On 3-fold flips in char $p > 0$.

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Outline of the talk

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1. Introduction
2. Extending sections in Char $p > 0$

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Birational classification over $\mathbb{C}$.

- In recent years there has been substantial progress in understanding the birational geometry of varieties over the complex numbers $\mathbb{C}$.
- It is known that the canonical ring $R(K_X)$ is finitely generated (for any smooth projective variety X; [BCHM], [Siu]).
- If X is of general type (i.e. K_X is big so that $h^0(mK_X) = O(m^{\dim X})$), then X has a minimal model which is given by a finite sequence of flips and divisorial contractions $X \dashrightarrow X_1 \dashrightarrow X_2 \ldots \dashrightarrow X_{\min}$. In particular $K_{X_{\min}}$ is nef.
- If K_X is not pseudo-effective (i.e. $K_X + \epsilon H$ is not big for any H ample and $0 < \epsilon \ll 1$), then after finitely many flips and divisorial contractions we obtain a Mori fiber space: $X_N \to Z$. In particular $-K_{X_N}$ is ample over Z and $\dim X > \dim Z$.

Birational classification in characteristic $p > 0$.

- By contrast, very little is known in characteristic $p > 0$.
- In dimension 2 the classification is analogous to the characteristic 0 case. More precisely we have
- Given a normal proj. surface X, $\Delta = \sum \delta_i \Delta_i \geq 0$ s.t.
  (1) X is $\mathbb{Q}$-factorial, $0 \leq \delta_i \leq 1$ or
  (2) $k = \overline{F}_p$ and $0 \leq \delta_i$ or
  (3) (X, Δ) is LC,
  then there exists a finite sequence of $K_X + \Delta$ negative contractions of irreducible curves
  $X \to \ldots \to X_i \to X_{i+1} \to \ldots X_N$ s.t.
  • X_N is a minimal model ($K_{X_N} + \Delta_N$ is nef) or
  • X_N is a Mori fiber space ($\exists\, f : X_N \to Z$ with $\rho(X_N/Z) = 1$, $\dim Z < \dim X_N$ and $-(K_{X_N} + \Delta_N)$ is f-ample).
- SLC abundance: If $K_X + \Delta$ is nef, then it is semiample.

Birational classification in characteristic $p > 0$.

- Resolution of singularities is expected to hold, but so far it is only known in dimension ≤ 3 (by Abhyankar, Cutkosky, Cossart and Piltant).
- There are many other technical difficulties (especially the failure of Kawamata-Viehweg vanishing).
- In this talk I will discuss recent progress for 3-folds.
- I begin by recalling a result of Keel. Let L be a nef line bundle (on a proper scheme), then the exceptional locus of L is $E(L) = \{Z \subset X | Z \cdot L^{\dim Z} = 0\}$.
- If L is semiample, then $f : X \to T = \mathrm{Proj} \oplus_{m \geq 0} H^0(\mathcal{O}_X(mL))$ contracts $E(L)$.
- If there is a proper morphism of algebraic spaces contracting $E(L)$, then L is **EWM** (endowed with map).

Birational classification in characteristic $p > 0$.

- Keel shows that in char $p > 0$ a nef line bundle L is semiample (EWM) iff $L|_{E(L)}$ is semiample (EWM). He then proves the following:
- **Base point free theorem:** If X is normal and $\mathbb{Q}$-factorial, $0 \leq \Delta < 1$, L is a nef and big Cartier divisor such that $L - (K_X + \Delta)$ is nef and big then L is EWM. (If $k = \bar{F}_p$ then L is semiample.)
- **Cone Theorem:** Assume moreover that $\kappa(K_X + \Delta) \geq 0$ then (1) $\overline{NE}_1(X) = \overline{NE}_1(X)_{(K_X+\Delta)_{\geq 0}} + \sum_{i \in \mathbb{N}} \mathbb{R}[C_i]$,
(2) $0 < -(K_X + \Delta) \cdot C_i \leq 3$ for almost all C_i,
(3) $\mathbb{R}[C_i]$ are discrete in $(K_X + \Delta)_{<0}$.
- This allows us to run a MMP as long as we can prove the existence of flips (and stay in the projective category).
- Flips exist (under "mild" restrictions).

Birational classification in characteristic $p > 0$.

