5.2 Equilibria and Display of Autonomous Differential Equations

MATHEMATICAL TECHNIQUES

♠ Find the equilibria of the following autonomous differential equations.

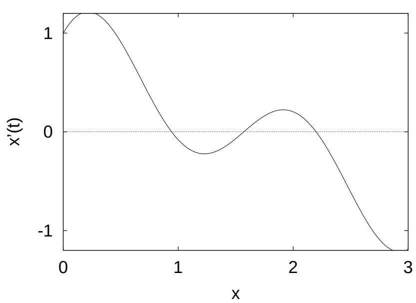
- EXERCISE **5.2.1**
  $\frac{dx}{dt} = 1 - x^2.$
- EXERCISE **5.2.2**
  $\frac{dx}{dt} = 1 - e^x.$
- EXERCISE **5.2.3**
  $\frac{dy}{dt} = y\cos(y).$
- EXERCISE **5.2.4**
  $\frac{dz}{dt} = \frac{1}{z} - 3.$

♠ Find the equilibria of the following autonomous differential equations that include parameters.

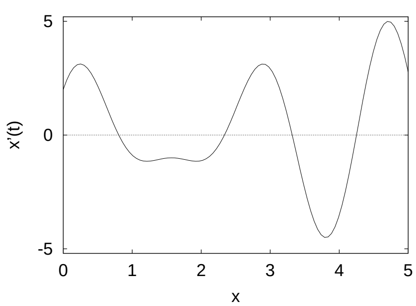
- EXERCISE **5.2.5**
  $\frac{dx}{dt} = 1 - ax.$
- EXERCISE **5.2.6**
  $\frac{dx}{dt} = cx + x^2.$
- EXERCISE **5.2.7**
  $\frac{dW}{dt} = \alpha e^{\beta W} - 1.$
- EXERCISE **5.2.8**
  $\frac{dy}{dt} = ye^{-\beta y} - ay.$

♠ From the following graphs of the rate of change as a function of the state variable, draw the phase-line diagram.

- EXERCISE **5.2.9**

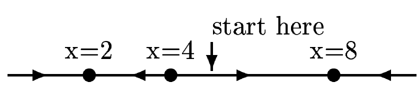


- EXERCISE **5.2.10**

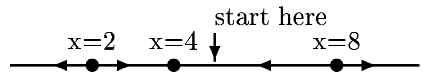


♠ From the following phase-line diagrams, sketch a solution starting from the specified initial condition.

- EXERCISE **5.2.11**



- EXERCISE **5.2.12**



♠ From the given phase-line diagram, sketch a possible graph of the rate of change of x as a function of x.

- EXERCISE **5.2.13**
  The phase line in exercise 5.2.11.
- EXERCISE **5.2.14**
  The phase line in exercise 5.2.12.

♠ Graph the rate of change as a function of the state variable and draw the phase-line diagram for the following differential equations.

- EXERCISE **5.2.15**
  $\frac{dx}{dt} = 1 - x^2$ (as in exercise 5.2.1). Graph for $-2 \le x \le 2$.
- EXERCISE **5.2.16**
  $\frac{dx}{dt} = 1 - e^x$ (as in exercise 5.2.2). Graph for $-2 \le x \le 2$.
- EXERCISE **5.2.17**
  $\frac{dy}{dt} = y\cos(y)$ (as in exercise 5.2.3). Graph for $-2 \le y \le 2$.
- EXERCISE **5.2.18**
  $\frac{dz}{dt} = \frac{1}{z} - 3$ (as in exercise 5.2.4). Graph for $0 < z \le 1$.

♠ Try to find the "equilibria" of the following autonomous differential equations. What goes wrong? Graph the "equilibria" as functions of time for $0 \le t \le 5$.

- EXERCISE **5.2.19**
  $\frac{dx}{dt} = x - t$.
- EXERCISE **5.2.20**
  $\frac{dx}{dt} = \ln(x) + t$.
- EXERCISE **5.2.21**
  $\frac{dx}{dt} = x^2 - t + 1$.
- EXERCISE **5.2.22**
  $\frac{dx}{dt} = x^2 - t^2$.

