DIVISORS ON PRINCIPALLY POLARIZED ABELIAN VARIETIES

Christopher D. Hacon

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Abstract. The purpose of this paper is to show how generalizations of generic vanishing theorems to a $\mathbb{Q}$-divisor setting can be used to study the geometric properties of pluritheta divisors on a principally polarized abelian variety (PPAV for short).

Introduction

Abelian varieties are some of the most studied higher dimensional algebraic varieties. They appear to be very simple, as they are constructed by quotienting $\mathbb{C}^n$ by an integral lattice. However, there are many open problems concerning their geometry. One of the techniques that has been used to investigate the properties of PPAVs is the study of the singularities of their theta divisor. This connection was first exploited by Andreotti and Mayer [AM], in their study of the Schottky problem. In another direction Kempf [Ke] showed that if A is the Jacobian variety of some curve then the theta divisor has only rational singularities. In a joint article [AD], Albarello and De Concini suggest that if (A,Θ) is an irreducible principally polarized abelian variety, then the singular locus of the theta divisor has codimension at least 3 in A. At any rate, Kollár [K] proved that the singularities of the theta divisor on a principally polarized abelian variety are mild in the following sense:

THEOREM (Kollár). *Let (A,Θ) be a PPAV, then the pair (A,Θ) is log canonical.*

This implies for example that if

$$\Sigma_k(\Theta)=\{x\in A \mid mult_x(\Theta)\geq k\ \},$$

then every component of $\Sigma_k(\Theta)$ has codimension at least k in A.

It is easy to construct examples in which the codimension is exactly k, by just considering a k-fold product of PPAVs. Ein and Lazarsfeld [EL] proved that this is essentially the only case in which such a phenomenon can occur.

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THEOREM (Ein, Lazarsfeld). *If $\Theta \subset A$ is an irreducible theta divisor, then Θ is normal and has only rational singularities.*

Consequently, if (A, Θ) is any PPAV for which $\Sigma_k(\Theta)$ contains an irreducible component of codimension k then (A, Θ) splits as a k-fold product of PPAVs. They also generalized the result of Kollár to $\mathbb{Q}$-divisors in the following sense:

THEOREM (Ein, Lazarsfeld). *If D is a divisor in the linear series $|m\Theta|$, then the pair $(A, \frac{1}{m}D)$ is log-canonical.*

In the same spirit as the first of the two theorems of Ein and Lazarsfeld, we prove the following:

THEOREM 1. *Let (A, Θ) be a principally polarized abelian variety and for $m \geq 1$ let D be a divisor in the linear series $|m\Theta|$ such that $\lfloor \frac{1}{m}D \rfloor = 0$. Then the pair $(A, \frac{1}{m}D)$ is log terminal.*

The condition that $\lfloor \frac{1}{m}D \rfloor = 0$ is equivalent to requiring that the multiplicity of every irreducible component of D is strictly less than m.

COROLLARY 2. *Let (A, Θ) be a principally polarized abelian variety, and for $m \geq 1$ let D be a divisor in the linear series $|m\Theta|$. If $k \geq 1$ is the greatest integer such that $\Sigma_{mk}(D)$ contains an irreducible component of codimension k, then (A, Θ) splits as a product of at least k PPAVs*

$$(A, \Theta) \cong (A_1, \Theta_1) \times ... \times (A_k, \Theta_k) \times (A', \Theta')$$

(the component (A', Θ') being possibly trivial). Moreover, the divisor D decomposes as follows

$$D = m \sum_{i=1}^{k} p_i^*(\Theta_i) + {p'}^* D'$$

where $D' \in |m\Theta'|$, and the pair $(A', \frac{1}{m}D')$ is log terminal.

REMARK. The component (A', Θ') is trivial precisely when $D' = 0$, ie when $\lfloor \frac{1}{m}D \rfloor = \frac{1}{m}D$. Similarly $(A, \Theta) \cong (A', \Theta')$ (ie $k = 0$) precisely when $\lfloor \frac{1}{m}D \rfloor = 0$. This is the case treated in theorem 1.

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0. Notation and conventions

f^*D pull-back

D_{red} reduced divisor associated to D

$\Omega^i_X < D >$ sheaf of holomorphic i-forms with logarithmic poles along D_{red}

$|D|$ linear series associated to the divisor D

$Bs|D|$ the base locus of the linear series associated to the divisor D

$\equiv$ numerical equivalence

$\omega_X = \Omega^n_X$ canonical sheaf of X

K_X linear equivalence class of a canonical divisor on X

$\omega_{X/Y} := \omega_X \otimes (f^*\omega_Y)^*$ relative canonical sheaf of a morphism $f : X \longrightarrow Y$

Unless otherwise stated X, Y will denote smooth complex projective varieties. If D is a $\mathbb{Q}$-divisor we will denote by $\lfloor D \rfloor$ and by $\lceil D \rceil$ the round down and the round up of D respectively.