Theorem

(Hacon-Xu) Let (X, Δ) *be a* $\mathbb{Q}$*-factorial* 3*-fold projective dlt pair,* $\Delta \in \{1 - \frac{1}{n} | n \in \mathbb{N}\}$ *and* $p > 5$ *then flips exist.*

• Recall that a **flipping contraction** is a small birational morphism $f : X \to Z$ with $\rho(X/Z) = 1$ such that $-(K_X + \Delta)$ is f-ample.
• The **flip** $f^+ : X^+ \to Z$ is another small birational morphism with $\rho(X^+/Z) = 1$ such that $K_{X^+} + \Delta^+$ is f^+-ample.
• If the flip exists (assuming Z is affine), then it is given by $X^+ = \mathrm{Proj} \oplus_{m \in \mathbb{N}} H^0(m(K_X + \Delta))$.
• So one must show that $R(K_X + \Delta)$ is finitely generated.

Corollary

If K_X *is pseudo-effective and terminal, then it has a minimal model.*

Birational classification in characteristic $p > 0$.

- There is an additional difficulty: we do not know that Z is projective. However, if X is projective, we can still guarantee that X^+ is projective.
- Similarly if $X \to Z$ is a divisorial contraction and X is projective, then we construct a projective "divisorial contraction" $X' \to Z$ as the relative minimal model of X over Z.
- Base point free theorem, existence of flips in characteristic ≤ 5 or for Δ with arbitrary coefficients, termination of klt flips and abundance are all open for 3-folds.

Outline of the talk

Vanishing over $\mathbb{C}$.

- The main tool in the subject is the Kawamata-Viehweg vanishing Theorem.
- Let (X, B) be a klt pair and D be a Cartier divisor such that $M := D - (K_X + B)$ is nef and big, then $H^i(X, \mathcal{O}_X(D)) = 0$ for $i > 0$.
- Recall: If X is smooth and $B = \sum b_i D_i$ has simple normal crossings support where $0 \leq b_i < 1$ then (X, B) is klt.
- M is nef if $M \cdot C \geq 0$ for any curve $C \subset X$ and nef and big if moreover $M^{\dim X} > 0$.
- If $H^1(X, \mathcal{O}_X(D)) = 0$, then the restriction $H^0(X, \mathcal{O}_X(D + S)) \to H^0(S, \mathcal{O}_S((D + S)|_S))$ is surjective and we may apply induction on $\dim X$.

Positive char approach

- Let k be algebraically closed field of characteristic $p > 0$.
- By examples of Raynaud and others, it is known that Kodaira (and hence also Kawamata-Viehweg) vanishing fails in positive characteristic and $\dim X \geq 2$.
- We therefore attempt to replace this vanishing by the systematic use of the Frobenius morphism and Serre vanishing.
- Consider $F : X \to X$ the Frobenius morphism (a finite morphism).
- By duality we identify $\mathcal{H}om(F_*\omega_X, \omega_X) \cong F_*\mathcal{H}om(\omega_X, \omega_X)$, and we let $Tr : F_*\omega_X \to \omega_X$ be the element corresponding to id_{ω_X}.
- We obtain homomorphisms $\ldots F_*^{e+1}\omega_X \to F_*^e\omega_X \to \ldots F_*\omega_X \to \omega_X$.

Canonical subsystems of adjoint bundles

- If X is smooth and $\mathcal{L}$ is a line bundle, then we also obtain homomorphisms $F^e_*(\omega_X \otimes \mathcal{L}^{p^e}) \cong (F^e_*\omega_X) \otimes \mathcal{L} \to \omega_X \otimes \mathcal{L}$.
- We Let $S^0(X, \omega_X \otimes \mathcal{L})$ be the image of $H^0(X, \omega_X \otimes \mathcal{L}^{p^e}) \to H^0(X, \omega_X \otimes \mathcal{L})$ for $e \gg 0$ sufficiently divisible.
- If $\mathcal{L}$ is ample, then $H^1(X, \omega_X \otimes \mathcal{L}^{p^e}) = 0$ for $e \gg 0$ so that $H^0(X, \omega_X(S) \otimes \mathcal{L}^{p^e}) \to H^0(S, \omega_S \otimes \mathcal{L}|_S^{p^e})$ is surjective.
- Thus $S^0(X, \sigma(X, S) \otimes \mathcal{L}(S)^{p^e}) \to S^0(S, \omega_S \otimes \mathcal{L})$ is surjective.
- Here $S^0(X, \sigma(X, S) \otimes \mathcal{L}(S)^{p^e})$ is the image of $H^0(\omega_X(S) \otimes \mathcal{L}^{p^e}) \subset H^0(\omega_X \otimes \mathcal{L}(S)^{p^e}) \to H^0(\omega_X \otimes \mathcal{L}(S))$.
- The above result generalizes to log pairs by work of Schwede:

