On 3-fold flips in char $p > 0$.

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Outline of the talk

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Birational classification over $\mathbb{C}$.

- In recent years there has been substantial progress in understanding the birational geometry of varieties over the complex numbers $\mathbb{C}$.
- It is known that the canonical ring $R(K_X)$ is finitely generated (for any smooth projective variety $X$; [BCHM], [Siu]).
- If $X$ is of general type (i.e. $K_X$ is big so that $h^0(mK_X) = O(m^{\dim X})$), then $X$ has a minimal model which is given by a finite sequence of flips and divisorial contractions $X \dashrightarrow X_1 \dashrightarrow X_2 \ldots \dashrightarrow X_{\text{min}}$. In particular $K_{X_{\text{min}}}$ is nef.
- If $K_X$ is not pseudo-effective (i.e. $K_X + \epsilon H$ is not big for any $H$ ample and $0 < \epsilon \ll 1$), then after finitely many flips and divisorial contractions we obtain a Mori fiber space: $X_N \rightarrow Z$. In particular $-K_{X_N}$ is ample over $Z$ and $\dim X > \dim Z$. 

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3-fold flips in char $p > 0$. 

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By contrast, very little is known in characteristic $p > 0$.

In dimension 2 the classification is analogous to the characteristic 0 case. More precisely we have

Given a normal proj. surface $X$, $\Delta = \sum \delta_i \Delta_i \geq 0$ s.t.

1. $X$ is $\mathbb{Q}$-factorial, $0 \leq \delta_i \leq 1$ or
2. $k = \overline{F}_p$ and $0 \leq \delta_i$ or
3. $(X, \Delta)$ is LC,

then there exists a finite sequence of $K_X + \Delta$ negative contractions of irreducible curves

$$X \rightarrow \ldots \rightarrow X_i \rightarrow X_{i+1} \rightarrow \ldots X_N$$

s.t.

- $X_N$ is a minimal model ($K_{X_N} + \Delta_N$ is nef) or
- $X_N$ is a Mori fiber space ($\exists f : X_N \rightarrow Z$ with $\rho(X_N/Z) = 1$, $\dim Z < \dim X_N$ and $-(K_{X_N} + \Delta_N)$ is $f$-ample).

SLC abundance: If $K_X + \Delta$ is nef, then it is semiample.
Birational classification in characteristic $p > 0$.

- Resolution of singularities is expected to hold, but so far it is only known in dimension $\leq 3$ (by Abhyankar, Cutkosky, Cossart and Piltant).
- There are many other technical difficulties (especially the failure of Kawamata-Viehweg vanishing).
- In this talk I will discuss recent progress for 3-folds.
- I begin by recalling a result of Keel. Let $L$ be a nef line bundle (on a proper scheme), then the exceptional locus of $L$ is $E(L) = \{ Z \subset X | Z \cdot L^{\dim Z} = 0 \}$.
- If $L$ is semiample, then $f : X \to T = \text{Proj} \oplus_{m \geq 0} H^0(\mathcal{O}_X(mL))$ contracts $E(L)$.
- If there is a proper morphism of algebraic spaces contracting $E(L)$, then $L$ is EWM (endowed with map).
Birational classification in characteristic $p > 0$.

- Keel shows that in char $p > 0$ a nef line bundle $L$ is semiample (EWM) iff $L|_{E(L)}$ is semiample (EWM). He then proves the following:

**Base point free theorem:** If $X$ is normal and $\mathbb{Q}$-factorial, $0 \leq \Delta < 1$, $L$ is a nef and big Cartier divisor such that $L - (K_X + \Delta)$ is nef and big then $L$ is EWM. (If $k = \bar{F}_p$ then $L$ is semiample.)

**Cone Theorem:** Assume moreover that $\kappa(K_X + \Delta) \geq 0$ then

1. $\overline{NE}_1(X) = \overline{NE}_1(X)_{(K_X + \Delta) \geq 0} + \sum_{i \in \mathbb{N}} \mathbb{R}[C_i]$, 
2. $0 < -(K_X + \Delta) \cdot C_i \leq 3$ for almost all $C_i$, 
3. $\mathbb{R}[C_i]$ are discrete in $(K_X + \Delta)_{<0}$.

- This allows us to run a MMP as long as we can prove the existence of flips (and stay in the projective category).
- Flips exist (under “mild” restrictions).
Birational classification in characteristic $p > 0$. 

