May 2000: Clay Mathematics Institute announced seven million dollar problems.

Chosen by seven mathematicians of international reputation, including Andrew Wiles.

Considered to be the most important unsolved problems in mathematics, and despite a lot of effort remain unsolved.

1900: Hilbert indicated what he thought were the 23 most important unsolved problems. (No prize money). Only one of these remains unsolved. Gave mathematical research a direction for the 20th century.

Clay prizes not intended to guide mathematics, but to focus attention on some important, difficult problems.

Millenium Prize Problems

- Riemann Hypothesis
- Poincare Conjecture
- Yang-Mills Theory
- P vs. NP
- Navier-Stokes Equations
- Birch, Swinnerton-Dyer Conjecture
- Hodge Conjecture

How do you collect?

Official rules on website www.claymath.org.

Clay Institute will determine if the problem is solved.

Solution must be published in a refereed mathematics journal.

Wait two years. If the proof is still “generally accepted” (i.e. no one has found anything wrong with it), then CMI will appoint a committee to verify the solution.

If solution depends heavily on previous work of others, they may get a cut.

Navier-Stokes equations:

$$\frac{\partial}{\partial t}u_i + \sum_{j=1}^{n} u_j \frac{\partial u_i}{\partial x_j} = \nu \Delta u_i - \frac{\partial p}{\partial x_i} + f_i(x,t) \quad (1)$$

$$\mathrm{div}\, u = \sum_{i=1}^{n} \frac{\partial u_i}{\partial x_i} = 0. \quad (2)$$

- u_i is the i-th component of the (unknown) velocity vector $u = u(x,t)$

- $\nu > 0$ is the (constant) viscosity of the fluid

- Δ is the Laplacian: $\Delta g = \sum_{i=1}^{n} \frac{\partial^2 g}{\partial x_i^2}$

- $p = p(x,t)$ is the (unknown) pressure

- $f_i(x,t)$ is the i-th component of a given external force

Solving the Navier-Stokes equations means finding (or showing the existence of)

- A real-valued function $p(x,t)$ defined on $\mathbb{R}^n \times [0,\infty)$, and

- A vector-valued function $u(x,t) : \mathbb{R}^n \times [0,\infty) \to \mathbb{R}^n$, satisfying (2) for all (x,t),

such that equation (1) is satisfied for all $(x,t) \in \mathbb{R}^n \times [0,\infty)$.

A solution pair (u, p) would describe the motion of a fluid that is

- incompressible ($div\ u = 0$), and
- homogeneous (constant density)

at all points x in space and at all times $t \geq 0$. The instantaneous velocity of a particle located at x at time t moved by the fluid is given by $u(x, t)$. The pressure of the fluid at that point at that moment in time is given by $p(x, t)$.

The viscosity of the fluid measures the frictional forces within the fluid. The higher the viscosity, the more resistant the fluid is to motion.

Fluids considered here are Newtonian, although non-Newtonian fluids are also important.

Some Conditions

A solution (u, p) is **physically reasonable** if:

- p and u are C^∞ functions on $\mathbb{R}^n \times [0, \infty)$, and
- there is a constant C such that for each $t \geq 0$, (bounded kinetic energy)

$$\int_{\mathbb{R}^n} |u(x,t)|^2 \, dx \leq C.$$

Decay conditions. Assume that for every α, K, and m, there exist constants $C_{\alpha K}$ and $C_{\alpha m K}$, such that for every (x, t):

$$|\partial_x^\alpha u_0(x)| \leq C_{\alpha K}(1 + |x|)^{-K}$$

$$|\partial_x^\alpha \partial_t^m f(x,t)| \leq C_{\alpha m K}(1 + |x| + t)^{-K}$$

Two Clay Navier-Stokes Problems

1. Let $\nu > 0$ and $n = 3$. Let $u_0(x)$ be a given smooth vector-valued function with $div\ u_0 = 0$, satisfying the decay condition. Let $f(x,t) \equiv 0$. Prove that there exists a physically reasonable solution to (1), (2) such that $u(x,0) = u_0(x)$.

2. Let $\nu > 0$ and $n = 3$. Find a smooth vector-valued function u_0, with $div\ u_0 = 0$, and a smooth function $f(x,t)$ satisfying the decay conditions for which there is no physically reasonable solution to (1), (2) satisfying $u(x,0) = u_0(x)$.

