Finite generation of canonical rings III

Christopher Hacon

University of Utah

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Outline of the talk

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1. Introduction
2. The MMP for 3-folds

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- K_X denotes (a choice of) the **canonical divisor** i.e. a formal linear combination of codimension 1 subvarieties of X such that $\omega_X = \mathcal{O}_X(K_X)$.
- We would like to use ω_X (or equivalently K_X) to understand the geometry of X.

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- If X and X' are birational, then $\forall m \geq 0$

$$H^0(X, \omega_X^{\otimes m}) \cong H^0(X', \omega_{X'}^{\otimes m}).$$

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- One would like to understand the structure of this ring and to use its features to classify complex projective varieties.

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Today I would like to discuss the following result:

Theorem (Birkar-Cascini-Hacon-M^{c}Kernan and Siu)

Let X be a smooth projective variety, then $R(X)$ is finitely generated.

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The two proofs are independent. Our proof is algebraic and uses the ideas of the MMP (minimal model program). Siu's proof is analytic and requires X to be of general type.

Birational geometry of surfaces

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- The morphism ν may be viewed as a surgery that replaces the point x by a rational curve $E \cong \mathbb{P}^1 = \mathbb{P}(T_x X)$.
- The exceptional curve E is called a -1 **curve** since

$$\omega_{\tilde{X}} \cdot E = \deg(\omega_{\tilde{X}}|_E) = -1.$$

Birational geometry of surfaces II

- By a Theorem of Castelnuovo, one may reverse this procedure i.e. given any surface X and any -1 curve $E \subset X$, there exists a morphism $\nu : X \to X_1$ such that $X = \mathrm{Bl}_x(X_1)$ for some point $x \in X_1$.

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- A **minimal surface** is a surface that contains no -1 curves.
- By Castelnuovo's Theorem, given any surface X there exists a birational morphism to a minimal surface $X \to X_{min}$ (given by finitely many point blow downs).

Birational geometry of surfaces III

- If $\kappa(X) = -1$, then X is covered by rational curves (i.e. by $\mathbb{P}^1$'s). The sections of $H^0(\omega_X^{\otimes m})$ must vanish along these curves for $m > 0$. Hence $R(X) \cong \mathbb{C}$.

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- If $\kappa(X) \geq 0$, then the minimal model $X \to X_{min}$ is unique and $K_{X_{min}}$ is nef. This means that for any curve $C \subset X_{min}$ we have

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- In fact one can show that $\omega_{X_{min}}$ is **semiample** i.e. there is an integer $m > 0$ such that the sections of $H^0(\omega_{X_{min}}^{\otimes m})$ have no common zeroes.

Birational geometry of surfaces IV

- Therefore, $\phi_{\omega_{X_{min}}^{\otimes m}} : X_{min} \to \mathbb{P}^N$ is a morphism, $\omega_{X_{min}}^{\otimes m} \cong \phi^* \mathcal{O}_{\mathbb{P}^N}(1)$.

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MMP for 3-folds

- By work of Mori, Kawamata, Kollár, Reid, Shokurov and others it is possible to generalize this picture to 3-folds:

Theorem

Assume $\dim(X) = 3$*. If* $\kappa(X) < 0$*, then* X *is covered by rational curves and* $R(X) \cong \mathbb{C}$*.*
If $\kappa(X) \geq 0$*, then there is a birational map* $X \dashrightarrow X_{min}$ *such that* $\omega_{X_{min}}$ *is nef (i.e.* $\omega_{X_{min}} \cdot C \geq 0$ *for any curve* $C \subset X_{min}$*).*

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- More precisely, if $\kappa(X) \geq 0$, there is a sequence of flips and divisorial contractions
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- It is known that $\omega_{X_{min}}$ is semiample (i.e. $\phi_{\omega_{X_{min}}^{\otimes m}} : X \to \mathbb{P}^N_{\mathbb{C}}$ is a morphism) and so $R(X)$ is finitely generated.

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- The minimal model X_{min} is not unique.
- The varieties X_i have mild singularities.

