On Generic vanishing in char $p > 0$.

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Varieties of m.A.d.

- Let X be a smooth projective variety over an algebraically closed field k and $a : X \to A$ a morphism to an abelian variety.
- If $\dim X = \dim a(X)$ then we say that X has maximal Albanese dimension (**m.A.d.**).
- Over $k = \mathbb{C}$ the geometry of varieties of m.A.d. is extreemely well understood.
- They admit good minimal models (Fujino). In fact since A contains no rational curves, a relative minimal model for X over A does the trick.
- The 4-th pluricanonical map gives the Iitaka fibration (3-rd if X is of general type; Jiang-Lahoz-Tirabassi).
- The main tools in studying geometry of m.A.d. varieties are Generic-Vanishing theorems and the Fourier-Mukai transform.

The Fourier Mukai transform

- Let $\hat{A}$ be the dual abelian variety and $\mathcal{P}$ the (normalized) Poincaré bundle on $A \times \hat{A}$ so that $\mathcal{P}|_{A\times x} = P_x$ is the topologically trivial line bundle parametrized by $x \in \hat{A}$.
- Let $R\hat{S} : D(A) \to D(\hat{A})$ be defined by $R\hat{S}(?) = Rp_{\hat{A},*}(Lp_A{}^*(?) \otimes \mathcal{P})$.
- For a coherent sheaf F and $x \in \hat{A}$ general, $R^i\hat{S}(F) \otimes k(x) = H^i(A, F \otimes P_x)$ (cohom. and base change).
- **Mukai:** $RS \circ R\hat{S} = (-1_A)^*[-g]$, and $R\hat{S} \circ RS = (-1_{\hat{A}})^*[-g]$.
- Eg. $R\hat{S}(k(x)) = Rp_{\hat{A},*}(\mathcal{P}|_{x\times\hat{A}}) = P_x$ and $R\hat{S}(P_x) = k(-x)[-g]$ (equivalent to the above isom.).
- If L is ample on $\hat{A}$ and $\phi_L : \hat{A} \to A$ the isogeny s.t. $\phi_L(x) = t_x^*L \otimes L^\vee$, then $RS(L)$ is a vector bundle of rank $h^0(L)$ s.t. $\phi_L^*(\hat{L}) = \bigoplus_{h^0(L)} L^\vee$.

The Fourier Mukai transform

- **Lemma**: If $F \in \mathrm{Coh}(A)$ and $h^i(A, F \otimes P_x) = 0$ for all i, $P_x \in \mathrm{Pic}^0(A)$, then $F = 0$.
- **Proof:** $R\hat{S}(F) = 0$ by Cohomology and Base Change so $F{=}0$ by $RS \circ R\hat{S} = (-1_A)^*[-g]$.
- We will be interested in computing

$$R\hat{S}(Ra_*\omega_X) = Rp_{\hat{A},*}(\omega_X \otimes (a \times \mathrm{id}_{\hat{A}})^*\mathcal{P})$$

and hence in the **cohomology support loci**

$$V^i(\omega_X) = \{x \in \hat{A} | h^i(\omega_X \otimes P_x) \neq 0\}$$

- These are governed by generic vanishing theorems.

Generic vanishing

Theorem (Green-Lazarsfeld)

Let X be a smooth complex projective variety, then every irreducible component of $V^i(\omega_X)$ is a (torsion) translate of a (reduced) subtorus of $\mathrm{Pic}^0(X)$ of codimension at least $i-(\dim(X)-\dim(a(X)))$. If X is m.A.d. then there are inclusions:

$$V^0(\omega_X)\supset V^1(\omega_X)\supset\ldots\supset V^{\dim(X)}(\omega_X).$$

($V^{\dim(X)}(\omega_X)=\{\mathcal{O}_X\}$ if $\hat{A}=\mathrm{Pic}^0(X)$.)

- Idea: Given an element of $H^i(\omega_X\otimes P_x)$ we try to deform it in the tangent direction $0\neq v\in H^1(X,\mathcal{O}_X)=T_x\hat{A}$.
- Using the $\partial\bar\partial$ lemma, Green and Lazarsfeld show that ϕ deforms to first order iff it deforms to all orders.
- Thus if $x\in V$ is a generic point of an irreducible component of $V^i(\omega_X)$ then V contains $T_x(V)$ and hence V is a reduced subtorus.