APPLICATIONS

♠ Suppose a population is growing at constant rate λ, but that individuals are harvested at a rate of h. The differential equation describing such a population is

$$\frac{db}{dt} = \lambda b - h.$$

For each of the following values of λ and h, find the equilibrium, draw the phase-line diagram and sketch one solution with initial condition below the equilibrium and another with initial condition above the equilibrium. Explain your result in words.

- EXERCISE **5.2.23**
  $\lambda = 2.0$, $h = 1000$.
- EXERCISE **5.2.24**
  $\lambda = 0.5$, $h = 1000$.

♠ Find the equilibria, graph the rate of change $\frac{db}{dt}$ as a function of b, and draw a phase-line diagram for the following models describing bacterial population growth.

- EXERCISE **5.2.25**
  The model in exercise 5.1.27. Check that your arrows are consistent with the behavior of $b(t)$ at $b = 10$ and $b = 1000$.
- EXERCISE **5.2.26**
  The model in exercise 5.1.28. Check that your arrows are consistent with the behavior of $b(t)$ at $b = 1000$ and $b = 5000$.

• EXERCISE **5.2.27**

The model in exercise 5.1.29. Check that your arrows are consistent with the behavior of $b(t)$ at $b = 100$ and $b = 300$.

• EXERCISE **5.2.28**

The model in exercise 5.1.30. Check that your arrows are consistent with the behavior of $b(t)$ at $b = 1000$ and $b = 3000$.

♠ Find the equilibria, graph the rate of change $\frac{dC}{dt}$ as a function of C, and draw a phase-line diagram for the following models describing chemical diffusion.

• EXERCISE **5.2.29**

The model in exercise 5.1.33. Check that the direction arrow is consistent with the behavior of $C(t)$ at $C = \Gamma$.

• EXERCISE **5.2.30**

The model in exercise 5.1.34. Check that the direction arrow is consistent with the behavior of $C(t)$ at $C = \Gamma$.

♠ Find the equilibria, graph the rate of change $\frac{dp}{dt}$ as a function of p, and draw a phase-line diagram for the following models describing competition.

• EXERCISE **5.2.31**

The model in exercise 5.1.37. What happens to a solution starting from a small, but positive, value of p?

• EXERCISE **5.2.32**

The model in exercise 5.1.38. What happens to a solution starting from a small, but positive, value of p?

♠ Find the equilibria and draw the phase-line diagram for the following differential equations, in addition to answering the questions.

• EXERCISE **5.2.33**

Suppose the population size of some species of organism follows the model

$$\frac{dN}{dt} = \frac{3N^2}{2 + N^2} - N$$

where N is measured in hundreds. Why might this population behave as it does at small values? This is another example of the Allee effect discussed in exercise 5.1.29.

• EXERCISE **5.2.34**

Suppose the population size of some species of organism follows the model

$$\frac{dN}{dt} = \frac{5N^2}{1 + N^2} - 2 * N$$

where N is measured in hundreds. What is the critical value below which this population is doomed to extinction (as in exercise 5.2.33)?

• EXERCISE **5.2.35**

The drag on a falling object is proportional to the square of its speed. In a differential equation

$$\frac{dv}{dt} = a - Dv^2$$

where v is speed, a is acceleration and D is drag. Suppose that $a = 9.8$ m/s^2 and that $D = 0.0032$ per meter (values for a sky-diver). Check that the units in the differential equation are consistent. What does the equilibrium speed mean?

• EXERCISE **5.2.36**

Consider the same situation as in exercise 5.2.35 but for a skydiver diving head down with her arms against her sides and her toes pointed, thus minimizing drag. The drag D is reduced to $D = 0.00048$ per meter. Find the equilibrium speed. How does it compare to the ordinary sky-diver?

