4.7 Applications of integrals

MATHEMATICAL TECHNIQUES

♠ Find the areas under the following curves. If you use substitution, draw a graph to compare the original area with that in transformed variables.

- EXERCISE **4.7.1**
  Area under $f(x) = 3x^3$ from $x = 0$ to $x = 3$.
- EXERCISE **4.7.2**
  Area under $g(x) = e^x$ from $x = 0$ to $x = \ln 2$.
- EXERCISE **4.7.3**
  Area under $h(x) = e^{x/2}$ from $x = 0$ to $x = \ln 2$.
- EXERCISE **4.7.4**
  Area under $f(t) = (1 + 3t)^3$ from $t = 0$ to $t = 2$.
- EXERCISE **4.7.5**
  Area under $G(y) = (3 + 4y)^{-2}$ from $y = 0$ to $y = 2$.
- EXERCISE **4.7.6**
  Area under $s(z) = \sin(z + \pi)$ from $z = 0$ to $z = \pi$.

♠ The definite integral can be used to find the area between two curves. In each case,

a. Sketch the graphs of the two functions over the given range, and shade the area between the curves.

b. Sketch the graph of the difference between the two curves. Note that the area **under** this curve matches the area **between** the original curves.

c. Find the area under the difference curve (remembering to use absolute value).

- EXERCISE **4.7.7**
  Find the area between $f(x) = 2x$ and $g(x) = x^2$ for $0 \le x \le 2$.
- EXERCISE **4.7.8**
  Find the area between $f(x) = e^x$ and $g(x) = x + 1$ for $-1 \le x \le 1$.
- EXERCISE **4.7.9**
  Find the area between $f(x) = 2x$ and $g(x) = x^2$ for $0 \le x \le 4$.
- EXERCISE **4.7.10**
  Find the area between $f(x) = x^2$ and $g(x) = x^3$ for $0 \le x \le 2$.
- EXERCISE **4.7.11**
  Find the area between $f(x) = e^x$ and $g(x) = \frac{e^{2x}}{2}$ for $0 \le x \le 1$.
- EXERCISE **4.7.12**
  Find the area between $f(x) = \sin(2x)$ and $g(x) = \cos(2x)$ for $0 \le x \le \pi$.

♠ Find the average value of the following functions over the given range. Sketch a graph of the function along with a horizontal line at the average to make sure that your answer makes sense.

- EXERCISE **4.7.13**
  x^2 for $0 < x < 3$.
- EXERCISE **4.7.14**
  $\frac{1}{x}$ for $0.5 < x < 2.0$.
- EXERCISE **4.7.15**
  $x - x^3$ for $-1 < x < 1$.
- EXERCISE **4.7.16**
  $\sin(2x)$ for $0 \le x \le \pi/2$.

♠ We have used little vertically oriented rectangles to compute areas. There is no reason why little horizontal rectangles cannot be used. Here are the steps to find the area under the curve $y = f(x)$ from $x = 0$ to $x = 1$ by using such horizontal rectangles.

a. Draw a picture with five horizontal rectangles, each of height 0.2, approximately filling the region to the right of the curve.

b. Calculate an upper and lower estimate of the length of each rectangle based on the length of the upper and lower boundaries.

c. Add these up to find upper and lower estimates of the area.

d. Think now of a very thin rectangle at height y. How long is the rectangle?

e. Write down a definite integral expression for the area.

f. Evaluate the integral and check that the answer is correct.

• EXERCISE **4.7.17**
With $f(x) = x^2$.

• EXERCISE **4.7.18**
With $f(x) = \sqrt{x}$.

♠ Archimedes developed the basic idea of integration to find the areas of geometric figures. Often, this involves using building regions out of small pieces with shapes more complicated than rectangles.

• EXERCISE **4.7.19**
Use the fact that the perimeter of a circle of radius r is $2\pi r$ to find the area of a circle with radius 1. Think of the circular region as being built out of little rings with some small width Δr.

• EXERCISE **4.7.20**
Use the fact that the area of a circle of radius r is πr^2 to find the volume of a cone of height 1 that has radius r at a height r. Think of the cone as being built of a stack of little circular disks with some small thickness Δr.