Generic vanishing

- Let $\phi \in \mathcal{A}^{n,i}$ and $w \in \mathcal{A}^{0,1}$ be the corresponding harmonic forms.
- ϕ deforms to order n if $\exists \phi_i$ s.t. $(\bar{\partial} + tw\wedge)(\phi + t\phi_1 + t^2\phi_2 + \ldots) = 0 \mod t^{n+1}$.
- ϕ deforms to first order if $\exists \phi_1$ s.t. $w \wedge \phi = \bar{\partial}\phi_1$.
- Since $\partial(w \wedge \phi) = 0$, then $w \wedge \phi = \bar{\partial}\partial c_0$.
- Replace ϕ_1 by ∂c_0.
- Now $w \wedge \partial c_0 = \partial(w \wedge c_0)$ and $\bar{\partial}(w \wedge \partial c_0) = w \wedge \bar{\partial}\partial c_0 = w \wedge (w \wedge \phi) = 0$, so that $w \wedge \phi_1 = w \wedge \partial c_0 = \bar{\partial}\partial c_1$.
- Let $\phi_2 = \partial c_1$ and repeat.
- We inductively get $w \wedge \partial c_{i-1} = \bar{\partial}\partial c_i$ and we let $\phi_{i+1} = \partial c_i$.

- Pink-Roessler prove a similar result in char $p > 0$ for varieties that lift to Witt vectors $w_2(k)$ with dim $\leq p$ and where the Picard variety has no supersingular factors.
- This uses a result of Deligne-Illusie: $h^r_D(X, P_x) \leq h^r_D(X, P_x^{\otimes p})$ where $h^r_D(X, P_x) = \sum h^{r-j}(X, \Omega^j_X \otimes P_x)$.
- From this they deduce that the loci $V^i(\Omega^j_X)$ are invariant under multiplication by p and this implies they are a finite union of torsion translates of subtori.

A conjecture of Green-Lazarsfeld

- We will be interested in char $p > 0$ versions of the generic vanishing statements: If X has m.A.d. then $V^i(\omega_X)$ has codimension $\geq i$.
- The following alternative point of view is a conjecture of Green-Lazarsfeld proven by Hacon and Popa-Pareschi

Theorem

Let X be a compact Kähler manifold of dimension d, $a : X \to A$ a morphism to an abelian variety with image $a(X) \subset A$ of dimension $d - k$. Then $R^i p_{\hat{A},*}((a \times \mathrm{id}_{\hat{A}})^* \mathcal{P}) = 0$ for $i \notin [d - k, d]$.

This result is in fact equivalent to generic vanishing as we will now explain.

A conjecture of Green-Lazarsfeld

- $Rp_{\hat{A}},*((a\times \mathrm{id}_{\hat{A}})^*\mathcal{P}) = Rp_{\hat{A}},*(Lp_X^*(D_X(\omega_X))\otimes(a\times \mathrm{id}_{\hat{A}})^*\mathcal{P}) = R\hat{S}(Ra_*(D_X(\omega_X))) = R\hat{S}(D_A(Ra_*(\omega_X)))$.
- By a theorem of Kollár $Ra_*\omega_X = \oplus_{i=0}^k R^i a_*\omega_X[-i]$.
- So the above result follows from $R\hat{S}(D_A(R^i a_*\omega_X)) = \mathcal{H}^0(R\hat{S}(D_A(R^i a_*\omega_X)))$.
- Let $F \in \mathrm{Coh}(A)$, then F is a GV_{-k} sheaf if

$$\mathrm{codim}\left(V^i(F) := \{P\in \hat{A} | h^i(F\otimes P)\neq 0\}\right) \geq i-k.$$

- And F is $\mathrm{WIT}_{\geq k}$ if

$$R^i\hat{S}(F) = 0 \qquad \text{for all } i<k.$$

- We then have

Characterization of generic vanishing

Theorem

Let $\dim A = g$ and $F \in \mathrm{Coh}(A)$, then the following are equivalent:

1. *F is GV_{-k}.*
2. *$D_A(F) = R\mathcal{H}om(F, \mathcal{O}_A)[g]$ is $WIT_{\geq -k}$.*
3. *$H^i(A, F \otimes \hat{L}^\vee) = 0$ for any $i > k$ and any sufficiently ample line bundle L on $\hat{A}$.*

- Since by Kollár $H^i(R^j a_* \omega_X \otimes M) = 0$ for any ample line bundle M, then it is easy to see that the theorem applies to $F = R^j a_* \omega_X$. (Recall $\phi_L^*(\hat{L}^\vee) = L^{\oplus h^0(L)}$).
- Thus $R\hat{S}(D_A(R^j a_* \omega_X))$ is a sheaf and $\mathrm{codim}\, V^i(R^j a_* \omega_X) \geq i$.
- Thus generic vanishing follows as a limit of Kollár vanishing.
- The result holds more generally for X klt (or using Nadel vanishing).

