## Math 2210 Section 1 Outline of Topics for Exam 3

This last test before the final exam covers Chapter 16, the topic of which is the integration of functions of several variables. We began with integrals of functions of two variables over sets in the plane. Later, we generalized this to integrals of functions of three variables on domains in 3-dimensional space. In both the two and three dimensional settings, we saw how to compute integrals in standard rectangular coordinates as well as in other coordinate systems (polar, cylindrical, spherical), which is more convenient for some problems. We also saw some applications of multiple integrals (computation of the mass and center-of mass of laminae and solid objects and the computation of the surface area of solids).

This summary is intended as a guide to preparing for the exam. You are permitted to bring one sheet of normal-sized paper as a formula sheet to the test. You can write anything you want on it: relevant formlas, definitions, or worked-out examples. Hopefully, this summary will help you in deciding what to include on your sheet. The best guide to the types of questions to expect is the homework (both the problems on COW and those from the text). If you are able to do these problems and understand what is going on with them, you should have no problem with the test. The problems from the text are more important for this test than they were for the previous test, because we did not have an online assignment for the material following Section 16.3. This review is not guaranteed to be complete: the fact that something is not listed here does not mean it will not be on the exam. On the other hand, just because a type of problem is mentioned below, this does not necessarily mean that it will appear on the test.

We began by defining the integral. This is of critical theoretical importance, but is not the way an integral is usually calculated, although Riemann sums can be used for approximation of an integral that cannot be calculated exactly. To define the definite integral of a function $f$ of two or more variables, we used the same procedure used for functions of a single variable. This works in the following way. We partition the set over which we are integrating (call it $S$) into smaller sets (originally rectangles for functions of two variables and rectangular boxes for functions of three variables). List these smaller sets: $S_1$, $S_2$, ... $S_n$. From each smaller set $S_k$ choose a point $p_k$. Now form the Riemann sum

$$\sum_{k=1}^{n} f(p_k)|S_k|$$

where $|S_k|$ is the area of $S_k$ if $S$ is two-dimensional, or if $S$ is three-dimensional, $|S_k|$ is the volume of $S_k$. Now we take the limit of the corresponding Riemann sums as the size (or mesh) of the partition goes to zero (meaning that all the sets $S_k$ become smaller and smaller). If this limit exists, this is the value of the integral. Symbolically,

$$\int_S \int f(x,y)\; dA = \lim_{|P|\to 0} \sum_{k=1}^{n} f(p_k)|S_k|,$$

for a double integral. For a triple integral, we would have a third integral sign, $dA$ would be replaced by $dV$, and $f$ would be a function of $x$, $y$ and $z$. When $f \geq 0$ and $S$ is a subset of the plane, $\int_S \int f(x,y)\, dA$ is the volume of the solid, under the graph of $z = f(x,y)$ and over the set $S$. Multiple integrals possess some of the same features as the single integrals familiar from first-year calculus (for example, the integral of the sum of two functions is the sum of the integrals, integrals can be split up into integrals over smaller sets, and a comparison property – see Section 16.1).

Multiple integrals are most easily computed as iterated integrals. This reduces the computation of a multiple integral to the calculation of several single integrals. This is useful because we can use the techniques of integration covered in first-year calculus to evaluate them. In other words, once the problem is set up as an iterated integral (often the hard step in such a problem), we can use techniques already at our disposal to compute them – antiderivatives of some basic functions (like sin, cos, exponential and powers), substitution, integration by parts, tables, partial fractions, etc. We started by integrating over rectangular regions in the plane. Rectangles are sets of the form:

$$R = \{(x,y) :\; a \leq x \leq b,\; c \leq y \leq d\}.$$

Then the integral of $f(x,y)$ over this set is

$$\int_R \int f(x,y)\; dA = \int_a^b \int_c^d f(x,y)\; dy\, dx = \int_c^d \int_a^b f(x,y)\; dx\, dy.$$

Note how the limits of integration correspond to the variables, by looking at what happens when we change the order of integration. The outer integral corresponds to the last variable ($y$ in the third integral above), and the inner integral corresponds to the inner variable.

