Chapter 5

Revised Exercises

5.1 Basic differential equations

MATHEMATICAL TECHNIQUES

♠ Identify the following as pure-time, autonomous, or nonautonomous differential equations. In each case, identify the state variable.

• EXERCISE **5.1.1**

$$\frac{dF}{dt} = F^2 + kt.$$

• EXERCISE **5.1.2**

$$\frac{dx}{dt} = \frac{x^2}{(x - \lambda)}.$$

• EXERCISE **5.1.3**

$$\frac{dy}{dt} = \mu e^{-t} - 1.$$

• EXERCISE **5.1.4**

$$\frac{dm}{dt} = \frac{e^{\alpha m} m^2}{(mt - \lambda)}.$$

♠ For the given time, value of the state variable, and values of the parameters, say whether the state variable is increasing, decreasing, or remaining unchanged.

• EXERCISE **5.1.5**

$t = 0$, $F = 1$, and $k = 1$ in the differential equation in exercise 5.1.1.

• EXERCISE **5.1.6**

$t = 0$, $x = 1$, and $\lambda = 2$ in the differential equation in exercise 5.1.2.

• EXERCISE **5.1.7**

$t = 1$, $y = 1$, and $\mu = 2$ in the differential equation in exercise 5.1.3.

• EXERCISE **5.1.8**

$t = 2$, $m = 0$, $\alpha = 2$ and $\lambda = 1$ in the differential equation in exercise 5.1.4.

♠ The following exercises compare the behavior of two similar-looking differential equations, the pure-time differential equation $\frac{dp}{dt} = t$ and the autonomous differential equation $\frac{db}{dt} = b$.

• EXERCISE **5.1.9**

Use integration to solve the pure-time differential equation starting from the initial condition $p(0) = 1$, find $p(1)$, and sketch the solution.

• EXERCISE **5.1.10**

Solve the pure-time differential equation starting from the initial condition $p(1) = 1$, find $p(2)$, and add the curve to your graph.

• EXERCISE **5.1.11**

Check that the solution of the autonomous differential equation starting from the initial condition $b(0) = 1$ is $b(t) = e^t$. Find $b(1)$ and sketch the solution.

- EXERCISE **5.1.12**

  Check that the solution of the autonomous differential equation starting from the initial condition $b(1) = 1$ is $b(t) = e^{t-1}$. Find $b(2)$ and add to the sketch of the solution.

- EXERCISE **5.1.13**

  Exercises 5.1.9 and 5.1.10 give the value of p one time unit after it took on the value 1. Why don't the two answers match? (This behavior is typical of pure-time differential equations).

- EXERCISE **5.1.14**

  Exercises 5.1.11 and 5.1.12 give the value of b one time unit after it took on the value 1. Why do the two answers match? (This behavior is typical of autonomous differential equations.)

♠ Check the solution of the autonomous differential equation, making sure that it also matches the initial condition.

- EXERCISE **5.1.15**

  Check that $x(t) = -\frac{1}{2} + \frac{3}{2}e^{2t}$ is a solution of the differential equation $\frac{dx}{dt} = 1 + 2x$ with initial condition $x(0) = 1$.

- EXERCISE **5.1.16**

  Check that $b(t) = 10e^{3t}$ is a solution of the differential equation $\frac{db}{dt} = 3b$ with initial condition $b(0) = 10$.

- EXERCISE **5.1.17**

  Check that $G(t) = 1 + e^t$ is a solution of the differential equation $\frac{dG}{dt} = G - 1$ with initial condition $G(0) = 2$.

- EXERCISE **5.1.18**

  Check that $z(t) = 1 + \sqrt{1 + 2t}$ is a solution of the differential equation $\frac{dz}{dt} = \frac{1}{z - 1}$ with initial condition $z(0) = 2$.

♠ Use Euler's method to estimate the solution of the differential equation at the given time, and compare with the value given by the exact solution. Sketch a graph of the solution along with the lines predicted by Euler's method.