Running a MMP

- A **Minimal Model program** is a sequence of geometric operations $X_i \dashrightarrow X_{i+1}$ that starting from a projective variety X produces a minimal model (if $\kappa(X) \geq 0$) or a covering family of rational curves (if $\kappa(X) < 0$).

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- We start with a projective variety X with mild singularities.
- If ω_X is nef we are done ($X = X_{min}$).
- If ω_X is not nef, then consider the **cone of effective curves**

$$\overline{NE}(X) = \{\sum r_i C_i | r_i \in \mathbb{R}^{\geq 0}\} / \equiv$$

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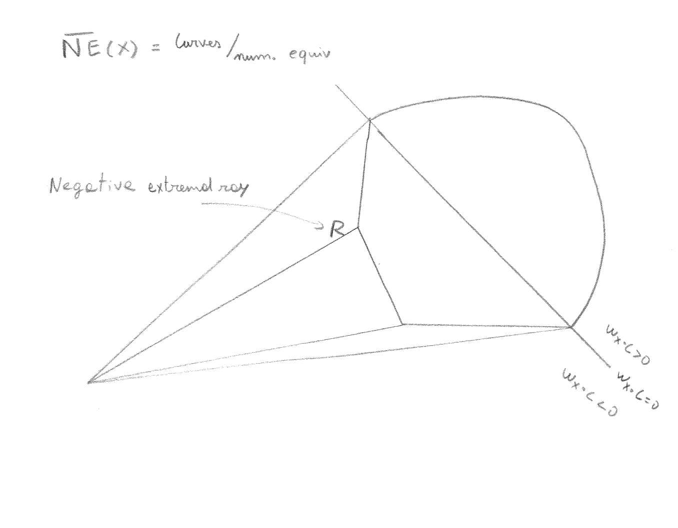
(here $C \equiv C'$ or $[C] = [C']$ if for any line bundle L on X, we have $L \cdot (C - C') = 0$).
- By the Cone Theorem, there is a negative extremal ray $R = \mathbb{R}^+[C]$ where $C \subset X$ is a curve such that $\omega_X \cdot C < 0$ and there is a morphism $\mathrm{Cont}_R : X \to Z$ such that a curve $D \subset X$ is contracted if and only if $[D] \in R^+[C]$.

The Cone Theorem



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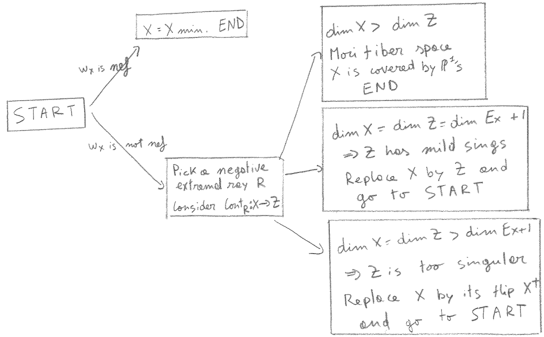
- Instead of replacing X by Z, we will replace X by its flip X^+.

MMP flow chart



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  1) $X \dashrightarrow X^+$ is an isomorphism on the complement of $\mathrm{Exc}(X \to Z)$,
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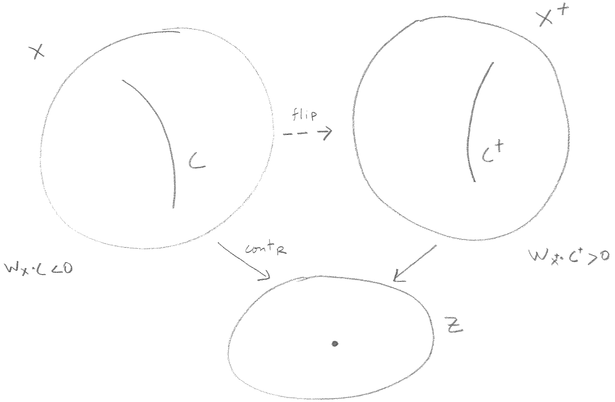
- In other words, the flip $X \dashrightarrow X^+$ is a surgery that replaces ω_X negative curves by ω_{X^+} positive curves.
- If the flip of $X \to Z$ exists, it is unique and given by $\mathrm{Proj}_Z R(X)$. So that to construct the flip we must show that $R(X)$ is finitely generated over Z (i.e. if $Z = \mathrm{Spec} A$, $R(X)$ is a finitely generated A-algebra).