Characterization of generic vanishing

- If $F \in \mathrm{Coh}(A)$ is GV_0 it also follows that $V^i(F) \supset V^{i+1}(F)$.
- So if F is GV_0 and $H^0(A, F \otimes P_x) = 0$ for all $P \in \hat{A}$, then $H^i(A, F \otimes P_x) = 0$ for all i and all $P \in \hat{A}$ and so $F = 0$.
- In general we expect that if F is GV_0 then F is determined by $H^0(A, F \otimes P_x) = 0$ for all $P \in \hat{A}$ (similarly for morphisms between GV_0 sheaves).

Generic fails vanishing in char. $p > 0$

- Since vanishing fails for non klt varieties and in char $p > 0$, we expect generic vanishing also fails in this context.

Theorem (Hacon-Kovács)

There exist a projective variety X such that either
(1) char > 0, then X is smooth, or
(2) char. $= 0$, X has isolated Gorenstein LC singularities,
and a separated projective morphism to an abelian variety $a : X \to A$ which is generically finite onto its image and such that $R^i p_{\hat{A}}, *((a \times \mathrm{id}_{\hat{A}})^* \mathcal{P}) \neq 0$ for some $i \neq \dim X$.

- In fact we show that if $R^i p_{\hat{A}}, *((a \times \mathrm{id}_{\hat{A}})^* \mathcal{P}) = 0$ for all $i \neq \dim X$, then $Ra_* \omega_X \cong a_* \omega_X$.
- Thus it suffices to construct $a : X \to A$ generically finite with $R^i a_* \omega_X \neq 0$ some $i \neq \dim X$.

Generic fails vanishing in char. $p > 0$

- If char > 0, then there are examples of Z a smooth 6-fold and L a very ample line bundle on Z such that $H^1(Z, K_Z + L) = 0$ (Lauritzen-Rao).
- We construct $a_Y : Y \to A$ finite with an isolated singularity isomorphic to a cone over Z.
- If $X \to Y$ is the resolution then we check that $R^1 a_* \omega_X \neq 0$ where $a : X \to A$.
- This forces $R^i p_{\hat{A}}, *((a \times \mathrm{id}_{\hat{A}})^* \mathcal{P}) \neq 0$ for some $i \neq \dim X$.
- I do not know of (but expect) examples where X is smooth and some $R^i a_* \omega_X$ fails generic vanishing.
- However we can prove a (weak) positive result.

Generic vanishing and Frobenius

- Consider $F : X \to X$ the Frobenius morphism.
- By duality we identify $\mathcal{H}om(F_*\omega_X, \omega_X) \cong F_*\mathcal{H}om(\omega_X, \omega_X)$, and let $Tr : F_*\omega_X \to \omega_X$ be the element corresponding to id_{ω_X}
- Iterating this we get maps $F^e_*\omega_X \to \omega_X$.
- It is well known that the image of these maps stabilizes (under mild hypothesis) to $\sigma(X) \otimes \omega_X$.
- If X is smooth (or F-pure) then $\sigma(X) = \mathcal{O}_X$.
- Tensoring by a line bundle L and taking global sections we get maps $H^0(F^e_*(\omega_X \otimes L^{p^e})) \to H^0(\omega_X \otimes L)$.
- The stable image is denoted by $S^0(\omega_X \otimes L) \subset H^0(\omega_X \otimes L)$.

Generic vanishing and Frobenius

- Given a morphism $a : X \to A$ we also obtain natural maps $F_*a_*\omega_X = a_*F_*\omega_X \to a_*\omega_X$.
- Let $\Phi^e : F^e_* a_*\omega_X \to a_*\omega_X$, then the image of Φ^e stabilizes for $e \gg 0$ (Blickle-Schwede).
- We call the stable image $S^0 a_*\omega_X$ and we consider the inverse system

$$\ldots \to F^{e+1}_* S^0 a_*\omega_X \to F^e_* S^0 a_*\omega_X \to \ldots$$

  with inverse limit $\Omega = \varprojlim F^e_* S^0 a_*\omega_X$.
- Thus $H^i(F^e_* S^0 a_*\omega_X \otimes \hat{L}^\vee) = H^i(F^e_* S^0 a_*\omega_X \otimes \hat{L}^\vee) = 0$ for any L sufficiently ample and $e \gg 0$, in fact $\phi_L^*(L^\vee) = L^{\oplus h^0(L)}$ and $F^{e*}L = L^{p^e}$ is sufficiently ample.
- Unluckily Ω is **quasi**-coherent and some difficulties arise. We show:

Generic vanishing and Frobenius

Theorem (Hacon-Patakfalvi)

There exists a quasi-coherent sheaf Λ on $\hat{A}$ such that $R\hat{S}(\Omega) = D_{\hat{A}}(\Lambda)$. In particular $\mathrm{codim} R^i\hat{S}(\Omega) \geq i$.