- EXERCISE **5.1.19**

  Estimate $x(2)$ if x obeys the differential equation $\frac{dx}{dt} = 1 + 2x$ with initial condition $x(0) = 1$. Use Euler's method with $\Delta t = 1$ for two steps. Compare with the exact answer in exercise 5.1.15.

- EXERCISE **5.1.20**

  Estimate $b(1.0)$ if b obeys the differential equation $\frac{db}{dt} = 3b$ with initial condition $b(0) = 10$. Use Euler's method with $\Delta t = 0.5$ for two steps. Compare with the exact answer in exercise 5.1.16.

- EXERCISE **5.1.21**

  Estimate $G(1.0)$ if G obeys the differential equation $\frac{dG}{dt} = G - 1$ with initial condition $G(0) = 2$. Use Euler's method with $\Delta t = 0.2$ for five steps. Compare with the exact answer in exercise 5.1.17.

- EXERCISE **5.1.22**

  Estimate $z(4.0)$ if z obeys the differential equation $\frac{dz}{dt} = \frac{1}{z - 1}$ with initial condition $z(0) = 2$. Use Euler's method with $\Delta t = 1.0$ for four steps. Compare with the exact answer in exercise 5.1.18.

♠ The derivation of the differential equation for p in the text requires combining two differential equations for a and b. Often, one can find a differential equation for a new variable derived from a single equation. In the following cases, use the chain rule to derive a new differential equation.

- EXERCISE **5.1.23**

  Suppose $\frac{dx}{dt} = 2x - 1$. Set $y = 2x - 1$ and find a differential equation for y. The end result should be simpler than the original equation.

- EXERCISE **5.1.24**

  Suppose $\frac{db}{dt} = 4b + 2$. Set $z = 4b + 2$ and find a differential equation for z. The end result should be simpler than the original equation.

- EXERCISE **5.1.25**

  Suppose $\frac{dx}{dt} = x + x^2$. Set $y = \frac{1}{x}$ and find a differential equation for y. This transformation changes a nonlinear differential equation for x into a linear differential equation for y (this is called a Bernoulli differential equation).

• EXERCISE **5.1.26**
Suppose $\frac{dx}{dt} = 2x + \frac{1}{x}$. Set $y = x^2$ and find a differential equation for y. This is another example of a Bernoulli differential equation.

APPLICATIONS

♠ The simple model of bacterial growth assumes that per capita reproduction does not depend on population size. The following problems help you derive models of the form

$$\frac{db}{dt} = \lambda(b)b$$

where the per capita reproduction λ is a function of the population size b.

• EXERCISE **5.1.27**
One widely used nonlinear model of competition is the "logistic" model, where per capita reproduction is a linearly decreasing function of population size. Suppose that per capita reproduction has a maximum at $\lambda(0) = 1$ and that it decreases with a slope of 0.002. Find $\lambda(b)$ and the differential equation for b. Is $b(t)$ increasing when $b = 10$? Is $b(t)$ increasing when $b = 1000$?

• EXERCISE **5.1.28**
Suppose that per capita reproduction decreases linearly from a maximum of $\lambda(0) = 4$ with slope 0.001. Find $\lambda(b)$ and the differential equation for b. Is $b(t)$ increasing when $b = 1000$? Is $b(t)$ increasing when $b = 5000$?

• EXERCISE **5.1.29**
In some circumstances, individuals reproduce better when the population size is large, and fail to reproduce when the population size is small (called the Allee effect). Suppose that per capita reproduction is an increasing linear function with $\lambda(0) = -2$ and a slope of 0.01. Find $\lambda(b)$ and the differential equation for b. Is $b(t)$ increasing when $b = 100$? Is $b(t)$ increasing when $b = 300$?

• EXERCISE **5.1.30**
Suppose that per capita reproduction increases linearly with $\lambda(0) = -5$ and a slope of 0.001. Find $\lambda(b)$ and the differential equation for b. Is $b(t)$ increasing when $b = 1000$? Is $b(t)$ increasing when $b = 3000$?