Generalization to log pairs

- If (X, Δ) is a pair such that $(p^e - 1)(K_X + \Delta)$ is Cartier for some $e > 0$ (i.e. p does not divide the index of $K_X + \Delta$). then we get $\Phi^e_\Delta : F^e_* \mathcal{O}_X((1 - p^e)(K_X + \Delta)) \to \mathcal{O}_X$ (by adding $(p^e - 1)\Delta$ and using Tr on the smooth locus).
- $\sigma(X, \Delta)$ denotes the image of Φ^e_Δ for $e > 0$ sufficiently divisible and $S^0(X, \sigma(X, \Delta) \otimes \mathcal{L})$ the image on global sections after tensoring by $\mathcal{L}$.
- $\sigma(X, \Delta)$ is an analog of the non LC ideal in characterisic 0
- At snc points $\sigma(X, \Delta) = \mathcal{O}_X$ iff $\Delta \leq 1$.
- (X, Δ) is **F-pure** if $\sigma(X, \Delta) = \mathcal{O}_X$ and **F-regular** if $\sigma(X, \Delta + \epsilon H) = \mathcal{O}_X$ for any H; $0 < \epsilon \ll 1$ (better: $\nexists \mathcal{I} \subset \mathcal{O}_X$ non-trivial s.t. $\Phi^e_\Delta(\mathcal{I} \cdot \mathcal{O}_X((1 - p^e)(K_X + \Delta))) \subset \mathcal{I}$).
- Analogs of LC and KLT.

Extension Theorem

Theorem (Schwede)

Let (X, Δ) be an F-pure pair such that p does not divide the index of $K_X + \Delta$. Assume that there is a normal non-F-regular center Z so that

$$\Phi^e_\Delta(\mathcal{I}_Z \cdot \mathcal{O}_X((1-p^e)(K_X+\Delta))) \subset \mathcal{I}_Z.$$

If $\mathcal{M}$ is Cartier and $\mathcal{M} - (K_X + \Delta)$ is ample, then

$$S^0(X, \sigma(X,\Delta) \otimes \mathcal{M}) \to S^0(Z, \sigma(Z, \Delta_Z) \otimes \mathcal{M}|_Z)$$

is surjective (where Δ_Z is defined by "adjunction").

Question: Can we arrange $S^0(Z, \sigma(Z, \Delta_Z) \otimes \mathcal{M}|_Z) \neq 0$?

Global F-regularity

- If $\mathcal{M}$ is sufficiently ample, then $S^0(X, \sigma(X, \Delta) \otimes \mathcal{M}) \cong H^0(X, \sigma(X, \Delta) \otimes \mathcal{M})$ is globally generated (in general the inclusion can be strict!).
- If $\mathcal{M}$ is big and semiample, let $f : X \to T = \mathrm{Proj}(\oplus_{m \geq 0} \mathcal{M}^m)$ Then (replacing $\mathcal{M}$ by a multiple)

$$H^0(X, \mathcal{O}_X((p^e - 1)(K_X + \Delta)) \otimes \mathcal{M}^{p^e}) \to H^0(X, \mathcal{M})$$

  is surjective provided that (X, Δ) is F-regular over T i.e. $F^e_* f_* \mathcal{O}_X((p^e - 1)(K_X + \Delta)) \to \mathcal{O}_T$ is surjective.
- By a result of Schwede-Smith X/T is of relative log Fano type.

Global F-regularity II

- If $\dim X = 2$, $p > 5$, (X, Δ) is klt, $\Delta \in \{1 - \frac{1}{n} | n \in \mathbb{N}\}$, $f : X \to T$ is birational and $-(K_X + \Delta)$ is f-ample then (X, Δ) is globally F-regular over T.
- Hara proved the case when $\Delta = 0$ and $f = id$.
- $p > 5$ is a necessary condition.
- The proof is by "classification"; uses Shokurov's theory of complements.
- We will now give a sketch of the proof of the existence of 3-fold flips which closely follows ideas of Shokurov.