• EXERCISE **5.2.37**

According to Torricelli's law of draining, the rate which a fluid flows out of a cylinder through a hole at the bottom is proportional to the square root of the depth of the water. Let y represent the depth of water in centimeters. The differential equation is

$$\frac{dy}{dt} = -c\sqrt{y}$$

where $c = 2.0\sqrt{\text{cm}}/\text{ sec}$. Show that the units are consistent. Use your phase-line diagram to sketch solutions starting from $y = 10.0$ and $y = 1.0$.

• EXERCISE **5.2.38**

Write a differential equation describing the depth in a cylinder (as in exercise 5.2.37) where water enters at a rate of 4.0 cm/sec but continues to drain out as above. Use your phase-line diagram to sketch solutions starting from $y = 10.0$ and $y = 1.0$.

• EXERCISE **5.2.39**

One of the most important differential equations in chemistry uses the **Michaelis-Menton** or **Monod** equation. Suppose S is the concentration of a substrate that is being converted into a product. Then

$$\frac{dS}{dt} = -k_1 \frac{S}{k_2 + S}$$

describes how substrate is used. Set $k_1 = k_2 = 1$. item How does this equation differ from Torricelli's law of draining (exercise 5.2.37)?

• EXERCISE **5.2.40**

Write a differential equation describing the amount of substrate if substrate is added at rate R but continues to be converted to product as before. Find the equilibrium, draw the phase-plane diagram and a representative solution with $R = 0.5$ and $R = 1.5$. Can you explain your results?

♠ Small organisms like bacteria take in food at rates proportional to their surface area but use energy at higher rates.

• EXERCISE **5.2.41**

Suppose that energy is used at a rate proportional to the mass. In this case,

$$\frac{dV}{dt} = a_1 V^{2/3} - a_2 V.$$

where V represents the volume in cm^3 and t is time measured in days. The first term says that surface area is proportional to volume to the 2/3 power. The constant a_1 gives the rate at which energy is taken in and has units of cm/day. a_2 is rate at which energy is used and has units of per day. Check the units. Find the equilibrium. What happens to the equilibrium as a_1 becomes smaller? Does this make sense? What happens to the equilibrium as a_2 becomes smaller? Does this make sense?

• EXERCISE **5.2.42**

Suppose that energy is used at a rate proportional to the mass to the 3/4 power (closer to what is observed). In this case,

$$\frac{dV}{dt} = a_1 V^{2/3} - a_2 V^{3/4}.$$

Find the units of a_2 if V is measured in cm^3 and t is measured in days. (They should look rather strange.) Find the equilibrium. What happens to the equilibrium as a_1 becomes smaller? Does this make sense? What happens to the equilibrium as a_2 becomes smaller? Does this make sense?

Chapter 6

Answers

5.2.1.

1. This equation is autonomous because the only variable on the right hand side is the state variable x.
2. We must solve $1 - x^2 = 0$.
3. Factoring gives $(1 - x)(1 + x) = 0$.
4. Solving each factor gives $x = 1$ or $x = -1$.

5.2.3.

1. This equation is autonomous because the only variable on the right hand side is the state variable y.
2. We must solve $y \cos(y) = 0$.
3. This is already in factored form.
4. Solving gives $y = 0$ or $y = \frac{\pi}{2} + \pi n$ for any integer value of n.

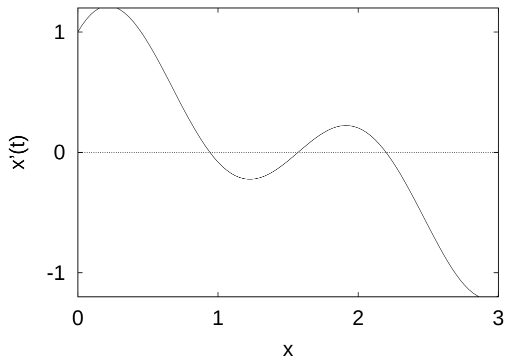
5.2.5.

