Shokurov's Rational connectedness conjecture

Christopher Hacon and James M^{c}Kernan

April 2005

The purpose of this lecture is to prove a conjecture of Shokurov which says that rational curves on varieties with "mild" singularities behave like rational curves on smooth varieties.

Here "mild singularities" is to be interpreted from the point of view of the MMP.

A **log pair** (X, Δ) is:

• a normal variety X;

• an effective $\mathbb{Q}$-Weil divisor $\Delta = \sum d_i D_i$ with $d_i \in \mathbb{Q}_{>0}$ such that

• $K_X + \Delta$ is $\mathbb{Q}$-Cartier (that is $\mathcal{O}_X(m(K_X + \Delta))$ is a line bundle when m is sufficiently divisible.)

Pick $\mu : X' \longrightarrow X$ a log resolution i.e. a proper birational morphism such that X' is smooth and

$$\mu^{-1}(\Delta) \cup \{exceptional\ set\}$$

is a divisor with simple normal crossings support. Let

$$K_{X'} + \Delta' = \mu^*(K_X + \Delta)$$

with $\Delta' = \sum a'_i \Delta'_i$. Then we say that

- (X, Δ) is KLT if $\lfloor \Delta' \rfloor < 0$ i.e. if $a'_i < 1 \forall i$;

- (X, Δ) is LC if $\lfloor (1-\epsilon)\Delta' \rfloor < 0$ for $0, \epsilon < 1$ i.e. if $a'_i \leq 1 \forall i$.

The locus of log canonical singularities is given by

$$LCS(X', \Delta') = \cup_{a_i \geq 1} \Delta_i$$

and $LCS(X, \Delta) = \mu(LCS(X', \Delta'))$.

The main result is the following:

Theorem: Let (X, Δ) be a log pair, $f : X \longrightarrow S$ a projective morphism such that $-K_X$ is relatively big and $\mathcal{O}_X(-m(K_X + \Delta))$ is relatively generated for some $m > 0$. Let $g : Y \longrightarrow X$ be any birational morphisms. Then every fiber of $\pi := f \circ g$ is rationally chain connected modulo $g^{-1}LCS(X, \Delta)$.

That is, for any two points of any fiber, there is a chain of curves connecting these points such that each curve is either rational or contained in $g^{-1}LCS(X, \Delta)$.

When $S = Spec\mathbb{C}$ we get the following result of Q. Zhang:

Theorem: Let (X, Δ) be a KLT pair such that $-(K_X + \Delta)$ is nef and big, then X is rationally connected (i.e. two general points can be joined by a rational curve).

When $X = S$ we have:

Theorem: Let (X, Δ) be a KLT pair and $g : Y \longrightarrow X$ a birational morphism, then the fibers of g are rationally chain connected.

In particular if (X, Δ) is a KLT pair, then:

• if $g : X \dashrightarrow Z$ is a rational map to a proper variety which is not everywhere defined, then Z contains a rational curve.

• X is rationally chain connected if and only if it is rationally connected.

The statement is sharp:

Let $f : S \longrightarrow C$ be a $\mathbb{P}^1$ bundle over an elliptic curve and E a section of minimal self intersection $E^2 < 0$. Contracting E we get a surface T which is rationally chain connected but not rationally connected. Notice that

$$K_S + tE$$

is KLT for $0 \leq t < 1$ and LC for $t = 1$ but $-(K_S + tE)$ is nef for $1 \leq t \leq 2$ and ample for $1 < t < 2$.

N.B. T is RCC but not RC, so RCC is not a birational property (the point is that $(T, \emptyset)$ is not KLT).

$S \to C$ is the MRC fibration: a surjective map with connected fibers which are RC, and the base is not uniruled (see the result of Graber-Harris-Starr).

Further consequences: Corollary: Let (X, Δ) be a projective log pair such that $-(K_X + \Delta)$ is semiample and $-(K_X + \Delta)$ is big. Then $\Pi_1(X)$ is a quotient of $\Pi_1(LCS(X, \Delta))$.

E.G. (Zhang): If (X, Δ) is KLT, $-(K_X + \Delta)$ is nef and big then X is simply connected.

Theorem: Let (X, Δ) be a KLT pair, $f : X \longrightarrow S$ a projective morphism with connected fibers such that $-K_X$ is relatively big and $-(K_X + \Delta)$ is relatively nef for some $m > 0$. Let $g : Y \longrightarrow X$ be any birational morphisms.Then
1) the natural map

$$\pi_* := (f \circ g)_* : CH^0(Y) \longrightarrow CH^0(S)$$

is an isomorphism.
2) π has a section over any curve.