Outline of the talk

Pl-flips and the restricted algebra

- By Shokurov's reduction to pl-flips it is enough to prove the existence of pl-flips.
- Thus we may assume that $f : X \to Z = \mathrm{Spec}(A)$ is a small birational morphism of normal varieties, $(X, \Delta = S + B)$ is plt where $S = \lfloor \Delta \rfloor$, $\rho(X/Z) = 1$ and $-(K_X + \Delta)$ and $-S$ are f ample.
- Let $R_S(K_X + \Delta) = \mathrm{Im}(R(K_X + \Delta) \to R(K_{S^n} + B_{S^n}))$ where $S^n \to S$ is the normalization and $K_{S^n} + B_{S^n} = (K_X + \Delta)|_{S^n}$.
- Then $R(K_X + \Delta)$ is fin. gen. iff $R(K_{S^n} + B_{S^n})$ is fin. gen.
- We have $K_X + \Delta = tS$; assume $t = 1$, then the kernel of $H^0(m(K_X + \Delta)) \to H^0(m(K_{S^n} + B_{S^n}))$ is in $H^0((m-1)(K_X + \Delta))$.
- If $R_S(K_X + \Delta) = R(K_{S^n} + B_{S^n})$ we would be done! So we must identify which elements of $R(K_{S^n} + B_{S^n})$ lift.

Normality of S

- In char 0, if $(X, S+B)$ is plt (near S) then S is normal and (S, B_S) is klt.
- In char $p > 0$, this is not known (pf. depends on KV-vanish.).
- We show that if $\dim X = 3$, $B \in \{1 - \frac{1}{n} | n \in \mathbb{N}\}$ and $p > 5$ this holds (more generally assuming that (S^n, B_{S^n}) is strongly F-regular then $(X, S+B)$ is F-pure near S (there are related results of Schwede and others)).
- Let $g : Y \to X$ be a log resolution, $g^{-1}_* S = S' \to S^n \to S$ the induced map.
- $K_Y + S' = g^*(K_X + S + B) + A_Y$, $K_{S'} = g'^*(K_{S^n} + B_{S^n}) + A_{S'}$.
- Pick $F \geq 0$ exceptional, g-anti-ample.
- Pick H sufficiently ample on X ($H|_{S^n}$ very ample).

Normality of S

- Let $L = g^*H + \lceil A_Y \rceil$ and $\Xi = S' + \{-A_Y\} + \epsilon F$ $(0 < \epsilon \ll 1)$.
- $L - (K_Y + \Xi) \cong g^*(H - (K_X + \Delta)) - \epsilon F$ is ample.
- $S^0(\sigma(Y, \Xi) \otimes \mathcal{O}_Y(L)) \to S^0(\sigma(S', \Xi) \otimes \mathcal{O}_{S'}(L))$ is onto.
- We will show that $S^0(\sigma(S', \Xi) \otimes \mathcal{O}_{S'}(L)) = H^0(\mathcal{O}_{S'}(L))$, thus $H^0(\mathcal{O}_X(H)) \cong H^0(\mathcal{O}_Y(L)) \twoheadrightarrow H^0(\mathcal{O}_{S'}(L)) \supset H^0(\mathcal{O}_{S^n}(H))$.
- As $H|_{S^n}$ is very ample, we are done.
- Pick E s.t. $S^n \supset E \supset g(F|_{S'})$ and $0 < \epsilon \ll \delta \ll 1$.
- $g^*(K_{S^n} + B_{S^n} + \delta E) \geq K_{S'} + \{-A_{S'}\} - \lceil A_{S'} \rceil + \epsilon F|_{S'}$, so $F^e_* \mathcal{O}_{S^n}((1 - p^e)(K_{S^n} + B_{S^n} + \delta E + p^e H|_{S^n}) \hookrightarrow$ $g_* F^e_* \mathcal{O}_{S'}((1 - p^e)(K_{S'} + \Xi_{S'}) + p^e L)$.
- As $K_{S^n} + B_{S^n}$ is F-regular, we have a surjection $F^e_* \mathcal{O}_{S^n}((1 - p^e)(K_{S^n} + B_{S^n} + \delta E + p^e H|_{S^n})) \to H^0(\mathcal{O}_{S^n}(H))$.