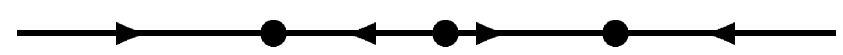
1. This equation is autonomous because the only variable on the right hand side is the state variable x.
2. We must solve $1 - ax = 0$.
3. There is only one term, so we don't need to factor.
4. Solving, we find $ax = 1$ which has solution $x = \frac{1}{a}$.

5.2.7.

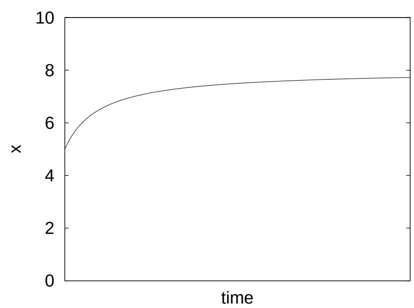
1. This equation is autonomous because the only variable on the right hand side is the state variable W.
2. We must solve $\alpha e^{\beta W} - 1 = 0$.
3. There is only one term, so we don't need to factor.
4. Solving, we find that $\alpha e^{\beta W} = 1$, which becomes $e^{\beta W} = \frac{1}{\alpha}$. Taking logarithms, $\beta W = -\ln(\alpha)$, which has solution $W = -\frac{\ln(\alpha)}{\beta}$.

5.2.9.

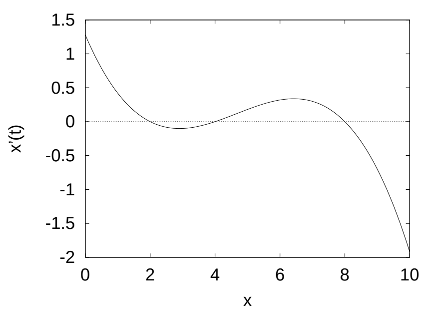




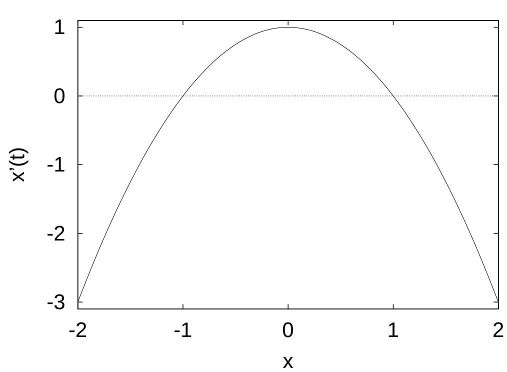
5.2.11.

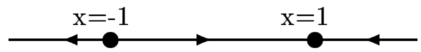


5.2.13.

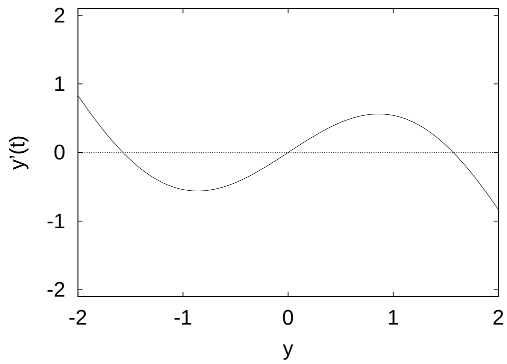


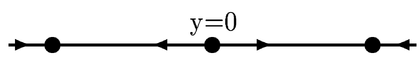
5.2.15.



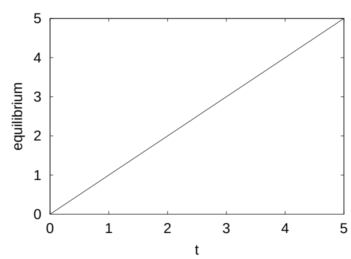


5.2.17.