♠ The derivation of the movement of chemical assumed that chemical moved as easily into the cell as out of it. If the membrane can act as a filter, the rates at which chemical enters and leaves might differ, or might depend on the concentration itself. In each of the following cases, draw a diagram illustrating the situation and write the associated differential equation. Let C be the concentration inside the cell, Γ the concentration outside, and β the constant of proportionality relating the concentration and the rate.

• EXERCISE **5.1.31**
Suppose that no chemical re-enters the cell. This should look like the differential equation for a population. What would be happening to the population?

• EXERCISE **5.1.32**
Suppose that no chemical leaves the cell. This should look like the differential equation for a volume. What would be happening to the volume?

• EXERCISE **5.1.33**
Suppose that the constant of proportionality governing the rate at which chemical enters the cell is three times as large as the constant governing the rate at which it leaves. Would the concentration inside the cell be increasing or decreasing if $C = \Gamma$? What would this mean for the cell?

• EXERCISE **5.1.34**
Suppose that the constant of proportionality governing the rate at which chemical enters the cell is half as large as the constant governing the rate at which it leaves. Would the concentration inside the cell be increasing or decreasing if $C = \Gamma$? What would this mean for the cell?

• EXERCISE **5.1.35**
Suppose that the constant of proportionality governing the rate at which chemical enters the cell is proportional to $1 + C$ (because the chemical helps to open special channels). Would the concentration inside the cell be increasing or decreasing if $C = \Gamma$? What would this mean for the cell?

• EXERCISE **5.1.36**

Suppose that the constant of proportionality governing the rate at which chemical enters the cell is proportional to $1 - C$ (because the chemical helps to close special channels). Would the concentration inside the cell be increasing or decreasing if $C = \Gamma$? What would this mean for the cell?

♠ Our model of changing gene frequencies included no interaction between bacterial types a and b because the per capita reproduction of each type is a constant. Write a pair of differential equations for a and b with the following forms for the per capita reproduction, and derive an equation for the fraction p of type a. Assume that the basic per capita reproduction for type a is $\mu = 2$, and that for type b is $\lambda = 1.5$.

• EXERCISE **5.1.37**

The per capital reproduction of each type is reduced by a factor of $1 - p$ (so that the per capita reproduction of type a is $2(1 - p)$). This is a case where a large proportion of type a reduces the reproduction of both types. Will type a take over?

• EXERCISE **5.1.38**

The per capital reproduction of type a is reduced by a factor of $1 - p$ and the per capita reproduction of type b is reduced by a factor of p. This is a case where a large proportion of type a reduces the reproduction of type a, and a large proportion of type b reduces the reproduction of type b. Do you think that type a will still take over.

♠ We will find later (with separation of variables) that the solution for Newton's law of cooling with initial condition $H(0)$ is

$$H(t) = A + (H(0) - A)e^{-\alpha t}.$$

For each set of given parameter values,

    a. Write and check the solution.

    b. Find the temperature at $t = 1$ and $t = 2$.

    c. Sketch of graph of your solution. What happens as t approaches infinity?

• EXERCISE **5.1.39**

Set $\alpha = 0.2/\text{min}$, $A = 10°\text{C}$ and $H(0) = 40$.

• EXERCISE **5.1.40**

Set $\alpha = 0.02/\text{min}$, $A = 30°\text{C}$ and $H(0) = 40$.

♠ Use Euler's method to estimate the temperature for the following cases of Newton's law of cooling. Compare with the exact answer.

• EXERCISE **5.1.41**

$\alpha = 0.2/\text{min}$ and $A = 10°\text{C}$ and $H(0) = 40$. Estimate $H(1)$ and $H(2)$ using $\Delta t = 1$. Compare with exercise 5.1.39.

• EXERCISE **5.1.42**

$\alpha = 0.02/\text{min}$ and $A = 30°\text{C}$ and $H(0) = 40$. Estimate $H(1)$ and $H(2)$ using $\Delta t = 1$. Compare with exercise 5.1.40. Why is the result so close?

♠ Use the solution for Newton's law of cooling (exercises 5.1.39 and 5.1.40 to find the solution expressing the concentration of chemical inside a cell as a function of time in the following examples. Find the concentration after 10 seconds, 20 seconds and 60 seconds. Sketch your solutions for the first minute.