Why is this problem important?

- Solving N-S would lead to a deeper understanding of fluid behavior.
- Motion of fluids arise in many physical, biological and engineering problems.
- In applications, solutions in special cases are approximated. Better analytic understanding would lead to better approximations, advances in engineering.

Some History

- Daniel Bernoulli applied calculus methods to fluids (1738). Results still important today.

- Euler formulated (did not solve) equations for viscosity-free fluid ($\nu = 0$ in (1)) (1700s). Still not known if solution exists for $n = 3$.

- Navier extended Euler's work to viscous fluids. Despite errors in derivation, his equations were correct (1822).

- Stokes (correctly) derived equation (1) (1842). (same Stokes as Stokes' Theorem)

Solving PDEs is hard. In most cases, explicit formulas for solutions cannot be found. Strategy is to demonstrate that a solution exists, even if an exact formula is not available. If this can be done, the solution can be approximated.

Even this is often very difficult. Showing that a solution exists involves showing that there are sufficiently differentiable function(s), so that the equation makes sense for them. Showing this "regularity" is often difficult.

Weak solutions. Idea: Find function(s) that share some properties with solutions, but are not necessarily regular. Weak solutions have fewer properties than "classical" solutions, so theoretically it should be easier to find one. The next step is to show that the weak solution is actually regular.

Let $\theta(x,t)$ be a smooth, vector-valued function compactly supported in $\mathbb{R}^3 \times (0,\infty)$. Multiply (1) by θ_i and integrate over $\mathbb{R}^3 \times [0,\infty]$. Since θ is zero for large x and t (and t near 0), when integrating by parts, there are no integrals over the boundary.

Result is (sum over i and use $div\, u = 0$):

$$-\int u \cdot \frac{\partial \theta}{\partial t} - \sum_{ij} \int u_i\, u_j \frac{\partial \theta_i}{\partial x_j} =$$

$$\nu \int u \cdot \Delta\theta + \int f \cdot \theta - \int p \cdot (div\, \theta)$$

where the integrals are all over $\mathbb{R}^3 \times [0,\infty]$.

If $\phi(x,t)$ is a smooth real-valued function compactly supported in $\mathbb{R}^3 \times (0,\infty)$, multiplying (2) by ϕ and integrating by parts gives:

$$\int u \cdot \nabla_x \phi \, dx \, dt = 0.$$

The derivatives have been pushed onto the test functions θ and ϕ. A weak solution then is a pair (u,p) such that these integral equations hold for all such θ and ϕ. Note that no differentiablity of u or p is required to be a weak solution, they need only satisfy some integrability conditions.

What progress has been made?

- In two dimensions, the problem can be solved.
- If $u_0(x)$ is "small", the problem can be solved.
- The problem ($n = 3$) can be solved on short time intervals. Given $u_0(x)$, there exists a time $T > 0$ such that the problem can be solved on $\mathbb{R}^n \times [0, T)$. In many cases, T is too small to be of much practical use.
- Weak solutions exist (Leray, 1934).
- Some results on the regularity of weak solutions. (Caffarelli, Nirenberg, Kohn, Lin, Scheffer, 1970s-1998)

For more information:

- CMI webpage www.claymath.org
- *The Millenium Problems: The Seven Greatest Unsolved Mathematical Puzzles of Our Time*, Keith Devlin, Basic Books.
- *Existence and Smoothness of the Navier-Stokes Equation*, Charles Fefferman, available on CMI webpage.
- *Mathematical Problems*, David Hilbert, reprinted in Bulletin of AMS, Volume 37, Number 4, October 2000.
- *The Hilbert Challenge*, Jeremy J. Gray, Oxford University Press, 2000.

- *The Honors Class: Hilbert's Problems and Their Solvers*, Benjamin H. Yandell, A K Paters, 2002.

- *A Mathematical Introduction to Fluid Mechanics*, Alexandre J. Chorin and Jerrold E. Marsden, 3rd edition, Springer, 1993.

- *Partial Differential Equations: Methods and Applications*, Robert C. McOwen, 2nd edition, Prentice-Hall, 2003.

- *Elementary Fluid Dynamics*, D.J. Acheson, Oxford, 1990.

- *An Introduction to Fluid Dynamics*, G.K. Batchelor, Cambridge, 1967.