**Theorem**

(Hacon-Xu) Let $(X, \Delta)$ be a $\mathbb{Q}$-factorial 3-fold projective dlt pair, $\Delta \in \{1 - \frac{1}{n} | n \in \mathbb{N}\}$ and $p > 5$ then flips exist.

- Recall that a **flipping contraction** is a small birational morphism $f : X \to Z$ with $\rho(X/Z) = 1$ such that $-(K_X + \Delta)$ is $f$-ample.
- The **flip** $f^+ : X^+ \to Z$ is another small birational morphism with $\rho(X^+/Z) = 1$ such that $K_{X^+} + \Delta^+$ is $f^+$-ample.
- If the flip exists (assuming $Z$ is affine), then it is given by $X^+ = \text{Proj} \bigoplus_{m \in \mathbb{N}} H^0(m(K_X + \Delta))$.
- So one must show that $R(K_X + \Delta)$ is finitely generated.

**Corollary**

If $K_X$ is pseudo-effective and terminal, then it has a minimal model.
Birational classification in characteristic $p > 0$.

- There is an additional difficulty: we do not know that $Z$ is projective. However, if $X$ is projective, we can still guarantee that $X^+$ is projective.

- Similarly if $X \to Z$ is a divisorial contraction and $X$ is projective, then we construct a projective "divisorial contraction" $X' \to Z$ as the relative minimal model of $X$ over $Z$.

- Base point free theorem, existence of flips in characteristic $\leq 5$ or for $\Delta$ with arbitrary coefficients, termination of klt flips and abundance are all open for 3-folds.
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2. Extending sections in Char $p > 0$
3. Proof of the existence of flips
The main tool in the subject is the Kawamata-Viehweg vanishing Theorem.

Let \((X, B)\) be a klt pair and \(D\) be a Cartier divisor such that \(M := D - (K_X + B)\) is nef and big, then \(H^i(X, \mathcal{O}_X(D)) = 0\) for \(i > 0\).

Recall: If \(X\) is smooth and \(B = \sum b_i D_i\) has simple normal crossings support where \(0 \leq b_i < 1\) then \((X, B)\) is klt.

\(M\) is nef if \(M \cdot C \geq 0\) for any curve \(C \subset X\) and nef and big if moreover \(M^{\dim X} > 0\).

If \(H^1(X, \mathcal{O}_X(D)) = 0\), then the restriction \(H^0(X, \mathcal{O}_X(D + S)) \rightarrow H^0(S, \mathcal{O}_S((D + S)|_S))\) is surjective and we may apply induction on \(\dim X\).
Let $k$ be algebraically closed field of characteristic $p > 0$.

By examples of Raynaud and others, it is known that Kodaira (and hence also Kawamata-Viehweg) vanishing fails in positive characteristic and $\dim X \geq 2$.

We therefore attempt to replace this vanishing by the systematic use of the Frobenius morphism and Serre vanishing.

Consider $F : X \to X$ the Frobenius morphism (a finite morphism).

By duality we identify $\mathcal{H}om(F_*\omega_X, \omega_X) \cong F_*\mathcal{H}om(\omega_X, \omega_X)$, and we let $Tr : F_*\omega_X \to \omega_X$ be the element corresponding to $\text{id}_{\omega_X}$.

We obtain homomorphisms

$$\ldots F_{e+1}^*\omega_X \to F_e^*\omega_X \to \ldots F_*\omega_X \to \omega_X.$$
If $X$ is smooth and $\mathcal{L}$ is a line bundle, then we also obtain homomorphisms $F^e_*(\omega_X \otimes \mathcal{L}^p) \cong (F^e_*\omega_X) \otimes \mathcal{L} \rightarrow \omega_X \otimes \mathcal{L}$.

We let $S^0(X, \omega_X \otimes \mathcal{L})$ be the image of $H^0(X, \omega_X \otimes \mathcal{L}^p) \rightarrow H^0(X, \omega_X \otimes \mathcal{L})$ for $e \gg 0$ sufficiently divisible.

If $\mathcal{L}$ is ample, then $H^1(X, \omega_X \otimes \mathcal{L}^p) = 0$ for $e \gg 0$ so that $H^0(X, \omega_X(S) \otimes \mathcal{L}^p) \rightarrow H^0(S, \omega_S \otimes \mathcal{L}|_S^p)$ is surjective.

Thus $S^0(X, \sigma(X, S) \otimes \mathcal{L}(S)^p) \rightarrow S^0(S, \omega_S \otimes \mathcal{L})$ is surjective.