♠ Some books define the natural log function with the definite integral as the function $l(a)$ for which

$$l(a) = \int_1^a \frac{1}{x} dx.$$

Using this definition, we can prove the laws of logs (chapter 6).

• EXERCISE **4.7.21**
Show that $l(6) - l(3) = l(2)$. (Use the summation property of the definite integral to write the difference as an integral, and then use the substitution $y = \frac{x}{3}$).

• EXERCISE **4.7.22**
Find the integral from a to $2a$ by following the same steps (make the substitution $y = \frac{x}{a}$). What law of logs does this correspond to?

• EXERCISE **4.7.23**
Show that $l(10^2) = 2 \cdot l(10)$. (Try the substitution $y = \sqrt{x}$ in $\int_1^{10^2} \frac{1}{x}\, dx$).

• EXERCISE **4.7.24**
Show that $l(a^b) = b \cdot l(a)$. (Try the substitution $y = \sqrt[b]{x}$ in $\int_1^{a^b} \frac{1}{x}\, dx$).

APPLICATIONS

♠ The average of a step function computed with the definite integral matches the average computed in the usual way. Test this in the following situations by finding the average of the values directly, and then as the integral of a step function.

• EXERCISE **4.7.25**
Suppose a math class has four equally weighted tests. A student gets 60 on the first test, 70 on the second, 80 on the third, and 90 on the last.

• EXERCISE **4.7.26**
A math class has twenty students. In a quiz worth 10 points, 4 students get 6, 7 students get 7, 5 students get 8, 3 students get 9 and 1 student gets 10.

♠ Suppose water is entering a tank at a rate of $g(t) = 360t - 39t^2 + t^3$ where g is measured in L/h and t is measured in h. The rate is 0 at times 0, 15 and 24.

• EXERCISE **4.7.27**
Find the total amount of water entering during the first 15 h, from $t = 0$ to $t = 15$. Find the average rate at which water entered during this time.

• EXERCISE **4.7.28**
Find the total amount and average rate from $t = 15$ to $t = 24$.
• EXERCISE **4.7.29**
Find the total amount and average rate from $t = 0$ to $t = 24$.
• EXERCISE **4.7.30**
Suppose that energy is produced at a rate of

$$E(t) = |g(t)|$$

in J/h. Find the total energy generated from $t = 0$ to $t = 24$. Find the average rate of energy production.

♠ Several very skinny 2.0 m long snakes are collected in the Amazon. Each has density of $\rho(x)$ given by the following formulas, where ρ is measured in g/cm and x is measured in centimeters from the tip of the tail. For each snake,

a. Find the minimum and maximum density of the snake. Where does the maximum occur?

b. Find the total mass of the snake.

c. Find the average density of the snake. How does this compare with the minimum and maximum?

d. Graph the density and average.

• EXERCISE **4.7.31**
$\rho(x) = 1.0 + 2.0 \times 10^{-8}x^2(300 - x)$
• EXERCISE **4.7.32**
$\rho(x) = 1.0 + 2.0 \times 10^{-8}x^2(240 - x)$

♠ A piece of E. coli DNA has about 4.7×10^6 nucleotides, and is about 1.6×10^6 nm long. The genetic code consists of 4 possible nucleotides, called A, C, G and T. For each of the following cases,

a. Use the given information to find the formula for the number of A's, C's, G's and T's per thousand as a function of distance along the DNA strand.

b. Find the total number of A's, C's, G's and T's in the DNA.

c. Find the mean number of A's, C's, G's and T's in the DNA per thousand.

• EXERCISE **4.7.33**
Suppose that the number of A's per thousand increases linearly from 150 at one end of the DNA strand to 300 at the other. The number of C's per thousand decreases linearly from 350 at one end to 200 at the other, and the number of G's per thousand increases linearly from 220 at one end to 320 at the other. The remainder is made up of T's.
• EXERCISE **4.7.34**
Suppose that the number of A's per thousand increases linearly from 200 at one end of the DNA strand to 250 at the other. The number of C's per thousand increases linearly from 250 at one end to 300 at the other, and the number of G's per thousand decreases linearly from 300 at one end to 200 at the other. The remainder is made up of T's.