• EXERCISE **5.1.43**

$\beta = 0.01/\text{sec}$, $C(0) = 5.0$ micromoles per cubic centimeter and $\Gamma = 2.0$ micromoles per cubic centimeter.

• EXERCISE **5.1.44**

$\beta = 0.1/\text{sec}$, $C(0) = 5.0$ micromoles per cubic centimeter and $\Gamma = 2.0$ micromoles per cubic centimeter.

♠ Recall that the solution of the updating equation $b_{t+1} = rb_t$ is $b_t = r^t b_0$. This is closely related to the differential equation $\frac{db}{dt} = \lambda b$.

• EXERCISE **5.1.45**

For what values of b_0 and r does this solution match $b(t) = 1.0 \times 10^6 e^{2t}$ (the solution of the differential equation with $\lambda = 2$ and $b(0) = 1.0 \times 10^6$) for all values of t?

- EXERCISE **5.1.46**

  For what values of b_0 and r does this solution match $b(t) = 100e^{-3t}$ (the solution of the differential equation with $\lambda = -3$ and $b(0) = 100$) for all values of t?

- EXERCISE **5.1.47**

  For what values of λ do solutions of the differential equation grow? For what values of r do solutions of the discrete-time dynamical system grow?

- EXERCISE **5.1.48**

  What is the relation between r and λ? That is, what value of r produces the same growth as a given value of λ?

♠ Use Euler's method to estimate the value of $p(t)$ for the given parameter values. Compare with the exact answer using the equation for the solution. Graph the solution, including the estimates from Euler's method.

- EXERCISE **5.1.49**

  Suppose $\mu = 2.0$, $\lambda = 1.0$ and $p(0) = 0.1$. Estimate the proportion after two minutes using a time step of $\Delta t = 0.5$.

- EXERCISE **5.1.50**

  Suppose $\mu = 2.5$, $\lambda = 3.0$ and $p(0) = 0.6$. Estimate the proportion after one minute using a time step of $\Delta t = 0.25$.

♠ The rate of change in the differential equation

$$\frac{dp}{dt} = (\mu - \lambda)p(1 - p)$$

is a measure of the strength of selection. For each set of parameter values, graph the rate of change as a function of p with $\mu = 2.0$. For what value of p is the rate of change fastest?

- EXERCISE **5.1.51**

  Graph the rate of change as a function of p with $\mu = 2.0$ and $\lambda = 1.0$.

- EXERCISE **5.1.52**

  Graph the rate of change as a function of p with $\mu = 0.2$ and $\lambda = 0.1$.

♠ Compute $\Delta p = p_{t+1} - p_t$ for the following discrete-time dynamical systems, and try to simplify to see whether the change resembles the related differential equation.

- EXERCISE **5.1.53**

  The discrete-time dynamical system describing selection,

  $$p_{t+1} = \frac{sp_t}{sp_t + r(1 - p_t)}.$$

  After a bit of algebra, the answer should look a bit like the differential equation $\frac{dp}{dt} = (s - r)p(1 - p)$.

- EXERCISE **5.1.54**

  The logistic discrete-time dynamical system

  $$p_{t+1} = rp_t(1 - p_t).$$

  After a bit of algebra, the answer should look a bit like the logistic differential equation $\frac{dp}{dt} = rp(1 - p)$.

Chapter 6

Answers

5.1.1. This differential equation is nonautonomous because the state variable F and the time t both appear on the right hand side. The parameter k does not affect the kind of differential equation.

5.1.3. This is a pure-time differential equation because the only variable on the right hand side is the time t (along with the parameter μ).
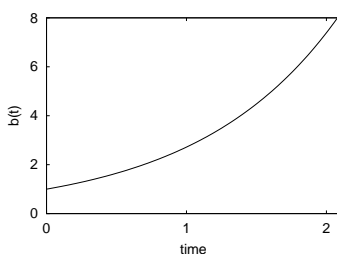
5.1.5. Substituting these values into the right hand side gives $\frac{dF}{dt} = 1^2 + 1 \cdot 0 = 1 > 0$. Therefore, F is increasing.