How to show that a variety X is RC.

It suffices to show that for any rational map $X \dashrightarrow Z$, Z is uniruled (i.e. covered by rational curves). In fact let

$$X \leftarrow X' \rightarrow Z$$

be the MRC fibration of X, then by Graber-Harris-Starr, Z is not uniruled so Z is a point i.e. X is RC.

Assuming that Z is smooth, by a result of Boucksom-Demailly-Paun-Peternell:
Theorem: Z is not uniruled if and only if K_Z is pseudoeffective.
(Recall $K_Z \in PSEF(Z)$ if for any $\epsilon > 0$, A ample on Z, one has $h^0(m(K_Z + \epsilon A)) > 0$ for all $m \gg 0$ sufficiently divisible.)

Conjecturally: $K_Z \in PSEF(Z)$ iff $\kappa(K_Z) \geq 0$.

Idea of the proof:

We have $\pi = (f \circ g) : Y \longrightarrow X \longrightarrow S$. Fix $s \in S$. We may replace Y by a higher birational model so that Y is smooth and $\pi^{-1}(s)$ is a divisor with simple normal crossings. For simplicity assume that (X, Δ) is KLT. We may also assume that

$$K_X + \Delta = 0$$

(just replace Δ by $\Delta + B/m$ where B is a general section of $|-m(K_X+\Delta)|$). We now write:

$$K_Y + \Gamma = g^*(K_X + \Delta) + E$$

where Γ, E have no common components and E is exceptional. Replacying Γ by an appropriate $\mathbb{Q}$-linearly equivalent divisor, we may assume that

$$\Gamma \geq A$$

where A is an ample divisor.

Pick G on S sufficiently singular at s so that for $t = 0, t_1, ..., t_k = 1$ we have :

$$LCS(Y, \Gamma + t\pi^* G) = \emptyset, F_1, F_1 \cup F_2,$$

$$..., \pi^{-1}(s) = F_1 \cup ... \cup F_k$$

The point is to show that F_1 is RC, F_2 is RCC modulo F_1 etc. (F_i is RCC modulo $V_i = F_i \cap (F_1 \cup ... \cup F_{i-1})$.)

So we consider the MRC fibration

$$F_i \leftarrow F_i' \rightarrow Z.$$

We must show that $V_i = F_i \cap (F_1 \cup ... \cup F_{i-1})$ dominates Z. Assume this is not the case. We know that K_Z is pseudoeffective. We claim that for H ample on Z, sections of $m(K_Z + \epsilon H)$ can be liftled to $m(K_{F_i'} + \{\Gamma\}|_{F_i'})$ and then to sections of $m(K_Y + \Gamma) + A'$ where A' is a sufficiently ample divisor on Y.

The point is that if $A' \leq g^*A''$, then

$$\pi_*\mathcal{O}_Y(m(K_Y + \Gamma) + A') \subset$$

$$\pi_*\mathcal{O}_Y(mE + g^*A'') = f_*\mathcal{O}_Y(A'')$$

is a fixed coherent sheaf containing lifts of sections of $m(K_Z + \epsilon H)$. But then as $\kappa(K_Z + \epsilon H) = \dim Z$, we see that Z is a point as required.

Sections of $m(K_Z + \epsilon H)$ can be lifted to $m(K_{F'_i} + \{\Gamma\}|_{F'_i})$:

This follows from the log additivity of the Kodaira dimension (cf. Campana, Lu). Let $z \in Z$ be a general point. We have

$$(K_Y + \{\Gamma\} + F)|_{F_z} = (K_Y + \Gamma + F)|_{F_z} = E|_{F_z} \geq 0;$$

$$\{\Gamma\} \geq \epsilon p^* H$$

$$(K_Y + \{\Gamma\} + F)|_{F_z} = K_F + \{\Gamma\}|_{F_z} \ is\ LC$$

Therefore, one has that

$$\kappa(K_F + \{\Gamma\}) \geq \kappa(K_Z + \epsilon H).$$

Sections of $m(K_{F_i'}+\{\Gamma\}|_{F_i'})$ can be lifted to sections of $m(K_Y+\Gamma)+A'$:
This follows from a generalization of the following result of Siu and Kawamata:
Theorem: Let $X \subset Y$ be a smooth divisor on a smooth variety, $\pi : Y \longrightarrow S$ a projective morphism, H a π-very ample divisor on Y. Then

$$\pi_*\mathcal{O}_Y(m(K_Y+X)+H) \longrightarrow \pi_*\mathcal{O}_X(m(K_Y+X)+H)$$

is surjective for all $m > 0$.