♠ Suppose water is entering a series of vessels at the given rate. In each case, find the total amount of water entering during the first second, and the average rate during that time. Compare the average rate with rate at the "average time", at $t = 0.5$ halfway through the time period from 0 to 1. In which case is the average rate greater than the rate at the average time? Graph the flow rate function, and mark the flow rate at the average time. Can you guess what it is about the shape of the graph that determines how the average rate compares with the rate at the average time?

• EXERCISE **4.7.35**
Water is entering at a rate of $t^3\text{cm}^3/\text{s}$.
• EXERCISE **4.7.36**
Water is entering at a rate of $\sqrt{t}$ cm^3/s.
• EXERCISE **4.7.37**
Water is entering at a rate of t cm^3/s.
• EXERCISE **4.7.38**
Water is entering at a rate of $4t(1 - t)$ cm^3/s.

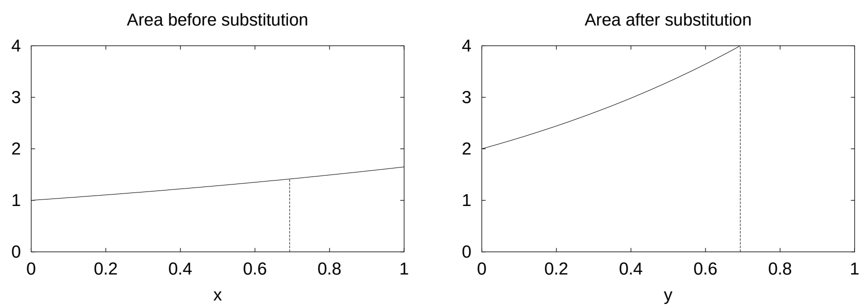
Chapter 5

Answers

4.7.1. $\int_0^3 3x^3\,dx = 3x^4/4|_0^3 = 60.75.$

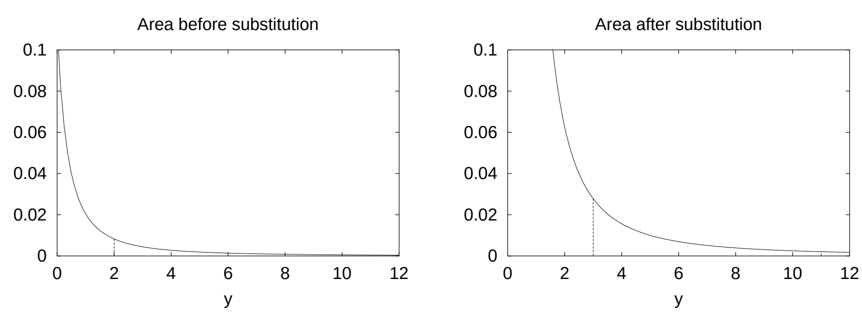
4.7.3. Use the substitution $y = x/2$. Then $dx = 2dy$ and the limits of integration are $y = 0$ to $y = \frac{\ln 2}{2}$.

$$\int_0^{\ln 2} e^{x/2} dx = \int_0^{\frac{\ln 2}{2}} 2e^y dy = 2(\sqrt{2} - 1) = 0.828$$

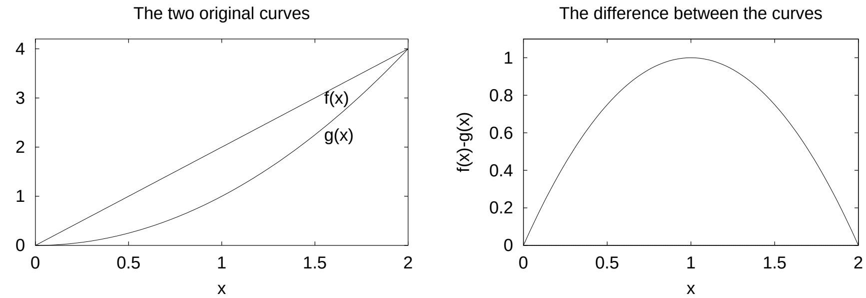


4.7.5. Set $z = (3 + 4y)$. Then $dy = dz/4$, and the limits of integration are from $z = 3$ to $z = 11$:

$$\int_0^2 (3 + 4y)^{-2} dy = \int_3^{11} \frac{z^{-2}}{4}\,dz = \frac{-z^{-1}}{4}|_3^{11} = 0.061.$$