- Here $\Lambda = \varinjlim (\hat{F}^e)^* R\hat{S}(S^0 a_* \omega_X)$ where $\hat{F} : \hat{A} \to \hat{A}$ is the dual isogeny to $F : A \to A$.
- It does **not** follow that $R\hat{S} D_A(\Omega) = D_{\hat{A}}(D_{\hat{A}}(\Lambda))$ is a sheaf!
- $\Lambda \otimes k(y) \cong \varinjlim H^0(A, F^e_*(S^0 a_* \omega_X) \otimes P_{-y})^\vee$.
- Thus $\Lambda = 0$ (and hence $\Omega = 0$) if $H^0(S^0 a_* \omega_X \otimes P) = 0$ for all $P \in \hat{A}$.
- $\exists$ closed subset $Z \subset \hat{A}$ s.t. if $i > 0$ and $p^e \cdot x \notin Z$ for all $e \geq 0$, then $\varinjlim H^i(A, F^e_*(S^0 a_* \omega_X) \otimes P_{-y})^\vee = 0$.
- However, $\overline{V^i(\Omega)} = \hat{A}$ (if non-empty).

Geometric consequences

Theorem (Hacon-Patakfalvi)

If $\max \dim S^0(X, mK_X) = 1$*, then* $a : X \to A$ *is surjective.*

- Recall that $S^0(X, mK_X) = \mathrm{Im}(H^0(F^e_*(\omega_X^{1+(m-1)p^e})) \to H^0(\omega_X^m))$ where $e \gg 0$.
- In characteristic 0, by a result of Kawamata, if $\max \dim H^0(X, mK_X) = 1$ then $a : X \to A$ is surjective.
- In characteristic $p > 0$, the groups $S^0(X, mK_X)$ are better behaved than $H^0(X, mK_X)$, however if E is an elliptic curve, then $S^0(X, mK_E) \neq 0$ iff E is not supersingular (so the above result is not optimal).

Geometric consequences

- The idea of the proof is as follows: Assume that $S^0(X, K_X) \neq 0$ (by hypothesis $S^0(X, mK_X) \neq 0$ for some $m > 0$; and one checks that $(m, p) \neq 1$).
- Then $0_{\hat{A}} \in V^0(S^0 a_* \omega_X)$ is an isolated point and so $R\hat{S}(S^0 a_* \omega_X)$ has an Artinian summand (supported at $0_{\hat{A}}$).
- One can show that $\Lambda = \varinjlim (\hat{F}^e)^* R\hat{S}(S^0 a_* \omega_X)$ has a non-zero Artinian summand $\mathcal{A}$.
- Thus $RS(D_{\hat{A}}(\mathcal{A}))$ is a summand of $\Omega = RS(D_{\hat{A}}(\Lambda))$.
- But $RS(D_{\hat{A}}(\mathcal{A}))$ is obtained by extensions of $\mathcal{O}_A$ and hence its support is A.

Theta divisors

Theorem

Let (A, Θ) be a PPAV and $D \in |m\Theta|$ then $(A, \frac{1}{m}D)$ is F-pure (and hence log canonical and $\mathrm{mult_x}(D) \leq m \dim A$ for all $x \in A$.

- We have that $\Theta - \frac{1-\epsilon}{m}D$ is ample and so $p^e\Theta + \frac{(1-p^e)(1-\epsilon)}{m}D$ is suff. ample, gen. and $H^i(A, - \otimes P) = 0$, $i > 0$, $P \in \hat{A}$.
- There is a surjection $F^e_*\mathcal{O}_A(p^e\Theta + \frac{(1-p^e)(1-\epsilon)}{m}D) \to \sigma(A, \frac{1-\epsilon}{m}D) \otimes \mathcal{O}_A(\Theta)$.
- Using the FM transform one checks that the induced map $H^0(F^e_*\mathcal{O}_A(p^e\Theta + \frac{(1-p^e)(1-\epsilon)m}{D}) \otimes P) \to H^0(\sigma(A, \frac{1-\epsilon}{m}D) \otimes \mathcal{O}_A(\Theta) \otimes P)$ is non-zero for general $P \in \hat{A}$.
- But then the general translate of Θ vanishes along the cosupport of $\sigma(A, \frac{1-\epsilon}{m}D)$ and so $\sigma(A, \frac{1-\epsilon}{m}D) = \mathcal{O}_A$.