5.1.7. Substituting these values into the right hand side gives $\frac{dy}{dt} = 2e^{-1} - 1 = -0.26 < 0$. Therefore, y is decreasing.

5.1.9. $p(t) = \frac{t^2}{2} + 1$. Then $p(1) = 1.5$.
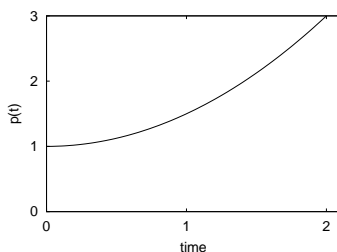
5.1.11. The derivative of e^t is e^t, and $e^0 = 1$, so this is the solution that matches the initial conditions. Also, $b(1) = e = 2.718$.

**5.1.13.** They don't match because the rate of change gets larger the longer you wait.

5.1.15. First, $x(t) = -\frac{1}{2} + \frac{3}{2}e^0 = 1$, so the initial condition matches. Next,

$$\frac{dx}{dt} = 3e^{2t} = 2(-\frac{1}{2} + \frac{3}{2}e^{2t}) + 1 = 1 + 2x$$

This checks.

5.1.17. First, $G(0) = 1 + e^0 = 2$, so the initial condition matches. Next, $\frac{dG}{dt} = e^t = (1 + e^t) - 1 = G - 1$. This checks.

5.1.19. $\hat{x}(1) = x(0) + x'(0) \cdot 1 = 1 + 3 = 4$. When $x = 4$, the°differential equation says that $dx/dt = 1 + 2 \cdot 4 = 9$, so that $\hat{x}(2) = \hat{x}(1) + x'(1) \cdot 1 = 4 + 9 = 13$. The exact answer is $-\frac{1}{2} + \frac{3}{2}e^{2 \cdot 2} = 81.4$ Euler's method is way off.
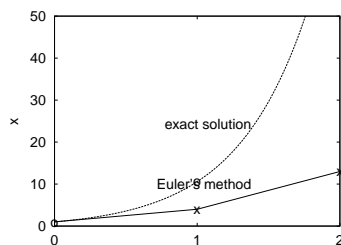
5.1.21.

$$
\begin{aligned}
\hat{G}(0.2) &= G(0) + G'(0) \cdot 0.2 = 2 + 1 \cdot 0.2 = 2.2 \\
\hat{G}(0.4) &= \hat{G}(0.2) + G'(0.2) \cdot 0.2 = 2.2 + 1.2 \cdot 0.2 = 2.44 \\
\hat{G}(0.6) &= \hat{G}(0.4) + G'(0.4) \cdot 0.2 = 2.44 + 1.44 \cdot 0.2 = 2.728 \\
\hat{G}(0.8) &= \hat{G}(0.6) + G'(0.6) \cdot 0.2 = 2.728 + 1.728 \cdot 0.2 = 3.0736 \\
\hat{G}(1.0) &= \hat{G}(0.8) + G'(0.8) \cdot 0.2 = 3.0736 + 2.0736 \cdot 0.2 = 3.48832.
\end{aligned}
$$

The exact answer is $1 + e^1 = 3.7182$ Euler's method is fairly close.
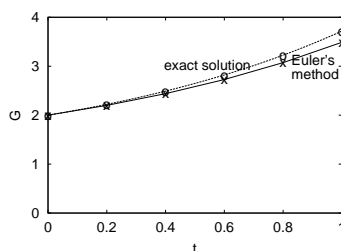
5.1.23. The derivative of y can be found with the chain rule, by thinking of $y(t)$ as the composition $F \circ x(t)$ where $F(x) = 2x - 1$. Then

$$
\frac{dy}{dt} = \frac{dF}{dx}\frac{dx}{dt} = 2\frac{dx}{dt} = 2(2x - 1).
$$

We need to rewrite the right hand side entirely in terms of y, and get

$$
\frac{dy}{dt} = 2y.
$$

This is simpler because the -1 term has disappeared.

