibrium $V = (\frac{a_2}{a_1})^3$, this is

$$\frac{2}{3} a_1 V^{-1/3} - a_2 = -\frac{1}{3} a_2 < 0.$$

Again, the equilibrium is stable, consistent with the inward pointing arrows on the phase-line diagram. This growing organism will reach a final size when it becomes too big to grow any more.

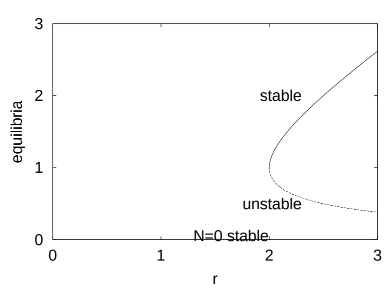
5.3.41. The equilibria are at $b = 0$ and $b = 1 - h$. The second equilibrium is positive only if $h < 1$. The rate of change function $f(b) = b(1 - b - h)$ has derivative $f'(b) = 1 - h - 2b$. Then $f'(0) = 1 - h$, so $b = 0$ is stable if $h > 1$ and unstable if $h < 1$. Also $f'(1 - h) = h - 1$, so $b = 1 - h$ is stable if $h < 1$ and unstable if $h > 1$ (where it is negative and makes no biological sense).



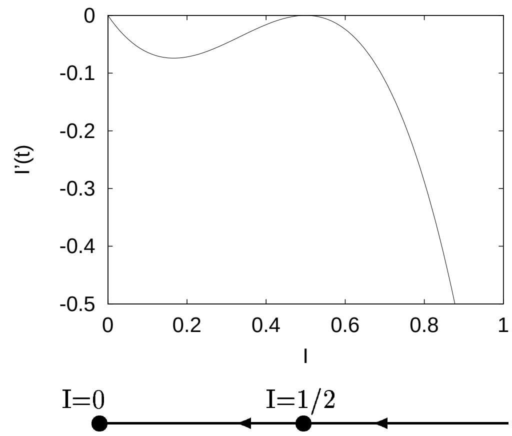
5.3.43. The equilibria are at $N = 0$, $N = \frac{r + \sqrt{r^2 - 4}}{2}$ and $N = \frac{r - \sqrt{r^2 - 4}}{2}$. The last two only exist when $r \geq 2$. The rate of change function $f(N) = \frac{rN^2}{1+N^2} - N$ has derivative

$$f'(N) = \frac{2rN}{1 + N^2} - \frac{2rN^3}{(1 + N^2)^2} - 1.$$

Then $f'(0) = -1$, so $I = 0$ is always stable. At $r = 3$, the equilibria are 2.618 and 0.382. Substituting in gives $f'(2.618) = -0.745$ and $f'(0.382) = 0.745$. The larger equilibrium is stable and the smaller one is unstable.



5.3.45. With $\alpha = 4$, the differential equation is $\frac{dI}{dt} = 4I^2(1 - I) - I$. The equilibria are at $I = 0$ and $I = 1/2$.



The derivative of the rate of change is $8I(1-I) - 4I^2 - 1$, which is negative at $I = 0$ and equal to 0 at $I = 1/2$. From our phase-line diagram, however, we see that the rate of change is always negative, implying that the equilibrium at $I = 1/2$ is half-stable, stable from the right and unstable to the left.