1. Preliminaries

Let X be a smooth complex projective variety and let D be a $\mathbb{Q}$-divisor on X. We will say that $f : Y \longrightarrow X$ is a log-resolution of the pair (X, D), if f is a proper birational morphism such that $f^{-1}D \cup \{$ *exceptional set of* f $\}$ is a divisor with normal crossing support. Given a log-resolution of the pair (X, D), we can define the multiplier ideal sheaf associated to the divisor D,

$$\mathcal{I}(D) := f_*(\mathcal{O}_Y(K_{Y/X} - \lfloor f^*D \rfloor)) \ .$$

The definition is independent of the choice of the log-resolution. Multiplier ideal sheaves may be defined in much grater generality. Their properties have been extensively studied eg. [N], [Sk], [De] and [E]. Under the above assumptions, we will say that the pair (X, D) is log canonical (respectively log terminal) if the multiplier ideal sheaf associated to the divisor $(1 - \epsilon)D$ is trivial, for $0 < \epsilon < 1$ (respectively $0 \leqq \epsilon < 1$).

As in [EV] given an invertible sheaf $\mathcal{L}$, an effective divisor

$$B = \sum b_i B_i$$

and a positive natural number N such that $\mathcal{L}^N = \mathcal{O}_X(B)$, we will denote by $\mathcal{L}^{(1)}$ the sheaf

$$\mathcal{L}^{(1)} := \mathcal{L} \otimes \mathcal{O}_X(-\lfloor \frac{1}{N} B \rfloor)$$

and by $B^{(1)}$ the divisor

$$B^{(1)} = \sum_{N \nmid b_i} B_i \quad .$$

We will study the loci

$$V^i(\omega_X \otimes \mathcal{L}^{(1)}) := \{\mathcal{P} \in Pic^0(X) \mid h^i(\omega_X \otimes \mathcal{L}^{(1)} \otimes \mathcal{P}) \neq 0\} \ .$$

Assume now that B is a divisor with normal crossing support, then the deformation theory of the cohomology groups $H^i(X, \omega_X \otimes \mathcal{L}^{(1)} \otimes \mathcal{P})$ behaves analogously to the deformation theory of the cohomology groups $H^i(X, \omega_X \otimes \mathcal{P})$. In fact, the geometry of these loci is governed by the following theorem which parallels the theory developed in [GL1] and in [GL2], (see Thm. 1.2 of [EL] for an analogous formulation).

THEOREM 1.1. *Let $T \subset Pic^0(X)$ be any irreducible component of $V^i(\omega_X \otimes \mathcal{L}^{(1)})$, and let $\mathcal{P} \in T$ be a general point. Then*

a. T is (a translate of) a sub-torus of $Pic^0(X)$

b. $codim_{Pic^0(X)} T \geq i - (dim X - dim\ alb_X(X))$

c. $Pic^0(X) \supset V^0(\omega_X \otimes \mathcal{L}^{(1)}) \supset V^1(\omega_X \otimes \mathcal{L}^{(1)}) \supset \ \dots\ \supset V^n(\omega_X \otimes \mathcal{L}^{(1)})$

d. If $0 \neq v \in H^1(X, \mathcal{O}_X) \cong T_{\mathcal{P}} Pic^0(X)$ is not tangent to T, then the sequence

$$H^{n-i-1}(X, (\mathcal{L}^{(1)} \otimes \mathcal{P})^*) \xrightarrow{\cup v} H^{n-i}(X, (\mathcal{L}^{(1)} \otimes \mathcal{P})^*) \xrightarrow{\cup v} H^{n-i+1}(X, (\mathcal{L}^{(1)} \otimes \mathcal{P})^*)$$

is exact. If $v \neq 0$ is tangent to T, then the maps in the above sequence vanish.

The statements a. b. c. are well known. For a., the reader may refer to [B] and [S], for b., to [D]; c. is analogous to Lemma 1.8 [EL]; the second part of d. is [EV] Lemma 12.6 c). As for the first assertion in d., one may proceed as in [GL2]. The main point is to represent the cohomology groups $H^i(X, (\mathcal{L}^{(1)} \otimes P)^*)$ in terms of harmonic $L^{(2)}$ forms with coefficients in an appropriate unitary local constant system cf. [EV] and [EV2] appendix D.