Here $S^0(X, \sigma(X, S) \otimes \mathcal{L}(S)^p)$ is the image of $H^0(\omega_X(S) \otimes \mathcal{L}^p) \subset H^0(\omega_X \otimes \mathcal{L}(S)^p) \rightarrow H^0(\omega_X \otimes \mathcal{L}(S))$.

The above result generalizes to log pairs by work of Schwede:
Generalization to log pairs

- If \((X, \Delta)\) is a pair such that \((p^e - 1)(K_X + \Delta)\) is Cartier for some \(e > 0\) (i.e. \(p\) does not divide the index of \(K_X + \Delta\)), then we get \(\Phi^e_\Delta : F^e_\ast \mathcal{O}_X((1 - p^e)(K_X + \Delta)) \to \mathcal{O}_X\) (by adding \((p^e - 1)\Delta\) and using \(Tr\) on the smooth locus).

- \(\sigma(X, \Delta)\) denotes the image of \(\Phi^e_\Delta\) for \(e > 0\) sufficiently divisible and \(S^0(X, \sigma(X, \Delta) \otimes \mathcal{L})\) the image on global sections after tensoring by \(\mathcal{L}\).

- \(\sigma(X, \Delta)\) is an analog of the non LC ideal in characteristic 0.

- At snc points \(\sigma(X, \Delta) = \mathcal{O}_X\) iff \(\Delta \leq 1\).

- \((X, \Delta)\) is **F-pure** if \(\sigma(X, \Delta) = \mathcal{O}_X\) and **F-regular** if \(\sigma(X, \Delta + \epsilon H) = \mathcal{O}_X\) for any \(H; 0 < \epsilon \ll 1\) (better: \(\not\exists I \subset \mathcal{O}_X\) non-trivial s.t. \(\Phi^e_\Delta(I \cdot \mathcal{O}_X((1 - p^e)(K_X + \Delta))) \subset I\)).

- Analogs of LC and KLT.
Extension Theorem

Theorem (Schwede)

Let \((X, \Delta)\) be an F-pure pair such that \(p\) does not divide the index of \(K_X + \Delta\). Assume that there is a normal non-F-regular center \(Z\) so that

\[
\Phi^e_\Delta(I_Z \cdot O_X((1 - p^e)(K_X + \Delta))) \subset I_Z.
\]

If \(\mathcal{M}\) is Cartier and \(\mathcal{M} - (K_X + \Delta)\) is ample, then

\[
S^0(X, \sigma(X, \Delta) \otimes \mathcal{M}) \rightarrow S^0(Z, \sigma(Z, \Delta_Z) \otimes \mathcal{M}|_Z)
\]

is surjective (where \(\Delta_Z\) is defined by "adjunction").

Question: Can we arrange \(S^0(Z, \sigma(Z, \Delta_Z) \otimes \mathcal{M}|_Z) \neq 0\)?
Global F-regularity

- If $\mathcal{M}$ is sufficiently ample, then
  \[ S^0(X, \sigma(X, \Delta) \otimes \mathcal{M}) \cong H^0(X, \sigma(X, \Delta) \otimes \mathcal{M}) \]
  is globally generated (in general the inclusion can be strict!).

- If $\mathcal{M}$ is big and semiample, let $f : X \to T = \text{Proj}(\oplus_{m \geq 0} \mathcal{M}^m)$
  Then (replacing $\mathcal{M}$ by a multiple)
  \[ H^0(X, \mathcal{O}_X((p^e - 1)(K_X + \Delta)) \otimes \mathcal{M}^{p^e}) \to H^0(X, \mathcal{M}) \]
  is surjective provided that $(X, \Delta)$ is $F$-regular over $T$ i.e.
  $F_* f_* \mathcal{O}_X((p^e - 1)(K_X + \Delta)) \to \mathcal{O}_T$ is surjective.

- By a result of Schwede-Smith $X/T$ is of relative log Fano type.
If \( \dim X = 2, \ p > 5, \ (X, \Delta) \) is klt, \( \Delta \in \{1 - \frac{1}{n} \mid n \in \mathbb{N}\} \), \( f : X \to T \) is birational and \( -(K_X + \Delta) \) is \( f \)-ample then \( (X, \Delta) \) is globally \( F \)-regular over \( T \).