Picture of a flip



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- If $X \to Z$ is a divisorial contraction, then $\rho(Z) = \rho(X) - 1 \geq 0$. If $X \dashrightarrow X^+$ is a flip, then $\rho(X^+) = \rho(X)$. Therefore, it suffices to show that there is no infinite sequence of flips.

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- In dimension 3, this is relatively easy to do.

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- The proof proceeds by induction on the dimension of X. We relate ω_X (equivalently K_X) to its restriction to a divisor S (i.e. a codimension 1 subvariety which typically belongs to $|mK_X|$).
- This is achieved by the adjunction formula which has the form $(K_X + S)|_S = K_S + \Delta_S$ for some $\mathbb{Q}$-divisor Δ_S.

A Theorem of Fujino-Mori

Mori and Fujino have shown that:

Theorem

Let (X, Δ) be any KLT $\mathbb{Q}$-factorial pair with $\Delta \in \mathrm{Div}_{\mathbb{Q}}(X)$. Then there exists a KLT $\mathbb{Q}$-factorial pair (Y, Γ) of general type with $\Gamma \in \mathrm{Div}_{\mathbb{Q}}(Y)$ such that

$$R(K_X + \Delta)^{(m)} \cong R(K_Y + \Gamma)^{(m)}$$

for any sufficiently divisible integer $m > 0$.

Recall that $K_Y + \Gamma$ is of general type if $\mathrm{tr.deg.}_{\mathbb{C}} R(K_Y + \Gamma)^{(m)} - 1 = \dim Y$.

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- As we have mentioned before, this implies that $R(K_X + \Delta)$ is finitely generated.
- Therefore, to show that $R(X)$ is finitely generated (for any smooth complex projective variety), it suffices to show that minimal models exist for KLT pairs of general type.

Existence of minimal models for KLT pairs of general type.

Theorem (Birkar-Cascini-Hacon-M^{c}Kernan)

Let (X, Δ) be any KLT pair of general type. Then there is a minimal model $f : X \dashrightarrow X_{min}$ (i.e. $K_{X_{min}} + f_*\Delta$ is semiample).

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Remark: We are unable to show termination of flips. But we can pick a sequence of flips that terminates.

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- (Birkar-Cascini-Hacon-McKernan) If flips exist in dimension $n+1$ (and minimal models exist in dimension n) then minimal models exist in dimension $n+1$. (Uses the ideas of the MMP and in particular techniques developed by Shokurov.)

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- By adjunction, we may write $(K_X + \Delta)|_S = K_S + \Delta_S$ where (S, Δ_S) is KLT.
- Shokurov's observation is that to show that $R(K_X + \Delta)$ is finitely generated over Z (and hence that the flip exists), it suffices to show that the restricted algebra

$$R_S(K_X + \Delta) := \mathrm{Im}(R(K_X + \Delta) \to R(K_S + \Delta_S))$$

is finitely generated (over Z).

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- Using a Diophantine approximation argument, we show that Θ is in fact a $\mathbb{Q}$-divisor and $\Theta=\Theta_m$ for some $m>0$.

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- In this way, we obtain a sequence of rational numbers $1 \geq t_1 \geq t_2 \cdots \geq 0$ and of flips/div. contractions $X_i \dashrightarrow X_{i+1}$ such that X_i is a minimal model for $(X, \Delta + t_i A)$.

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- Note that to prove the above result on the finiteness of minimal models, we must use a compactness argument and hence it is critical to work with $\mathbb{R}$-divisors.