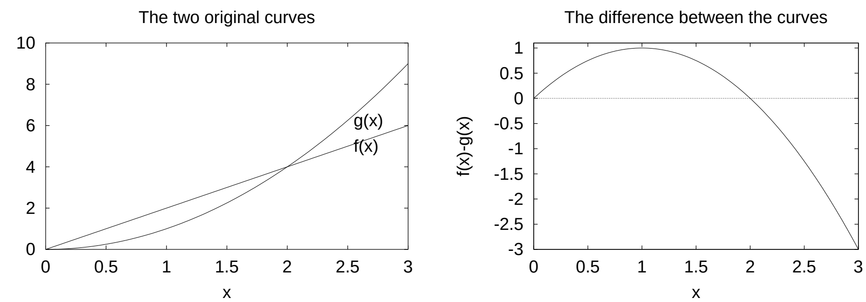
4.7.7.



c. $2x - x^2 > 0$ for all $0 \le x \le 2$. Therefore,

$$\int_0^2 2x - x^2\,dx = x^2 - \frac{x^3}{3}|_0^2 = (4 - 8/3) = 1.33.$$

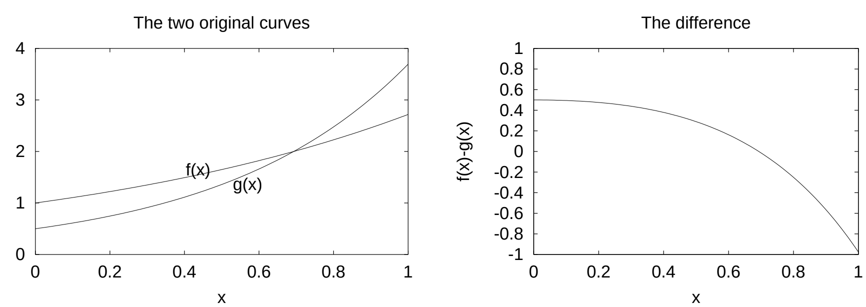
4.7.9.



c. $2x - x^2 > 0$ for $0 \le x < 2$. but $2x - x^2 < 0$ for $2 < x \le 3$. Therefore,

$$\begin{aligned}\int_0^3 |2x - x^2| dx &= \int_0^2 (2x - x^2)\,dx - \int_2^3 (x^2 - 2x)\,dx \\ &= x^2 - \frac{x^3}{3}|_0^2 + \frac{x^3}{3} - x^2|_2^3 \\ &= 4/3 + (9 - 9 - 8/3 + 4) = 8/3.\end{aligned}$$

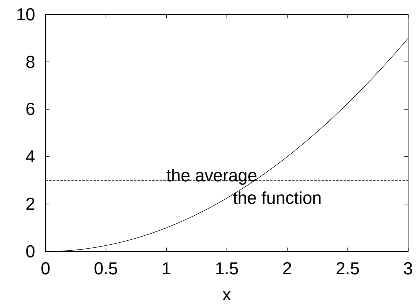
4.7.11.



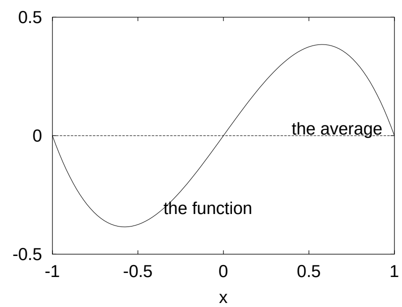
c. $e^x - \frac{e^{2x}}{2} > 0$ if $x < \ln(2)$. Therefore,

$$\begin{aligned}\int_0^1 |e^x - \frac{e^{2x}}{2}| dx &= \int_0^{\ln(2)} (e^x - \frac{e^{2x}}{2}) dx - \int_{\ln(2)}^1 (\frac{e^{2x}}{2} - e^x) dx \\ &= (e^x - \frac{e^{2x}}{4})|_0^{\ln(2)} - (\frac{e^{2x}}{4} - e^x)|_{\ln(2)}^1 \\ &= -e + \frac{e^2}{4} + \frac{5}{4} = 0.378\end{aligned}$$

