Math 2210 Section 1 Exam 1 Solutions

1) The position vector of a particle moving in space is given by

$$\vec{r}(t) = (t^2 + t)\vec{i} + (\frac{1}{t+1})\vec{j} - e^{2t}\vec{k}$$

What is the velocity vector $\vec{v}(t)$?

The velocity vector is the derivative of the position vector, so

$$\vec{v}(t) = (2t+1)\vec{i} - (t+1)^{-2}\vec{j} - 2e^{2t}\vec{k}$$

What is the acceleration vector $\vec{a}(t)$?

The acceleration is the derivative of the velocity (or the second derivative of the position):

$$\vec{a}(t) = 2\vec{i} + 2(t+1)^{-3}\vec{j} - 4e^{2t}\vec{k}$$

What is the angle between $\vec{v}(t)$ and $\vec{a}(t)$ when $t = 0$?

We will use the fact that for any two vectors $\vec{u}$ and $\vec{v}$, the cosine of the angle θ between them is given by the formula:

$$\vec{u} \cdot \vec{v} = |\vec{u}|\,|\vec{v}|\cos\theta$$

We apply this formula with $\vec{u} = \vec{v}(0)$ and $\vec{v} = \vec{a}(0)$. Using the results from above:

$$\vec{v}(0) = \vec{i} - \vec{j} - 2\vec{k} \text{ and } \vec{\mathrm{a}}(0) = 2\vec{\mathrm{i}} + 2\vec{\mathrm{j}} - 4\vec{\mathrm{k}}$$

Also, $|\vec{v}(0)| = \sqrt{6}$, and $|\vec{a}(0)| = \sqrt{24} = 2\sqrt{6}$. So,

$$\cos\theta = \frac{\vec{v}(0) \cdot \vec{a}(0)}{|\vec{v}(0)|\,|\vec{a}(0)|} = \frac{8}{\sqrt{6}(2\sqrt{6})} = \frac{2}{3},$$

and therefore $\theta = \cos^{-1}\frac{2}{3}$.

2) Find an equation for the plane passing through the points $(2,4,3)$, $(1,3,-3)$, and $(-4,3,4)$.

Call the three points P_0, P_1 and P_2. To write the equation for a plane, we need to know one point of the plane and a vector normal to the plane. In this problem, we are given three points in the plane, so all we need to do is find a normal vector. To do this, we first find two vectors in the plane and then compute their cross product, which will be our normal vector.

$$\vec{P_0P_1} = -\vec{i} - \vec{j} - 6\vec{k} \text{ and } \vec{\mathrm{P_0P_2}} = -6\vec{\mathrm{i}} - \vec{\mathrm{j}} + \vec{\mathrm{k}}$$

$$\vec{n} = \vec{P_0P_1} \times \vec{P_0P_2} = \det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & -1 & -6 \\ -6 & -1 & 1 \end{bmatrix} = -7\vec{i} + 37\vec{j} - 5\vec{k}.$$

Using P_0 as our point in the plane, an equation for the plane is:

$$-7(x-2) + 37(y-4) - 5(z-3) = 0.$$

3) A curve is described by the vector equation $\vec{r}(t) = (2\cos t)\vec{i} + (3\sin t)\vec{j}$.

Find the unit tangent vector $\vec{T}(t)$.

The unit tangent vector $\vec{T}'(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}$.

$$\vec{r}'(t) = (-2\sin t)\vec{i} + (3\cos t)\vec{j},$$

and $|\vec{r}'(t)| = \sqrt{4\sin^2 t + 9\cos^2 t} = \sqrt{4 + 5\cos^2 t}$. Therefore,

$$\vec{T}(t) = \frac{1}{\sqrt{4+5\cos^2 t}}((-2\sin t)\vec{i} + (3\cos t)\vec{j}).$$

Recall that the curvature κ is given by the following formula, where x and y are the coordinate functions of the curve:

$$\kappa = \frac{|x'y'' - y'x''|}{[(x')^2 + (y')^2]^{\frac{3}{2}}}$$

Find the maximum and minimum values for κ.

Using the formula, we get that

$$\kappa = \frac{|6\sin^2 t + 6\cos^2 t|}{(4\sin^2 t + 9\cos^2 t)^{\frac{3}{2}}} = \frac{6}{(4 + 5\cos^2 t)^{\frac{3}{2}}}.$$

This has a maximum of $\frac{3}{4}$ when $\cos t = 0$, and a minimum of $\frac{2}{9}$ when $\cos t = 1$.

4) Show that the vector $a\vec{i} + b\vec{j}$ is perpendicular to the line $ax + by = c$.

The line $ax + by = 0$ is parallel to $ax + by = c$, so $a\vec{i} + b\vec{j}$ is perpendicular to the line $ax + by = c$ if and only if it is perpendicular to $ax + by = 0$. The line $ax + by = 0$ passes through the points $(0, 0)$ and $(b, -a)$. So the line is parallel to the vector $b\vec{i} - a\vec{j}$. Now compute the dot product of these two vectors:

$$(a\vec{i} + b\vec{j}) \cdot (b\vec{i} - a\vec{j}) = ab - ab = 0,$$

so these vectors, and therefore the line and the vector, are perpendicular.

5) For each of the following equations, describe the graph. For example, if the graph is a sphere, you should say that it is a sphere and give its radius and its center.
$r = 4$ in cylindrical coordinates

The graph is the right circular cylinder with radius 4, with the z-axis for its axis of symmetry.

$\phi = \frac{\pi}{4}$ in spherical coordinates.

The graph is a circular cone with vertex at the origin and the axis is the z-axis. The angle between any vector in the cone originating at the origin and the positive z-axis is $\frac{\pi}{4}$.

6) Find parametric equations for the line formed by the intersection of the planes $x + y - z = 6$ and $2x - y + 3z = 2$.

The line is in both planes, so it must be perpendicular to the normal of each plane. Since the cross product is also perpendicular to each of these normals, the line and this cross product must be parallel. The vector $\vec{n}_1 = \vec{i} + \vec{j} - \vec{k}$ is normal to the first plane, and $\vec{n}_2 = 2\vec{i} - \vec{j} + 3\vec{k}$ is normal to the second. Then

$$\vec{n}_1 \times \vec{n}_2 = \det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & -1 \\ 2 & -1 & 3 \end{bmatrix} = 2\vec{i} - 5\vec{j} - 3\vec{k}.$$

Now we need to find a point on the line. Let $z = 0$. Now solve the system of equations formed by setting $z = 0$ in the equations for the two planes: $x + y = 6$, and $2x - y = 2$. Solving, we obtain $(\frac{8}{3}, \frac{10}{3})$. Therefore, the point $(\frac{8}{3}, \frac{10}{3}, 0)$ is on the line. Now, since we know a point on the line and a vector parallel to the line, we can write the parametric equations:

$$\begin{aligned} x = & \ \tfrac{8}{3} + 2t \\ y = & \ \tfrac{10}{3} - 5t \\ z = & \ -3t \end{aligned}$$