5.1.25. The derivative of y can be found with the chain rule, by thinking of $y(t)$ as the composition $F \circ x(t)$ where $F(x) = 1/x$. Then

$$
\frac{dy}{dt} = \frac{dF}{dx}\frac{dx}{dt} = -\frac{1}{x^2}\frac{dx}{dt} = -\frac{1}{x^2}(x + x^2).
$$

We need to rewrite the right hand side entirely in terms of y, and get
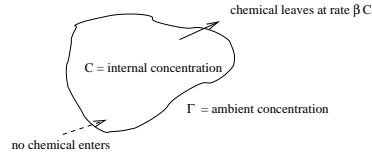
$$
\frac{dy}{dt} = -y^2\left(\frac{1}{y} + \frac{1}{y^2}\right) = -y + 1.
$$

This differential equation is linear.

5.1.27. A line with intercept 1 and slope -0.002 has equation $\lambda(b) = 1 - 0.002b$, so that the differential equation is $\frac{db}{dt} = (1 - 0.002b)b$. When $b = 10$, $\frac{db}{dt} = 9.8 > 0$, so this population would increase. When $b = 1000$, $\frac{db}{dt} = -1000 < 0$, so this population would decrease.
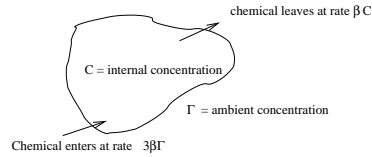
5.1.29. A line with intercept -2 and slope 0.01 has equation $\lambda(b) = -2 + 0.01b$, so that the differential equation is $\frac{db}{dt} = (-2 + 0.01b)b$. When $b = 100$, $\frac{db}{dt} = -200 < 0$, so this population would decrease. When $b = 300$, $\frac{db}{dt} = 300 > 0$, so this population would increase.

5.1.31.

chemical leaves at rate βC

C = internal concentration

Γ = ambient concentration

no chemical enters

The equation is $\frac{dC}{dt} = -\beta C$. This is the same as the differential equation for a shrinking population. The population size (or the chemical concentration) decays exponentially to 0 because there is no source of chemical.

5.1.33.

chemical leaves at rate βC

C = internal concentration

Γ = ambient concentration

Chemical enters at rate $3\beta\Gamma$

The equation is $\frac{dC}{dt} = -\beta C + 3\beta\Gamma$. If the concentrations were equal, the derivative would be positive, meaning that the internal concentration would increase, and become larger than the external concentration.

5.1.35.

chemical leaves at rate βC

C = internal concentration

Γ = ambient concentration

Chemical enters at rate $\beta(1+C)\Gamma$
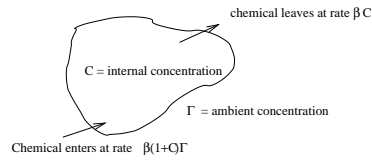
The equation is $\frac{dC}{dt} = -\beta C + \beta(1 + C)\Gamma$. If the concentrations were equal, the derivative would be positive, meaning that the internal concentration would increase, and become larger than the external concentration.

5.1.37. The differential equations are

$$
\begin{aligned}
\frac{da}{dt} &= 2(1-p)a \\
\frac{db}{dt} &= 1.5(1-p)b.
\end{aligned}
$$

Then

$$
\begin{aligned}
\frac{dp}{dt} &= \frac{b\frac{da}{dt} - a\frac{db}{dt}}{(a+b)^2} \\
&= \frac{2(1-p))ab - 1.5(1-p))ab}{(a+b)^2} \\
&= \frac{(2-1.5)(1-p)ab}{(a+b)^2} \\
&= 0.5p(1-p)^2.
\end{aligned}
$$

Even though the differential equation is different, the rate of change of p is always positive, and type a still takes over.