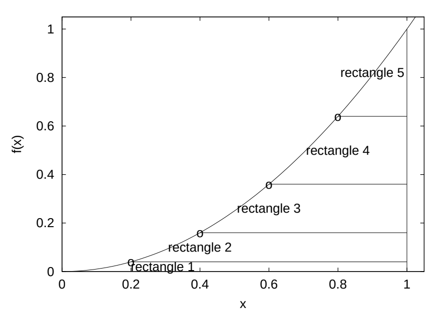
4.7.13. $\int_0^3 x^2 = \frac{x^3}{3}|_0^3 = 9$. The average is the integral divided by the width of the interval, or 9/3=3.



4.7.15. $\int_{-1}^1 x - x^3 = \frac{x^2}{2} - \frac{x^4}{4}|_{-1}^1 = 0$. The average is the integral divided by the width of the interval, or 0/2=0.



4.7.17.



a.

b. Rectangle 1: lower size estimate is $0.8 \cdot f(0.2) = 0.0032$; upper size estimate is $1.0 \cdot f(0.2) = 0.004$, Rectangle 2: lower size estimate is $0.6 \cdot (f(0.4 - f(0.2)) = 0.072$; upper size estimate is $0.8 \cdot (f(0.4) - f(0.2)) = 0.096$. Rectangle 3: lower size estimate is $0.4 \cdot (f(0.6 - f(0.4)) = 0.08$; upper size estimate is $0.6 \cdot (f(0.6) - f(0.4)) = 0.08$. Rectangle 4: lower size estimate is $0.2 \cdot (f(0.8 - f(0.6)) = 0.056$; upper size estimate is $0.4 \cdot (f(0.8) - f(0.6)) = 0.112$. Rectangle 5: lower size estimate is 0.0; upper size estimate is $0.2 \cdot (f(1.0) - f(0.8)) = 0.072$.

c. Lower estimate is 0.211; upper estimate is 0.364.

d. A rectangle at height y goes from the point where $x^2 = y$, or $x = \sqrt{y}$, to $x = 1$. Its length is $1 - \sqrt{y}$.

e. Area=$\int_0^1 (1 - \sqrt{y})\, dy$.

f. $\int_0^1 (1 - \sqrt{y})\, dy = (y - 2y^{3/2}/3)|_0^1 = 1/3$. It checks.

4.7.19. A little ring at radius r will have area approximately equal to $2\pi r \Delta r$ because it looks like a curved rectangles with length equal to the perimeter $2\pi r$ and width Δr. If we pick n rings with width Δr (so that $n\Delta r = 1$), the total area will be approximated by the Riemann sum $\sum_{i=0}^n 2\pi r \Delta r$. In the limit, this is the definite integral $\int_0^1 2\pi r\, dr = \pi r^2|_0^1 = \pi$.

4.7.21.

$$l(6) - l(3) = \int_1^6 \frac{1}{x} dx - \int_1^3 \frac{1}{x} dx = \int_3^6 \frac{1}{x} dx = \int_1^2 \frac{1}{y} dy = l(2)$$

where we set $y = \frac{x}{3}$, so $dy = \frac{dx}{3}$ and the limits of integration go from 1 to 2.

4.7.23. $\int_1^{10^2} 1/x\, dx = l(10^2)$. Substituting $y = \sqrt{x}$, we find that

$$\frac{dy}{dx} = \frac{1}{2} x^{-1/2} = \frac{1}{2} \frac{y}{x}.$$

Then the integrand becomes

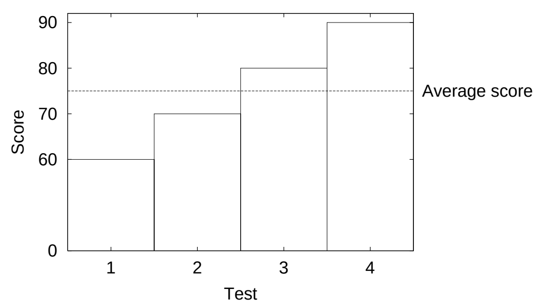
$$\frac{1}{x}\, dx = \frac{2}{y}\, dy$$

and the limits of integration go from 1 to 10. So

$$\int_1^{10^2} \frac{1}{x}\, dx = \int_1^1 0 \frac{2}{y}\, dy = 2l(10).$$

This matches the law of logs.