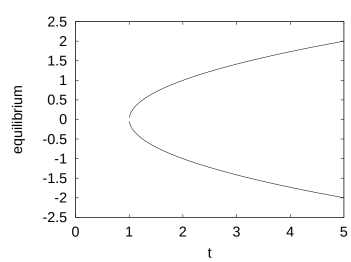




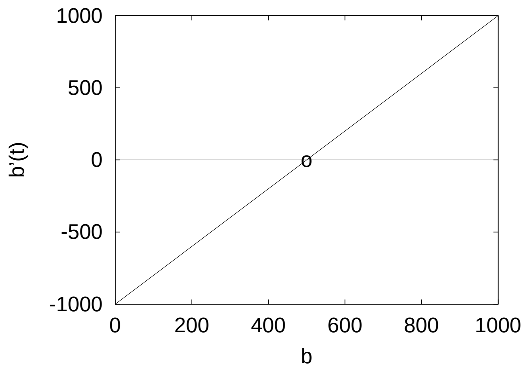
5.2.19. This is not an autonomous equation because t appears on the right hand side. If we go ahead and solve $x - t = 0$, we find $x = t$. The equilibrium is an ever-increasing function of t.

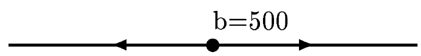


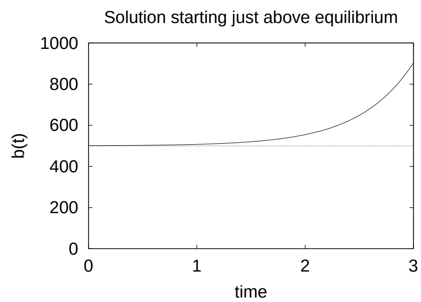
5.2.21. This is not an autonomous equation because t appears on the right hand side. If we go ahead and solve $x^2 = t - 1$, we find $x = \pm\sqrt{t-1}$. This has no solution if $t < 1$, and two solutions for $t > 1$.

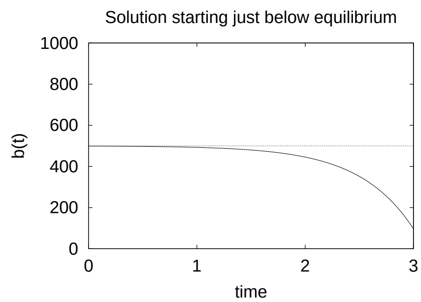


5.2.23. The equilibrium is $b^* = 500$.



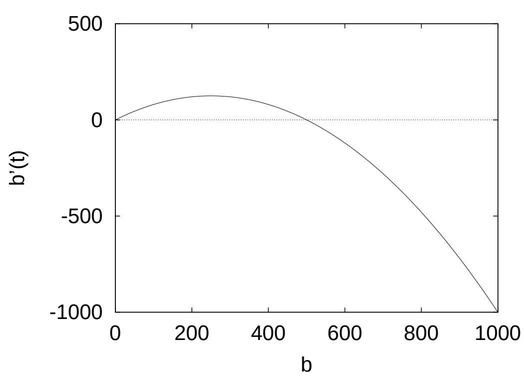


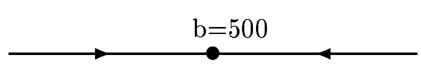




This population can outgrow the harvest if it starts at a large enough value. If it starts too small, the harvest will drive it to extinction.

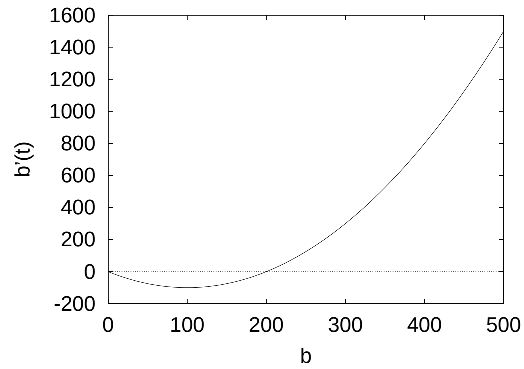
5.2.25. We found that the population obeys the autonomous differential equation $\frac{db}{dt} = (1 - 0.002b)b$. This is in factored form, and has equilibria at $b = 500$ and at $b = 0$.