- Hara proved the case when \( \Delta = 0 \) and \( f = id \).
- \( p > 5 \) is a necessary condition.
- The proof is by ”classification”; uses Shokurov’s theory of complements.
- We will now give a sketch of the proof of the existence of 3-fold flips which closely follows ideas of Shokurov.
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Pl-flips and the restricted algebra

By Shokurov’s reduction to pl-flips it is enough to prove the existence of pl-flips.

Thus we may assume that $f : X \rightarrow Z = \text{Spec}(A)$ is a small birational morphism of normal varieties, $(X, \Delta = S + B)$ is plt where $S = \lfloor \Delta \rfloor$, $\rho(X/Z) = 1$ and $-(K_X + \Delta)$ and $-S$ are $f$ ample.

Let $R_S(K_X + \Delta) = \text{Im}(R(K_X + \Delta) \rightarrow R(K_{S^n} + B_{S^n}))$ where $S^n \rightarrow S$ is the normalization and $K_{S^n} + B_{S^n} = (K_X + \Delta)|_{S^n}$.

Then $R(K_X + \Delta)$ is fin. gen. iff $R(K_{S^n} + B_{S^n})$ is fin. gen.

We have $K_X + \Delta = tS$; assume $t = 1$, then the kernel of $H^0(m(K_X + \Delta)) \rightarrow H^0(m(K_{S^n} + B_{S^n}))$ is in $H^0((m - 1)(K_X + \Delta))$.

If $R_S(K_X + \Delta) = R(K_{S^n} + B_{S^n})$ we would be done! So we must identify which elements of $R(K_{S^n} + B_{S^n})$ lift.
Normality of $S$

- In char 0, if $(X, S + B)$ is plt (near $S$) then $S$ is normal and $(S, B_S)$ is klt.
- In char $p > 0$, this is not known (pf. depends on KV-vanish.).
- We show that if $\dim X = 3$, $B \in \{1 - \frac{1}{n} | n \in \mathbb{N}\}$ and $p > 5$ this holds (more generally assuming that $(S^n, B_{S^n})$ is strongly $F$-regular then $(X, S + B)$ is $F$-pure near $S$ (there are related results of Schwede and others)).
- Let $g : Y \to X$ be a log resolution, $g_*^{-1}S = S' \to S^n \to S$ the induced map.
- $K_Y + S' = g^*(K_X + S + B) + A_Y$, $K_{S'} = g'^*(K_{S^n} + B_{S^n}) + A_{S'}$.
- Pick $F \geq 0$ exceptional, $g$-anti-ample.
- Pick $H$ sufficiently ample on $X$ ($H|_{S^n}$ very ample).
Normality of $S$

- Let $L = g^* H + [A_Y] + [A_Y] + \epsilon F$ (0 < $\epsilon$ ≪ 1).
- $L - (K_Y + \Xi) \cong g^* (H - (K_X + \Delta)) - \epsilon F$ is ample.
- $S^0(\sigma(Y, \Xi) \otimes O_Y(L)) \to S^0(\sigma(S', \Xi) \otimes O_{S'}(L))$ is onto.
- We will show that $S^0(\sigma(S', \Xi) \otimes O_{S'}(L)) = H^0(O_{S'}(L))$, thus $H^0(O_X(H)) \cong H^0(O_Y(L)) \to H^0(O_{S'}(L)) \supset H^0(O_{S^n}(H))$.
- As $H|_{S^n}$ is very ample, we are done.
- Pick $E$ s.t. $S^n \supset E \supset g(F|_{S'})$ and 0 < $\epsilon$ ≪ $\delta$ ≪ 1.
- $g^*(K_{S^n} + B_{S^n} + \delta E) \geq K_{S'} + \{-A_{S'}\} - [A_{S'}] + \epsilon F|_{S'}$, so $F^e_* O_{S^n}((1 - p^e)(K_{S^n} + B_{S^n} + \delta E + p^e H|_{S^n}) \hookrightarrow g^* F^e_* O_{S'}((1 - p^e)(K_{S'} + \Xi_{S'}) + p^e L)$.
- As $K_{S^n} + B_{S^n}$ is $F$-regular, we have a surjection $F^e_* O_{S^n}((1 - p^e)(K_{S^n} + B_{S^n} + \delta E + p^e H|_{S^n})) \to H^0(O_{S^n}(H)).
The restricted algebra

- For any $f : Y \to X$, let $N_{i,Y} = \text{Mob}(i(K_X + \Delta))$ and $M_{i,S'} = N_{i,Y}|_{S'}$, $D_{i,S'} = \frac{1}{i} M_{i,S'}$ and $D_{S'} = \lim D_{i,S'}$.