Now for a concrete example: Let $R = \{(x,y) :\; 2 \leq x \leq 3,\; -1 \leq y \leq 2\}$, and $f(x,y) = xy - y$. Then:

$$\int_R\int xy - y\, dA = \int_2^3\int_{-1}^2 xy - y\, dy\, dx$$

Since the integrand is simpler in $x$ than it is in $y$, it will be easier to integrate in the variable $x$ first.

$$\int_{-1}^2\int_2^3 xy - y\, dx\, dy = \int_{-1}^2 (\frac{1}{2}x^2y - yx)|_2^3\, dy$$

$$= \int_{-1}^2 (\frac{9}{2}y - 3y) - (2y - 2y)\, dy = \int_{-1}^2 \frac{3}{2}y\, dy = \frac{3}{4}y^2|_{-1}^2 = 3 - \frac{3}{4} = \frac{9}{4}.$$

Notice that, in the first integration, the variable $y$ (the variable of the outer integration) is treated as a constant. Also, after evaluating the inner integral, the result is no longer a function of that variable. After completing the integration in $x$, what remains should be only a function of the variable(s) not yet integrated. This example also demonstrates that a good choice of the order of integration can simplify the calculation. The first two modules from COW Assignment 5 have many examples of these types of problems.

We next looked at double integrals over nonrectangular regions. One crucial step in evaluating such an integral is deciding which variable to integrate in first. The set $S$ is called $y$-simple if each line parallel to the $y$-axis intersects $S$ in a single line segment, a point or the empty set. Similarly, $S$ is $x$ simple if each line parallel to the $x$-axis intersects $S$ in a line segment, a point or the empty set. If a set is $y$-simple (or $x$-simple), you can integrate in $y$ (or $x$) first. The reason for this is that if $S$ is $y$-simple, the lower boundary of $S$ is described by a function of $x$, say $\phi_1(x)$. Similarly, the upper boundary of $S$ is the graph of another function $\phi_2(x)$. These functions tell us where to start and stop integrating with respect to $y$. Then if $S$ lies between $x = a$ and $x = b$, the integral of $f(x, y)$ over $S$ is

$$\int_a^b\int_{\phi_1(x)}^{\phi_2(x)} f(x, y)\, dy\, dx.$$

If the set $S$ is $x$-simple, the left and right boundaries can be described by functions of $y$. For pictures of $x$-simple and $y$-simple sets, and examples of double integrals over such sets, see pp. 695-699.

In many problems, the set over which we are integrating is both $x$- and $y$-simple. For these problems, there is a choice about which variable to integrate in first. Often, this choice makes a big difference in solving the problem – for one choice, evaluating the integral may be difficult or impossible, but integrating in the opposite order may be routine. For this reason, being able to reverse the order of integration of a multiple integral is an important skill. Two modules from COW Assignment 5 dealt with this topic: this is a good place to review.

It is easier to describe some regions in the plane by polar coordinates than it is by Cartesian coordinates. For integrals over such sets, it is more convenient to integrate in polar coordinates. To set up the integral, we followed the same procedure as with rectangular coordinates (Riemann sums, etc.), but instead of partitioning the set into small rectangles, we divided it into "polar rectangles". When computing the area of a polar rectangle, there is a factor of $\bar{r}$, the average radius of the polar rectangle. For this reason, when we take the limit of the Riemann sums, a factor of $r$ appears. In other words, the "area element" for rectangular coordinates, $dx\,dy$, is transformed to $r\,dr\,d\theta$. So recalling that polar and Cartesian coordinates are related by the formulas $x = r\cos\theta$ and $y = r\sin\theta$, we get the following formula for changing from rectangular to polar integration:

$$\int_R\int f(x,y)\;dx\,dy = \int_R\int f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta.$$