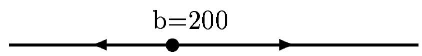




The arrow points up at $b = 10$, consistent with an increasing population, and down at $b = 1000$, consistent with a decreasing population.

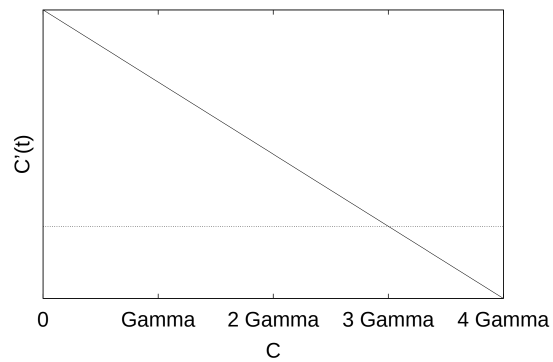
5.2.27. We found that the population obeys the autonomous differential equation $\frac{db}{dt} = (-2 + 0.01b)b$. This is in factored form, and has equilibria at $b = 200$ and at $b = 0$.





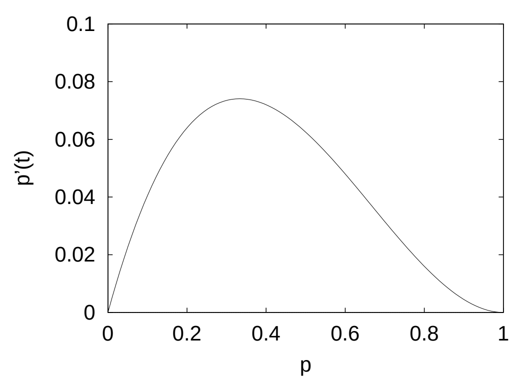
The arrow points down at $b = 100$, consistent with a decreasing population, and up at $b = 300$, consistent with an increasing population.

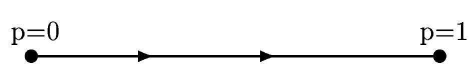
5.2.29. We found that the population obeys the autonomous differential equation $\frac{dC}{dt} = -\beta C + 3\beta\Gamma$. As long as $\beta \neq 0$, this has equilibria at $C = 3\Gamma$.



The arrow points up at $C = \Gamma$, consistent with an increasing concentration.

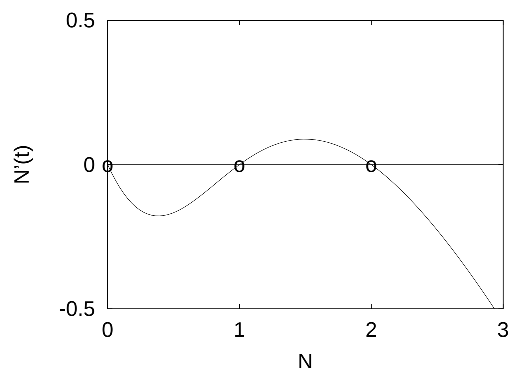
5.2.31. We found that the population obeys the autonomous differential equation $\frac{dp}{dt} = 0.5p(1-p)^2$. This has equilibria at $p = 0$ and $p = 1$.

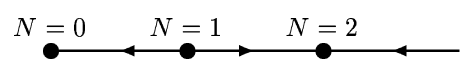




All the arrows points up, except at the equilibria, so the solution moves up to $p = 1$, meaning that a takes over.

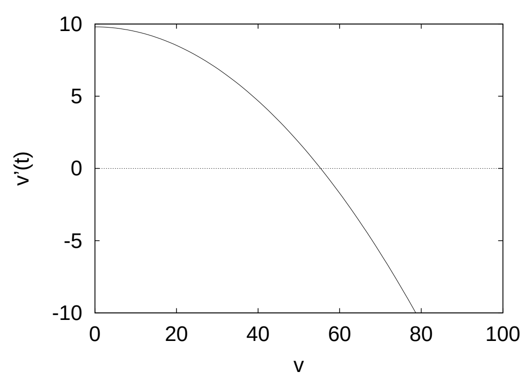
5.2.33. The equilibria are at $N = 0$, and the solution of $\frac{3N}{2+N^2} - 1 = 0$, which occurs where $N^2 - 3N + 2 = 0$. This factors to have solutions at $N = 1$ and $N = 2$. To see whether N is increasing or decreasing between the equilibria, we need to check whether $\frac{dN}{dt}$ is positive or negative. We find that $f(1/2) = -1/6 < 0$, $f(3/2) = 3/34 > 0$ and $f(3) = -6/11 < 0$. Therefore, the graph of the rate of change and the phase-line diagram must be the following.