- For $Y$ sufficiently high, $N_{i,Y}$ and $M_{i,S'}$ are free and $[A_Y]$ (resp. $[A_{S'}]$) saturated i.e. $|N_{i,Y} + [A_Y]| = |N_{i,Y}| + [A_Y]$. (This is because $[A_Y] \geq 0$ is exceptional and so $|f^*(i(K_X + \Delta)) + [A_Y]| = |f^*(i(K_X + \Delta))| + [A_Y]$.)

- $R_S(K_X + \Delta)$ is finitely generated iff
  - there exists $\bar{S} \to S$ such that $M_{i,S'}$ descends to $\bar{S}$ for all $i \gg 0$ (given $h : S' \to \bar{S}$, then $M_{i,S'} = h^* M_{i,\bar{S}}$ and
  - an index $\bar{i}$ s.t. $D_{i,\bar{S}} = D_{\bar{i},\bar{S}}$ whenever $\bar{i}|i$.

- $\mu : \bar{S} \to S$ is just the terminalization of $(S, B_S)$ i.e. the "smallest" resolution such that $\text{mult}_x(B_\bar{S}) < 1$ for any $x \in \bar{S}$ and $K_\bar{S} + B_\bar{S} = \mu^*(K_S + B_S)$ (nb. $B_\bar{S} = -A_\bar{S} \geq 0$).
So we must show that if $M = M_{i,S'}$ is free and $[A_{S'}]$ saturated, then it descends to $\bar{S}$.

Since $[A_{S'}]$ is the exceptional set of $S' \to \bar{S}$, we need $M \cap [A_{S'}] = \emptyset$.

Suppose that $\exists \bar{N} \in |N|$ s.t. $C = \bar{N} \cap S' \in |M|$ is smooth then let $\Xi = S' + \{-A_Y\} + \epsilon F$ and arguing as above, one sees that $S^0(\sigma(Y, \Xi + \bar{N}) \otimes \mathcal{O}_Y(N + [A_Y])) \to S^0(\sigma(C, B_C) \otimes \mathcal{O}_C(N + [A_Y]))$ is surjective.

Since $C$ is an affine curve, the RHS is generated, but the LHS vanishes along $[A_Y]$ and so $C \cap [A_{S'}] = 0$, i.e. $M \cap [A_{S'}] = \emptyset$ as required.
Descending to $\overline{S}$ (sing. $C$).

- If $C$ is singular, technical issues arise.
- In char 0, this creates a 0-dimensional non-klt center at the singular points of $C$ and we expect to be able to lift sections from these centers and conclude as above.
- In char $p > 0$ we use $F$-seshadri constants (developed by Mustata and Schwede).
- Since $C$ moves (in $|M \otimes m^2_x|$), one sees that $\epsilon(x, M_i, S') \geq 2$ (so $\epsilon_F(x, M_i, S') \geq 1$) and we are able to lift sections from these points.
- Thus all $M_i, S'$ descend to $\overline{S}$. Note that the limit $D_{\overline{S}}$ is nef (over $\mathbb{Z}$) and $(\overline{S}, B_{\overline{S}})$ is a weak log Fano, thus $D_{\overline{S}}$ is a semiample $\mathbb{R}$-divisor.
- Let $a : \overline{S} \rightarrow S^+$ be the induced morphism. We claim that $D_{\overline{S}} = a^* D_{S^+}$ where $D_{S^+} \in \text{Div}_\mathbb{Q}(S^+)$.
$D_{\bar{S}}$ descends to $S^+$. 

To see this, we pick $C \in \text{Div}(S^+)$ so that $||C - jD_{S^+}|| \ll 1$ (by Diophantine approximation).

Proceeding as above one checks that 

$\left\lceil \frac{j}{i} M_{i,S'} + A_{S'} \right\rceil - M_{j,S'} \cdot g^* C = 0$ and hence 

$\left\lceil \frac{j}{i} M_{i,S'} + A_{S'} \right\rceil - M_{j,S'}$ is exceptional over $S^+$. 

Finally, the usual saturation arguments (with Kawamata Viehweg vanishing replaced by the $S^0$ extension results) show that $\frac{1}{i} M_{i,\bar{S}} = D_{\bar{S}}$ for all $i > 0$ sufficiently divisible.