2. Proofs

Proof of theorem 1. Let D be a divisor in the linear series $|m\Theta|$. Consider its decomposition into irreducible reduced components

$$D = \sum d_i D_i \ .$$

We may assume that $1 \leqq d_i \leqq m-1$. Now consider a log resolution $f : X \longrightarrow A$ of the pair (A, D). Define the sheaf

$$\mathcal{L}^{(1)} := \mathcal{O}_X(f^*(\Theta) - \lfloor \frac{1}{m} f^* D \rfloor)$$

and observe that (cf. [E])

$$f_*(\omega_X \otimes \mathcal{L}^{(1)}) = f_*(\omega_{X/A} \otimes f^*(\Theta) - \lfloor \frac{1}{m} f^* D \rfloor) = \mathcal{I}(\frac{1}{m} D) \otimes \mathcal{O}_A(\Theta)$$

$$R^i f_*(\omega_X \otimes \mathcal{L}^{(1)}) = 0 \ \ \forall\ i > 0 \ .$$

We must show that the multiplier ideal $\mathcal{I} := \mathcal{I}(\frac{1}{m} D)$ is trivial. As before, let

$$\begin{aligned} V_i &:= \{\mathcal{P} \in Pic^0(X) \mid h^i(\omega_X \otimes \mathcal{L}^{(1)} \otimes \mathcal{P}) \neq 0 \ \} \\ &= \{\mathcal{P} \in Pic^0(A) \mid h^i(\mathcal{O}_A(\Theta) \otimes \mathcal{I}(\frac{1}{m} D) \otimes \mathcal{P}) \neq 0 \ \}, \end{aligned}$$

and consider S an irreducible component of V_0 of dimension k.

By Theorem 1.1.a, we have that S is (a translate of) a subtorous of $Pic^0(X)$, so we may consider the dual abelian variety $C := S^*$ and the dual map

$$\pi : A \longrightarrow C \ .$$

The component S is not empty, as otherwise, we would have (Theorem 1.1.c) $H^i(X, \omega_X \otimes \mathcal{L}^{(1)} \otimes \mathcal{P}) = 0$ for all $\mathcal{P} \in Pic^0(X)$ and all integers $0 \leqq i \leqq n$. Therefore, by a result of Mukai [M], $\omega_X \otimes \mathcal{L}^{(1)} \otimes \mathcal{P} = 0$ and this is a contradiction.

PROPOSITION 2.1. *Let F be any component of the divisor D such that $\pi(F) = C$. Then*

$$F \subset Bs|\Theta \otimes \mathcal{P} \otimes \mathcal{I}(\frac{1}{m}D)|$$

for general $\mathcal{P} \in S$.

Proof. Let $B := f^*D = \sum b_i B_i$, then the divisor $B^{(1)} = \sum_{m \nmid b_i} B_i$ contains the proper transforms of all the components of D. Now consider

$$H^0(X, \Omega_X^{n-1} < B^{(1)} > \otimes \mathcal{L}^{(1)} \otimes \mathcal{P}(-B^{(1)})) \xrightarrow{\wedge\omega} H^0(X, \omega_X \otimes \mathcal{L}^{(1)} \otimes \mathcal{P}) \qquad (1)$$

where $\omega \in H^0(X, \Omega_X^1)$ is a global holomorphic 1-form. By [T], this is the conjugate of the last map in the complex in theorem 1.1,d . The conjugate $v = \overline{\omega} \in H^1(X, \mathcal{O}_X)$ may be identified with a vector in the tangent space $T_{\mathcal{P}}(Pic^0(X))$. Recall that if $v \notin T_{\mathcal{P}}S$, then (1) is onto. In order to recover the required information from (1), we must make the appropriate choice of the holomorphic 1-form ω. To this end, consider any component F of D such that $\pi(F) = C$. Let E be the proper transform of F under the map $f : X \longrightarrow A$. E is a component of $B^{(1)}$. Let p a general smooth point of the divisor E, and choose local parameters $z_1, ..., z_n$ on an appropriate open set U, such that E is defined on U by the equation $z_1 = 0$. Choose $\omega \in H^0(X, \Omega_X^1)$ such that $\omega(p) = dz_1$. By choosing $p \in E$ generically, we may assume that $\overline{\omega}$ is not tangent to S, and so (1) is onto. Let s be a section of $H^0(X, \Omega_X^{n-1} < B^{(1)} > \otimes \mathcal{L}^{(1)} \otimes \mathcal{P}(-B^{(1)}))$. Then restricting to the open set U, we may write