The restricted algebra

- For any $f : Y \to X$, let $N_{i,Y} = \mathrm{Mob}(i(K_X + \Delta))$ and $M_{i,S'} = N_{i,Y}|_{S'}$, $D_{i,S'} = \frac{1}{i} M_{i,S'}$ and $D_{S'} = \lim D_{i,S'}$.
- For Y sufficiently high, $N_{i,Y}$ and $M_{i,S'}$ are free and $\lceil A_Y \rceil$ (resp. $\lceil A_{S'} \rceil$) saturated i.e. $|N_{i,Y} + \lceil A_Y \rceil| = |N_{i,Y}| + \lceil A_Y \rceil$. (This is because $\lceil A_Y \rceil \geq 0$ is exceptional and so $|f^*(i(K_X + \Delta)) + \lceil A_Y \rceil| = |f^*(i(K_X + \Delta))| + \lceil A_Y \rceil$.)
- $R_S(K_X + \Delta)$ is finitely generated iff
  • there exists $\bar{S} \to S$ such that $M_{i,S'}$ descends to $\bar{S}$ for all $i \gg 0$ (given $h : S' \to \bar{S}$, then $M_{i,S'} = h^* M_{i,\bar{S}}$ and
  • an index $\bar{i}$ s.t. $D_{i,\bar{S}} = D_{\bar{i},\bar{S}}$ whenever $\bar{i}|i$.
- $\mu : \bar{S} \to S$ is just the terminalization of (S, B_S) i.e. the "smallest" resolution such that $\mathrm{mult}_x(B_{\bar{S}}) < 1$ for any $x \in \bar{S}$ and $K_{\bar{S}} + B_{\bar{S}} = \mu^*(K_S + B_S)$ (nb. $B_{\bar{S}} = -A_{\bar{S}} \geq 0$).

Descending to $\bar{S}$.

- So we must show that if $M = M_{i,S'}$ is free and $\lceil A_{S'} \rceil$ saturated, then it descends to $\bar{S}$.
- Since $\lceil A_{S'} \rceil$ is the exceptional set of $S' \to \bar{S}$, we need $M \cap \lceil A_{S'} \rceil = \emptyset$.
- Suppose that $\exists \bar{N} \in |N|$ s.t. $C = \bar{N} \cap S' \in |M|$ is smooth then let $\Xi = S' + \{-A_Y\} + \epsilon F$ and arguing as above, one sees that $S^0(\sigma(Y, \Xi + \bar{N}) \otimes \mathcal{O}_Y(N + \lceil A_Y \rceil)) \to S^0(\sigma(C, B_C) \otimes \mathcal{O}_C(N + \lceil A_Y \rceil))$ is surjective.
- Since C is an affine curve, the RHS is generated, but the LHS vanishes along $\lceil A_Y \rceil$ and so $C \cap \lceil A_{S'} \rceil = 0$, i.e. $M \cap \lceil A_{S'} \rceil = \emptyset$ as required.

Descending to $\bar{S}$ (sing. C).

- If C is singular, technical issues arise.
- In char 0, this creates a 0-dimensional non-klt center at the singular points of C and we expect to be able to lift sections from these centers and conclude as above.
- In char $p > 0$ we use F-seshadri constants (developed by Mustata and Schwede).
- Since C moves (in $|M \otimes \mathfrak{m}_x^2|$), one sees that $\epsilon(x, M_{i,S'}) \geq 2$ (so $\epsilon_F(x, M_{i,S'}) \geq 1$) and we are able to lift sections from these points.
- Thus all $M_{i,S'}$ descend to $\bar{S}$. Note that the limit $D_{\bar{S}}$ is nef (over Z) and $(\bar{S}, B_{\bar{S}})$ is a weak log Fano, thus $D_{\bar{S}}$ is a semiample $\mathbb{R}$-divisor.
- Let $a : \bar{S} \to S^+$ be the induced morphism. We claim that $D_{\bar{S}} = a^* D_{S^+}$ where $D_{S^+} \in \mathrm{Div}_{\mathbb{Q}}(S^+)$.

$D_{\bar{S}}$ descends to S^+.

- To see this, we pick $C \in \mathrm{Div}(S^+)$ so that $||C - jD_{S^+}|| \ll 1$ (by Diophantine approximation).
- Proceeding as above one checks that $(\lceil \frac{j}{i} M_{i,S'} + A_{S'} \rceil - M_{j,S'}) \cdot g^* C = 0$ and hence $\lceil \frac{j}{i} M_{i,S'} + A_{S'} \rceil - M_{j,S'}$ is exceptional over S^+.
- Finally, the usual saturation arguments (with Kawamata Viehweg vanishing replaced by the S^0 extension results) show that $\frac{1}{i} M_{i,\bar{S}} = D_{\bar{S}}$ for all $i > 0$ sufficiently divisible.