This (along with the change of variable formulas) is worth committing to memory or adding to your formula sheet. As with the double integrals in rectangular coordinates, it is also important to choose which variable to integrate in first with polar coordinates. There are the notions of $r$-simple and $\theta$-simple sets, to help in this determination. See Example 2 in Section 16.4. A natural question is: how can one determine when polar coordinates should be used? If a region is described in polar coordinates, it is a pretty safe bet that integrating in polar coordinates is a good idea. Also, if the set is "circular" in that its boundary is made up of circular arcs, polar coordinates will probably be more convenient than Cartesian. For practice with polar integrals, see the problems from Section 16.4.

In Section 16.5, some physical applications of double integrals were presented. Specifically, we saw how to compute the mass and the center of mass of a nonhomogeneous lamina $L$. A lamina is a very thin object (thin enough that it can be considered two-dimensional). Suppose the density (mass per unit area) is given by $\delta(x,y)$. In the parts of $L$ where $\delta$ is large the object is dense, and has more mass. We approximated the mass of $L$ by partitioning $L$ into small rectangles and approximating the mass of each rectangle by multiplying the density at a point of the rectangle by the area of the rectangle (if density is constant, the mass of such a rectangle is exactly the density times the area of the rectangle, so if the rectangle is small and the density varies continuously, our approximation should be pretty good). By summing up over all of the rectangles we get a Riemann sum, and taking a limit, we produce a double integral. The mass of the lamina $L$ with density $\delta(x, y)$ is

$$m = \int_L \int \delta(x, y)\, dy\, dx.$$

By a similar approximation argument (starting with discrete systems of point masses), the center of mass of a lamina $L$ with mass $m$ is the point

$$(\bar{x}, \bar{y}) = \left( \frac{\int_S \int x\delta(x, y)\, dA}{m}, \frac{\int_S \int y\delta(x, y)\, dA}{m} \right).$$

The center of mass of a lamina $L$ can be thought of as a balance point. If you put your finger right underneath the center of mass of $L$, you could lift it without it tipping over. Another way to think of it is that $L$ and the point mass with the same mass as $L$ concentrated at $(\bar{x}, \bar{y})$ would tend to produce the same rotations about any line.

A geometric application of double integrals is the calculation of the surface area of solids (Section 16.6). Picture a surface in 3-space, for example, the graph of a differentiable function $z = f(x, y)$. A surface is a two-dimensional object (lying in 3-space) so it makes sense to talk about its area (rather than its length or volume). Suppose we want to calculate the area of a portion $G$ of a surface and that the surface is described by $z = f(x, y)$. Denote by $S$ the projection of $G$ into the $xy$-plane. Then $G$ is the graph of the function $f$ over the region $S$. The area of the surface is then

$$A(G) = \int_S \int \sqrt{f_x^2 + f_y^2 + 1}\, dA.$$

This formula was obtained by approximating the area of the surface in the following manner. First, partition $S$ into small rectangles $R_1$, $R_2$, $\ldots R_n$. Let $(x_k, y_k)$ be the lower left corner of the rectangle $R_k$ (for each integer $k$ between 1 and $n$). Find the equation $z = T_k(x, y)$ that describes the tangent plane to the surface at the point $(x_k, y_k, f(x_k, y_k))$. The graph of $T_k(x, y)$ over the rectangle $R_k$ is a parallelogram that is a close approximation to the surface when the rectangles are small. We can compute the area of these parallelograms (using the cross product), and by adding all of their areas together, we get an approximation to the surface area of $G$. By taking a limit, we produce the double integral above. For example, the area of the part of the surface $z = \sqrt{4 - x^2 - y^2}$ directly over the unit square $S = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 1\}$ is

$$A = \int_0^1 \int_0^1 \sqrt{\frac{x^2}{4 - x^2 - y^2} + \frac{y^2}{4 - x^2 - y^2} + 1}\, dA.$$