We show that:

Theorem: Let $X \subset Y$ be a smooth divisor on a smooth variety, $\pi : Y \longrightarrow S$ a projective morphism, H a sufficiently π-ample divisor on Y, $A = (\dim Y)H$. Assume that

1) Γ is a $\mathbb{Q}$-divisor with simple normal crossings support such that Γ contains X with coefficient 1 and (Y, Γ) is LC;

2) $k \in \mathbb{Z}_{>0}$ is such that $k\Gamma$ is integral;

3) $C \geq 0$ is an int. divisor not containing X;

4) Given $D = k(K_Y + \Gamma)$, $D' = D|_X = K_X + \Delta$ and $G = D + C$, then $D' + \epsilon H'$ is π-$\mathbb{Q}$-effective for any $\epsilon > 0$ and $\pi_*\mathcal{O}_Y(mG) \longrightarrow \pi_*\mathcal{O}_Z(mG)$ is non-zero for any $Z \in LCC(Y, \ulcorner\Gamma\urcorner)$.

Then, for all $m > 0$, the image of

$$\pi_*\mathcal{O}_Y(mG + H + A) \longrightarrow \pi_*\mathcal{O}_X(mG + H + A)$$

contains the image of the sheaf

$$\pi_*\mathcal{O}_X(mD + H)$$

considered as a subsheaf of $\pi_*\mathcal{O}_X(mG + H + A)$ by the inclusion induced by any divisor in $mC + |A|$ not containing X.

We apply this with $X = F_i$, $\Gamma = F_i + \{\Gamma_i\}$, $C = k(\Gamma_i - F_i - \{\Gamma_i\})$. Then $mG = m(K_Y + \Gamma_i) = g^*(K_X + \Delta + t_i G) + mE$ is π-generated at all $Z \in LCC(Y, \Gamma)$.

The statement is necessarily technical. After all, let $Y \longrightarrow \mathbb{P}^2$ be the blow up of a point P and X be the strict transform of a line through this point, and E the exceptional divisor. Then we don't expect to lift sections of $\mathcal{O}_X(mE)$ to $\mathcal{O}_Y(mE)$. Our additional hypothesis are natural in the sense that they take into account the base locus on Y.

There is another application of this extension result (for $S = Spec\mathbb{C}$). Following ideas of Tsuji, we can show that:

Theorem: For any positive integer n, there exists an integer r_n such that if X is a smooth projective variety of general type and dimension n, then $\phi_{rK_X} : X \dashrightarrow \mathbb{P}(H^0(\mathcal{O}_X(rK_X)))$ is birational for all $r \geq r_n$.

Tsuji's idea: It suffices to show that for $x \in X$ very general, there is a $\mathbb{Q}$-divisor $G \sim_{\mathbb{Q}} \lambda K_X$ such that G has an isolated log canonical center at x and

$$\lambda = \frac{A}{(\mathrm{vol}(K_X))^{1/n}} + B$$

where A, B are positive constants depending only on $n = \dim(X)$. Then, for an analogous bound, one has that the map ϕ_{rK_X} is birational. If $\mathrm{vol}(K_X) \geq 1$, the theorem is clear. If $\mathrm{vol}(K_X) < 1$, one then shows that X belongs to a birationally bounded family and hence there is a uniform lower bound for the volume of K_X.

In fact, let Z be the image of ϕ_{rK_X}, one sees that

$$\deg(Z) \leq \mathrm{vol}(rK_X) = r^n \mathrm{vol}(K_X) \leq (An+B)^n.$$

It is straightforward to produce a $\mathbb{Q}$-divisor $G \sim \lambda_1 K_X$ with nontrivial log canonical center V_x at x. The point now is to cut down this divisor to a point. Since X is of gen. type and $x \in X$ is general, we may assume that V_x is of general type. By Kawamata's Subadjunction, one expects that $(1+\lambda_1)K_X|_{V_x} \geq K_{V_x}$. The main difficulty is to lift these sections to X. If K_X is ample (eg. assuming the MMP), this is immediate. In general, it is a very delicate statement. Using our extension result we are able to achieve this on an appropriate log resolution $Y \longrightarrow X$......