Math 2210 Section 1 Exam 3 Solutions

1)a) Compute the integral $\int_{-1}^{2}\int_{0}^{2} 3x^2y^2 - y\,dx\,dy$.

$$\int_{-1}^{2}\int_{0}^{2} 3x^2y^2 - y\,dx\,dy = \int_{-1}^{2}(x^3y^2 - xy)\Big|_0^2\,dy =$$

$$\int_{-1}^{2} 8y^2 - 2y\,dy = \left(\frac{8}{3}y^3 - y^2\right)\Big|_{-1}^{2} = \left(\frac{64}{3} - 4\right) - \left(\frac{-8}{3} - 1\right) = 21.$$

b) Suppose $f(x, y) \geq 0$ for all points (x, y) in the set S, the region in the first quadrant bounded by the y-axis and the curves $y = 3x$ and $y = -x^2+2$. Write a double integral (but do not integrate) that is equal to the volume of the region under the graph $z = f(x, y)$ and over the set S. Hint: A sketch of S might be helpful.

The volume under the curve is the double integral of the function over the set S. A sketch of the set shows that the curve $y = -x^2 + 2$ is the upper boundary of S, and the line $y = 3x$ is the lower boundary. Therefore, if we choose to integrate in y first (although one could integrate in x first), the lower limit of integration is $3x$ and the upper limit is $-x^2 + 2$. Since S is in the first quadrant and bounded by the y-axis, 0 is the lower limit of integration in x. The x-coordinate of the point of intersection of the line and the curve will give us the upper limit of integration for x. To find this point of intersection, we need to solve the equation $3x = -x^2 + 2$. This is a quadratic equation, so by using the quadratic formula, and choosing the positive root (the other root gives a point not in the first quadrant), we get that $x = (-3 + \sqrt{17})/2$. Call this number $\bar{x}$. Then our integral is:

$$\int_{0}^{\bar{x}}\int_{3x}^{-x^2+2} f(x, y)\,dy\,dx.$$

2) Replace the integral $\int_{-1}^{2}\int_{0}^{\sqrt{y+1}} g(x, y)\,dx\,dy$ with an equivalent integral by changing the order of integration.

A sketch shows that the region of integration is bounded by the line $y = 2$, the y-axis and the curve $x = \sqrt{y+1}$. Solving this last equation for y, we obtain that $y = x^2 - 1$. Over this region, x varies 0 to $\sqrt{3}$. For each x in this range, y varies from the curve $y = x^2 - 1$ to 2. Therefore, when we change the order of integration, we get

$$\int_0^{\sqrt{3}} \int_{x^2-1}^{2} g(x,y)\; dy\, dx.$$

3) Compute the surface area of the graph of $z^2 = x^2 + y^2$ above the triangle in the xy-plane with vertices $(0,0)$, $(4,0)$ and $(0,4)$.

The surface we want to measure is the graph of $z = f(x,y) = \sqrt{x^2+y^2}$ over the triangle T in the xy plane bounded by the x-axis, the y-axis and the line $y = -x + 4$. Then $f_x = x/\sqrt{x^2+y^2}$ and $f_y = y/\sqrt{x^2+y^2}$. The general formula for computing the surface area is

$$A = \int_T\int \sqrt{f_x^2 + f_y^2 + 1}\; dy\, dx.$$

So, in this situation using the calculations above, the area is

$$A = \int_0^4 \int_0^{-x+4} \sqrt{\frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2} + 1}\; dy\, dx =$$

$$\int_0^4 \int_0^{-x+4} \sqrt{\frac{x^2+y^2+x^2+y^2}{x^2+y^2}}\; dy\, dx = \sqrt{2}\int_0^4 \int_0^{-x+4} \; dy\, dx.$$

This last integral can be evaluated directly, or noticing that it is the area of the triangle, we get that $A = 8\sqrt{2}$.

4) a) Compute the mass of the lamina L with density $\delta(x,y) = \sqrt{x^2+y^2}$ where L is the semicircle $\{(x,y) : x^2+y^2 \le 1,\ 0 \le y \le 1\}$.

The mass of the object is given by the formula $\int_L\int \delta(x,y)\; dy\, dx$. It is convenient to compute this integral in polar coordinates. Then, $\delta(x,y) = \sqrt{x^2+y^2} = r$, and $L = \{(r,\theta) : 0 \le r \le 1,\ 0 \le \theta \le \pi\}$. Therefore:

$$m = \int_0^{\pi}\int_0^1 r\, r\, dr\, d\theta = \int_0^{\pi}\int_0^1 r^2\; dr\, d\theta = \frac{1}{3}\int_0^{\pi} d\theta = \frac{\pi}{3}$$

4 b) Compute the center of mass of L.

The center of mass of L is (m is the mass of L):

$$(\bar{x}, \bar{y}) = \left(\frac{\int_L \int x\,\delta(x,y)\ dy\,dx}{m}, \frac{\int_L \int y\,\delta(x,y)\ dy\,dx}{m} \right).$$

Again using polar coordinates to compute the integrals:

$$\int_L \int x\,\delta(x,y)\ dy\,dx = \int_0^\pi \int_0^1 (r\cos\theta)\,r\ r\,dr\,d\theta = \int_0^\pi \int_0^1 r^3\cos\theta\ dr\,d\theta =$$

$$\frac{1}{4}\int_0^\pi \cos\theta\ d\theta = \frac{1}{4}(\sin\theta)|_0^\pi = 0$$

and

$$\int_L \int y\,\delta(x,y)\ dy\,dx = \int_0^\pi \int_0^1 r^3 \sin\theta\ dr\,d\theta =$$

$$\frac{1}{4}\int_0^\pi \sin\theta\ d\theta = \frac{1}{4}(-\cos\theta)|_0^\pi = 1/2$$

Therefore, (dividing the result of the last integral by $m = \pi/3$), $(\bar{x}, \bar{y}) = (0, \frac{3}{2\pi})$.

5) Calculate the volume of the solid S lying inside the sphere $x^2 + y^2 + z^2 = 16$, outside the cone $z = \sqrt{x^2 + y^2}$, and above the xy-plane.

It is easiest to do this in spherical coordinates. The equation of the given sphere is then $\rho = 4$. The equation of the cone is $\phi = \frac{\pi}{4}$. To see this algebraically, change the equation for the cone to spherical coordinates and do some algebra:

$$\rho\cos\phi = \sqrt{\rho^2\sin^2\phi\cos^2\theta + \rho^2\sin^2\phi\sin^2\theta} = \rho\sin\phi,$$

so we must have that $\cos\phi = \sin\phi$, and since ϕ is between 0 and π, $\phi = \frac{\pi}{4}$. Then in S, θ varies from 0 to 2π, ϕ varies from $\frac{pi}{4}$ to $\frac{pi}{2}$ (since S is above the xy-plane), and ρ varies from 0 to 4. So,

$$V = \int_0^4 \int_0^{2\pi} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \rho^2 \sin\phi\ d\phi\,d\theta\,d\rho = \int_0^4 \int_0^{2\pi} -\rho^2\cos\phi|_{\frac{\pi}{4}}^{\frac{\pi}{2}}\ d\theta\,d\rho =$$

$$\int_0^4 \int_0^{2\pi} \frac{\sqrt{2}}{2}\rho^2\ d\theta\,d\rho = \pi\sqrt{2}\int_0^4 \rho^2\ d\rho = \pi\sqrt{2}\frac{\rho^3}{3}|_0^4 = \frac{64\pi\sqrt{2}}{3}.$$