We then turned to triple integrals. The theory of the triple integral in rectangular coordinates was presented above. If the set $S$ over which we wish to integrate is $z$-simple, meaning that every vertical line (parallel to the $z$-axis) intersects $S$ in a single line segment (or a single point or not at all), the triple integral of $f$ over $S$ can be written as

$$\int_{S_{xy}} \int \left( \int_{\psi_1(x,y)}^{\psi_2(x,y)} f(x, y, z)\, dz \right) dA.$$

where $S_{xy}$ is the projection of $S$ into the $xy$-plane, $\psi_1(x, y)$ is the lower boundary of $S$ over $S_{xy}$ and $\phi_2(x, y)$ is the upper boundary. See Figure 3 on page 717. If the set $S_{xy}$ is then $y$-simple and is in between the vertical lines $x = a_1$ and $x = a_2$, we can rewrite the outer double integral above as the iterated integral (again see Figure 3, p. 717):

$$\int_{a_1}^{a_2} \int_{\phi_1(x)}^{\phi_2(x)} \int_{\psi_1(x,y)}^{\psi_2(x,y)} f(x, y, z)\, dz\, dy\, dx.$$

There is nothing special about our original set $S$ being $z$-simple. If the set is $x$-simple, we could integrate in $x$ first, and then the outer integral would be over the projection of $S$ into the $yz$-plane. Note that the limits of the outer integral are constants, the limits of the middle integral are functions of the outer variable, and the limits of the inner intergral are functions of the outer two variables. Assuming the functions that arise in triple integrals are easily integrated, the hard part of the problem is setting up the limits of integration. For some problems, the limits will be given; for others, like problems 9-18 on p.721, the limits must be determined from the information given in the problem.

We can use triple integrals to compute the mass and center of mass of a nonhomogeneous solid $S$ with density $\delta(x, y, z)$. These formulas are the same formulas we had for the laminae extended to one more dimension. The mass is

$$m = \int_S\int\int \delta(x,y,z)\;dV$$

and the center of mass is the point

$$\left(\frac{\int_S\int\int x\delta(x,y,z)\;dV}{m}, \frac{\int_S\int\int y\delta(x,y,z)\;dV}{m}, \frac{\int_S\int\int z\delta(x,y,z)\;dV}{m}\right).$$

The last section concerned the evaluation of triple integrals in other systems of coordinates. We first looked at cylindrical coordinates. The derivation of the integral in cylindrical coordinates is similar to the integral in polar coordinates (this is because cylindrical coordinates are simply polar coordinates, plus the $z$ variable). The volume element $dx\,dy\,dz$ becomes $r\,dr\,d\theta\,dz$ under a change from rectangular to cylindrical coordinates. So we have the following formula:

$$\int_S\int\int f(x,y,z)\;dx\,dy\,dz = \int_S\int\int f(r\cos\theta, r\sin\theta, z)\;r\,dr\,d\theta\,dz.$$

We then considered triple integrals in spherical coordinates. We partitioned our region in 3-space into "spherical wedges" and formed a Riemann sum. When computing the volume of these sets some factors appeared that we did not have for Cartesian coordinates (like the factor $r$ for polar coordinates). As before, we took the limit of the Riemann sums, and produced a triple integral in spherical coordinates. The volume element was transformed from $dx\,dy\,dz$ into $\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$, and the formula for transforming from rectangular to spherical coordinates is:

$$\int_S\int\int f(x,y,z)\;dV = \int_S\int\int f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\rho^2\sin\phi\,d\rho\,d\theta\,d\phi.$$

Both of these formulas (as well as the formulas for the coordinate changes) are worth adding to your formula sheet. Cylindrical coordinates are especially useful when the problem has symmetry with respect to an axis (like a cylinder), and spherical coordinates are convenient when there is symmetry with respect to a point (like a sphere). The best way to get comfortable with integrals in these systems is by doing the suggested exercises from 16.8 and studying the examples in this section of the text.