4.7.25.



The average is (60+70+80+90)/4=75. The function is

$$f(x) = \begin{cases} 60 & \text{for } 0 \le x < 1 \\ 70 & \text{for } 1 \le x < 2 \\ 80 & \text{for } 2 \le x < 3 \\ 90 & \text{for } 3 \le x < 4 \end{cases}$$

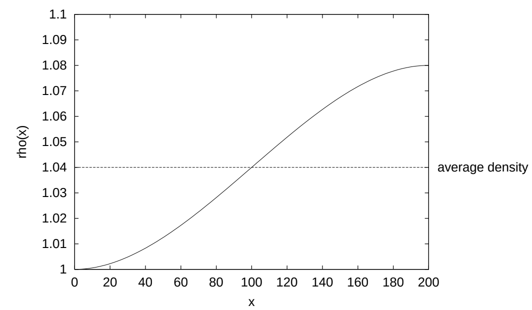
$$\begin{aligned} \text{Total score} &= \int_0^4 f(x) dx = 60 + 70 + 80 + 90 = 300 \\ \text{Average score} &= \frac{\text{Total score}}{\text{width of interval}} = \frac{300}{4} = 75. \end{aligned}$$

4.7.27. $\int_0^{15} g(t)\, dt = 9281.25$ L. Divide this answer by 15 h to find 618.75 L/h.

4.7.29. $\int_0^{24} g(t)\, dt = 6912.0$ L. Over the full 24 h, the average rate is 288.0 L/h.

4.7.31.

- **a.** The critical points are at $x = 0$ and $x = 200$ (both endpoints). Because $\rho(0) = 1.0$ and $\rho(200) = 1.08$, the first is the minimum and the second is the maximum.
- **b.** $\int_0^{200} \rho(x) dx = 208$ gm.
- **c.** The average is 1.04 g/cm, which lies right between the minimum and the maximum.

d. 

4.7.33.

a. Let x represent distance along the strand. Then the formula for the line giving the number of A's can be found by finding the slope as

$$\text{slope} = \frac{300 - 150}{4.7\times 10^6} = 3.19\times 10^{-5}.$$

Using the point $A(0) = 150$, we find $A(x) = 150 + 3.19\times 10^{-5}x$. Similarly, $C(x) = 350 - 3.19\times 10^{-5}x$, $G(x) = 220 + 2.13\times 10^{-5}x$. Because $A(x) + C(x) + G(x) + T(x) = 1000$,

$$T(x) = 1000 - A(x) - C(x) - G(x) = 280 - 2.13\times 10^{-5}x.$$

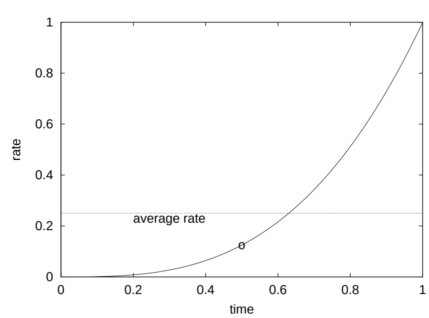
b. The totals can be found by integrating,

$$\int_0^{4.7\times 10^6} A(x) = 1.057\times 10^9.$$

However, this must be divided by 1000 to get the actual number, 1.057×10^6. The total number of C's is 1.292×10^6, the total number of G's is 1.269×10^6 and the total number of T's is 1.055×10^6.

c. The mean numbers per thousand are: A, 225; C, 275; G, 270; T, 230.

4.7.35. $\int_0^1 t^3\, dt = 0.25\text{cm}^3$. The average rate is $0.25\text{cm}^3/\text{s}$. The rate at time 0.5 is 0.125, less than the average rate during the first second. This seems to be because the function is concave up (has positive second derivative).



4.7.37. $\int_0^1 t\, dt = 0.5\text{cm}^3$. The average rate is $0.5\text{cm}^3/\text{s}$. The average matches the rate at the average time, probably because this function is linear.