$$s = \sum \eta_i z_1 a_i \ ,$$

where $\eta_1 = dz_2 \wedge ... \wedge dz_n$, $\eta_i = \frac{dz_1}{z_1} \wedge ... \wedge dz_{i-1} \wedge dz_{i+1} \wedge ... \wedge dz_n$ for $2 \leqq i \leqq n$, and $a_i \in \Gamma(U, \mathcal{L}^{(1)} \otimes \mathcal{P})$. Therefore,

$$(s \wedge \omega)_{(p)} = (a_1 z_1 \ dz_1 \wedge ... \wedge dz_n)_{(p)} = 0 \ ,$$

so $p \in Bs|\omega_X \otimes \mathcal{L}^{(1)} \otimes \mathcal{P}|$. Moreover, p is a general point of E, so $E \in Bs|\omega_X \otimes \mathcal{L}^{(1)} \otimes \mathcal{P}|$. Since

$$H^0(X, \omega_X \otimes \mathcal{L}^{(1)} \otimes \mathcal{P}) \cong H^0(A, \mathcal{O}_A(\Theta) \otimes \mathcal{P} \otimes \mathcal{I}) \ ,$$

and E is the proper transform of F, it follows that

$$F \subset Bs|\Theta \otimes \mathcal{P} \otimes \mathcal{I}(\frac{1}{m}D)| \ \square$$

COROLLARY 2.2. *The divisor*

$$\Theta - \sum_{\pi(D_i)=C} D_i$$

is algebraically equivalent to an effective divisor, and hence is nef.

Now consider the divisor

$$H := D - \sum_{\pi(D_i)=C} d_i D_i \ .$$

Using the corollary, we may rewrite the divisor H as a sum of an ample divisor, a nef divisor and effective divisor

$$H \equiv \Theta + (m-1)(\Theta - \sum_{\pi(D_i)=C} D_i) + \sum_{\pi(D_i)=C} (m-1-d_i)D_i \ .$$

On an PPAV, any effective divisor is nef, and the sum of an ample divisor with a nef divisor is always ample. Therefore the divisor H is ample.

The theorem follows, since as H is ample and contained in the pullback of a divisor on C, the map π must be an isomorphism, and hence $S = Pic^0(A)$. So

$$H^0(A, \mathcal{O}_A(\Theta) \otimes \mathcal{P} \otimes \mathcal{I}(\frac{1}{m}D)) \neq 0$$

for all $\mathcal{P} \in Pic^0(A)$. This means that all translates of Θ must vanish along the cosupport of $\mathcal{I}$, but then $\mathcal{I} = \mathcal{O}_A$ $\square$

Proof of corollary 2. In order to prove corollary 2, we will need the following lemmas:

LEMMA 2.2. *The components of the divisor $D \in |m\Theta|$ have multiplicity at most equal to m.*

Proof. Suppose that there exists a component F of D of multiplicity $f > m$, and let

$$D = fF + \sum d_i D_i$$

be a decomposition of D into distinct irreducible components. We have the following numerical equivalence of $\mathbb{Q}$-divisors

$$\Theta - F \equiv_{\mathbb{Q}} \frac{1}{m}(D - mF) \ .$$

Since $f \geq m+1$, it follows (after an easy calculation) that $\frac{f-m}{m} \geq \frac{1}{m+1}\frac{f}{m}$. Consequently

$$(\Theta - F).C = \frac{1}{m}(D - mF).C = \frac{1}{m}((f-m)F + \sum d_i D_i).C \geq$$

$$\geq \frac{1}{m}(\frac{f}{m+1}F + \sum \frac{d_i}{m+1}D_i).C = (\frac{1}{m+1})\Theta.C$$

for any curve C, and hence $(\Theta - F)$ is ample. By Riemann-Roch, $h^0(\Theta - F) > 0$, so $h^0(\Theta - F) = h^0(\Theta) = 1$. This means that $(\Theta - F)^n = (\Theta)^n$. On an abelian variety this is a contradiction unless the component F is empty $\square$

Now we may assume that the divisor D decomposes as follows

$$D = \sum_{i=1}^{\delta} d_i D_i + m\Delta \ ,$$

where $1 \leqq d_i \leqq m-1$, and $\Delta = \sum_{d_i=m} D_i$ is the reduced Cartier divisor whose components are the components of D of multiplicity exactly m.