5.1.39.

a. $H(t) = 10 + 30e^{-0.2t}$. $H(0) = 40$, matching the initial condition.
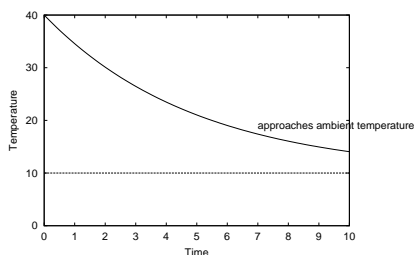
$$\frac{dH}{dt} = -6.0e^{0.2t}.$$

This is supposed to match $\alpha(A - H(t))$, which is

$$\alpha(A - H(t)) = 0.2(10 - (10 + 30e^{-0.2t})) = 0.2(-30e^{-0.2t}) = -6.0e^{0.2t}.$$
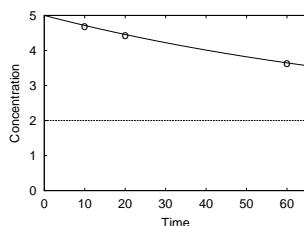
It checks.

b. $H(1) = 34.56$ and $H(2) = 30.1$.

c. The term $30e^{-0.2t}$ approaches 0, so $\lim_{t \to \infty} H(t) = 10$, the ambient temperature.

5.1.41. $H'(0) = 0.2(10 - 40) = -6.0$. Then $\hat{H}(1.0) = H(0) - 6.0 \cdot 1.0 = 34.0°$C. $H'(1.0) = 0.2(10 - 34) = -4.8$. Then $\hat{H}(2.0) = \hat{H}(1.0) - 4.8 \cdot 1.0 = 29.2°$C. This is a bit low.

5.1.43.

$$
\begin{aligned}
C(t) &= \Gamma + e^{-\beta t}(C(0) - \Gamma) = 2.0 + e^{-0.01t}(5.0 - 2.0) = 2.0 + 3.0e^{-0.01t} \\
C(10) &= 2.0 + 3.0e^{-0.1} = 4.714 \\
C(20) &= 2.0 + 3.0e^{-0.2} = 4.456 \\
C(60) &= 2.0 + 3.0e^{-0.6} = 3.646.
\end{aligned}
$$

5.1.45. When $b_0 = 1.0 \times 10^6$ and $\ln(r) = 2$ or $r = 7.39$.

5.1.47. The population described by the differential equation grows when $\lambda > 0$. The population described by the discrete-time dynamical system grows when $r > 1$.
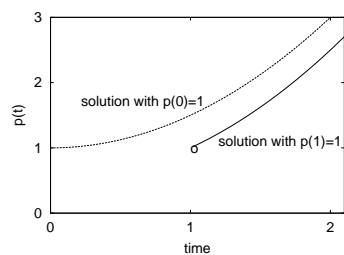
5.1.49. We need to take four steps.

$$
\begin{aligned}
\hat{p}(0.5) &= 0.1 + 0.1 \cdot 0.9 \cdot 0.5 = 0.145 \\
\hat{p}(1.0) &= 0.145 + 0.145 \cdot 0.855 \cdot 0.5 = 0.207 \\
\hat{p}(1.5) &= 0.207 + 0.207 \cdot 0.793 \cdot 0.5 = 0.289 \\
\hat{p}(2.0) &= 0.289 + 0.289 \cdot 0.711 \cdot 0.5 = 0.392.
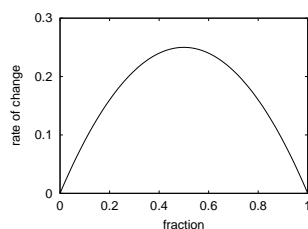\end{aligned}
$$

The exact solution is

$$p(t) = \frac{0.1e^{2.0t}}{0.1e^{2.0t} + 0.9e^{1.0t}}$$

so that $p(2) = 0.45$.

5.1.51.

The rate of change is greatest when $p = 0.5$.

5.1.53. The change in p is

$$
\begin{aligned}
\Delta p &= p_{t+1} - p_t \\
&= \frac{sp_t}{sp_t + r(1 - p_t)} - p_t \\
&= \frac{sp_t - spt^2 - rp_t(1 - p_t)}{sp_t + r(1 - p_t)} = \frac{(s - r)p_t(1 - p_t)}{sp_t + r(1 - p_t)}.
\end{aligned}
$$

The numerator looks like the rate of change in the differential equation, but the denominator is new.