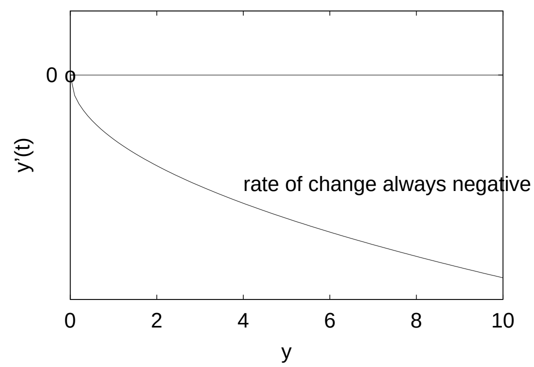


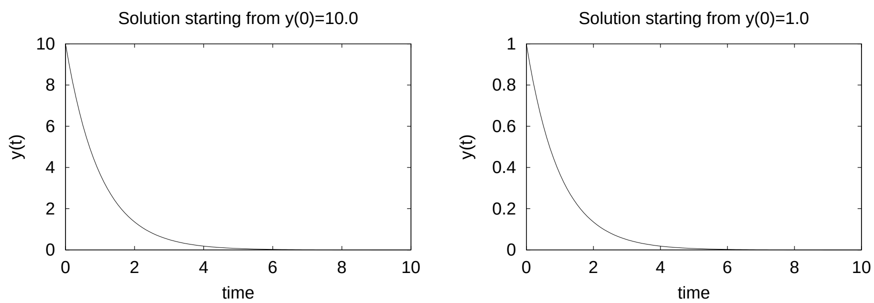
This population dies out if it drops below $N = 1$. Perhaps they cannot find mates when the population gets below one hundred.

5.2.35. Everything has units of meters per second squared. The equilibrium is the solution of $9.8 - 0.0032v^2 = 0$, or $v^* = \sqrt{9.8/0.0032} = 55.3$ m/s. This is the terminal velocity of a sky-diver in free fall.

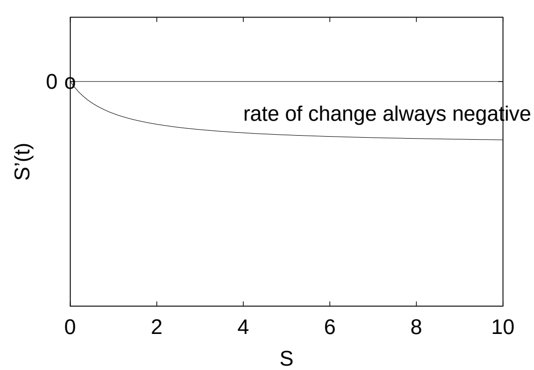


5.2.37. Both sides have units of cm/sec. This checks. The equilibrium is $y^* = 0$, meaning that all water has drained out of the cylinder.





5.2.39. The equilibrium is at $S^* = 0$. Eventually, all substrate will be used. In both cases the rate is always negative. However, the graph of the rate for Torricelli's law of draining is much steeper near a value of 0 for the state variable.



5.2.41. Everything has units of cm^3/day. The equilibrium is

$$V^* = \left(\frac{a_1}{a_2}\right)^3.$$

The equilibrium gets smaller for smaller values of a_1 because this animal is less effective at collecting food. The equilibrium gets larger when a_2 becomes smaller because this animal is more efficient at using energy.