LEMMA 2.3. *There exist PPAVs* (A_0, Θ_0) *and* (A', Θ'), *and projections* $p_0 : A \longrightarrow A_0$, $p' : A \longrightarrow A'$ *such that the map*

$$(p_0 \times p') : A \longrightarrow A_0 \times A'$$

is an isomorphism of PPAVs. Moreover

$$\Delta = p_0^* \Delta_0$$

$$D_i = {p'}^* D_i'$$

where $\Delta_0 = \Theta_0$ *and* $D' = \sum_{i=1}^{\delta} d_i D_i' \equiv m\Theta'$.

Proof. We will use the notation and the results of [LB] chapters 3.3 and 4.3. The divisor $m(\Theta - \Delta)$ is linearly equivalent to the effective divisor $\sum d_i D_i$, hence $\mathcal{O}_A(m(\Theta - \Delta))$ and $\mathcal{O}_A(\Theta - \Delta)$ are positive semidefinite line bundles. Therefore, there exists a topologically trivial line bundle $P \in Pic^0(A)$ such that $h^0(\mathcal{O}_A(\Theta - \Delta) \otimes P) \neq 0$. Define $p_0 : A \longrightarrow A_0 := A/K(\Delta)_0$ and $p' : A \longrightarrow A' := A/K(\Theta - \Delta + P)_0$. By [LB] 3.3.2, there exist positive definite line bundles $\mathcal{M}'$ on A' and $\mathcal{M}_0$ on A_0 such that

$$\mathcal{O}_A(\Delta) = p_0^* \mathcal{M}_0 \quad \textit{and} \quad H^0(A, \mathcal{O}_A(\Delta)) = p_0^*(H^0(A_0, \mathcal{M}_0))$$

$$\mathcal{O}_A(\Theta - \Delta) \otimes P = {p'}^* \mathcal{M}' \quad \textit{and} \quad H^0(A, \mathcal{O}_A(\Theta - \Delta) \otimes P) = {p'}^*(H^0(A', \mathcal{M}'))$$

for some $P \in Pic^0(A)$. By the decomposition theorem [LB] 4.3.1, the map

$$(p_0, p') : A \longrightarrow A_0 \times A'$$

is an isomorphism of the PPAVs $(A, \Theta \otimes P)$ and $(A_0 \times A', {p_0}^* \mathcal{M}_0 \otimes {p'}^* \mathcal{M}')$. Finally, since $K(\Theta - \Delta)_0 = K((\Theta - \Delta) \otimes P)_0 = K(m(\Theta - \Delta))_0$, it follows that

$$H^0(A, \mathcal{O}_A(m(\Theta - \Delta))) = {p'}^*(H^0(A', m\mathcal{M}')),$$

so $\Delta = p_0^* \Delta_0$ and $\sum d_i D_i = {p'}^* D'$ $\square$

Suppose that there exists S an irreducible component of $\Sigma_{mk}(D)$ of codimension k. Then there exist positive integers k_0 and k' such that $mk = k_0 + k'$ and

$$S \subset \Sigma_{k_0}(m\Delta_0) \times \Sigma_{k'}(\sum_{i=1}^{\delta} d_i D_i') \ .$$

From the second of the two theorems due to Ein and Lazarsfeld, we know that

$$codim_{A_0} \Sigma_{k_0}(m\Delta_0) = codim_{A_0} \Sigma_{\lceil \frac{k_0}{m} \rceil}(\Delta_0) \geq \lceil \frac{k_0}{m} \rceil \ .$$

Also, by theorem 1, we know that if $k' > 0$, then

$$codim_{A'} \Sigma_{k'}(\sum_{i=1}^{\delta} d_i D'_i) \geq \lfloor \frac{k'}{m} \rfloor + 1 \ .$$

Therefore,

$$k = codim_A S \geq codim_{A_0} \Sigma_{k_0}(m\Delta_0) + codim_{A'} \Sigma_{k'}(\sum_{i=1}^{\delta} d_i D'_i) \geq$$

$$\geq \lceil \frac{k_0}{m} \rceil + \lfloor \frac{k'}{m} \rfloor + 1 = k + 1 \ .$$

This implies that $k' = 0$, $mk = k_0$ and $codim_{A_0} \Sigma_k(\Delta_0) = k$. The corollary now follows directly from the first of the two theorems due to Ein and Lazarsfeld $\square$

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Christopher Derek Hacon, UCLA, Department of Mathematics, Los Angeles, CA 90095, USA

E-mail address: chhacon@math.ucla.edu