Finite generation of canonical rings II

Christopher Hacon

University of Utah

March, 2008

Outline of the talk

Outline of the talk

Outline of the talk

1. Introduction
2. Surfaces
3. Higher dimensional preview

Outline of the talk

Notation

- I am interested in the birational geometry of complex projective varieties.

Notation

- I am interested in the birational geometry of complex projective varieties.
- $X \subset \mathbb{P}^N_{\mathbb{C}}$ is defined by homogeneous polynomials $P_1, \cdots, P_t \in \mathbb{C}[x_0, \cdots, x_N]$.

Notation

- I am interested in the birational geometry of complex projective varieties.
- $X \subset \mathbb{P}^N_{\mathbb{C}}$ is defined by homogeneous polynomials $P_1, \cdots, P_t \in \mathbb{C}[x_0, \cdots, x_N]$.
- We will assume that X is irreducible and smooth.

Notation

- I am interested in the birational geometry of complex projective varieties.
- $X \subset \mathbb{P}^N_{\mathbb{C}}$ is defined by homogeneous polynomials $P_1, \cdots, P_t \in \mathbb{C}[x_0, \cdots, x_N]$.
- We will assume that X is irreducible and smooth.
- In the previous talk I discussed the geometry of **curves** i.e. $\dim X = 1$.

Notation

- I am interested in the birational geometry of complex projective varieties.
- $X \subset \mathbb{P}^N_{\mathbb{C}}$ is defined by homogeneous polynomials $P_1, \cdots, P_t \in \mathbb{C}[x_0, \cdots, x_N]$.
- We will assume that X is irreducible and smooth.
- In the previous talk I discussed the geometry of **curves** i.e. $\dim X = 1$.
- In this case X is a smooth orientable compact manifold of $\dim_{\mathbb{R}} X = 2$ and so X is a Riemann Surface.

Riemann Surfaces

Curves

- Curves are topologically determined by their genus.

Curves

- Curves are topologically determined by their genus.
- $g = 0$: X is a **rational curve** i.e. $X = \mathbb{P}^1$.

Curves

- Curves are topologically determined by their genus.
- $g = 0$: X is a **rational curve** i.e. $X = \mathbb{P}^1$.
- $g = 1$: There is a 1-parameter family of **elliptic curves**.

Curves

- Curves are topologically determined by their genus.
- $g = 0$: X is a **rational curve** i.e. $X = \mathbb{P}^1$.
- $g = 1$: There is a 1-parameter family of **elliptic curves**.
- $g \geq 2$: **Curves of general type**. These form a $3g - 3$-dimensional family.

Curves

- Curves are topologically determined by their genus.
- $g = 0$: X is a **rational curve** i.e. $X = \mathbb{P}^1$.
- $g = 1$: There is a 1-parameter family of **elliptic curves**.
- $g \geq 2$: **Curves of general type**. These form a $3g - 3$-dimensional family.
- There are infinitely many ways to embed $X \subset \mathbb{P}^N_{\mathbb{C}}$.

Curves II

- Embeddings $X \subset \mathbb{P}^N_{\mathbb{C}}$ are obtained as follows.

Curves II

- Embeddings $X \subset \mathbb{P}^N_{\mathbb{C}}$ are obtained as follows.
- If L is a line bundle and $s_0, \ldots, s_N$ are a basis of $H^0(X, L)$, then $x \to [s_0(x) : \ldots : s_N(x)]$ defines a map $\phi_L : X \dashrightarrow \mathbb{P}^N_{\mathbb{C}}$ on the complement of the common zeroes of the s_i.

Curves II

- Embeddings $X \subset \mathbb{P}^N_{\mathbb{C}}$ are obtained as follows.
- If L is a line bundle and $s_0, \ldots, s_N$ are a basis of $H^0(X, L)$, then $x \to [s_0(x) : \ldots : s_N(x)]$ defines a map $\phi_L : X \dashrightarrow \mathbb{P}^N_{\mathbb{C}}$ on the complement of the common zeroes of the s_i.
- If $s_0, \ldots, s_N$ separate points and tangent directions on X, then ϕ_L is an embedding.

Curves II

- Embeddings $X \subset \mathbb{P}^N_{\mathbb{C}}$ are obtained as follows.
- If L is a line bundle and $s_0, \ldots, s_N$ are a basis of $H^0(X, L)$, then $x \to [s_0(x) : \ldots : s_N(x)]$ defines a map $\phi_L : X \dashrightarrow \mathbb{P}^N_{\mathbb{C}}$ on the complement of the common zeroes of the s_i.
- If $s_0, \ldots, s_N$ separate points and tangent directions on X, then ϕ_L is an embedding.
- One natural choice for L is $\omega_X^{\otimes k}$, where ω_X is the line bundle whose sections $s \in H^0(X, \omega_X)$ may be locally written as $s|_U = f(x)dx$. (One may think of these as rational/meromorhic functions on X with prescribed poles.)

Curves II

- Embeddings $X \subset \mathbb{P}^N_{\mathbb{C}}$ are obtained as follows.
- If L is a line bundle and $s_0, \dots, s_N$ are a basis of $H^0(X, L)$, then $x \to [s_0(x) : \dots : s_N(x)]$ defines a map $\phi_L : X \dashrightarrow \mathbb{P}^N_{\mathbb{C}}$ on the complement of the common zeroes of the s_i.
- If $s_0, \dots, s_N$ separate points and tangent directions on X, then ϕ_L is an embedding.
- One natural choice for L is $\omega_X^{\otimes k}$, where ω_X is the line bundle whose sections $s \in H^0(X, \omega_X)$ may be locally written as $s|_U = f(x)dx$. (One may think of these as rational/meromorhic functions on X with prescribed poles.)
- Note that the genus of X is then given by $g = \dim H^0(X, \omega_X)$.

Curves III

- The **canonical ring** $R(X) = \bigoplus_{k \geq 0} H^0(X, \omega_X^{\otimes k})$ is a fundamental invariant of X.

Curves III

- The **canonical ring** $R(X) = \bigoplus_{k \geq 0} H^0(X, \omega_X^{\otimes k})$ is a fundamental invariant of X.
- If $g = 0$, then $\omega_X = \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(-2)$ so that $H^0(X, \omega_X^{\otimes k}) = 0$ for all $k > 0$ i.e. $R(X) \cong \mathbb{C}$.

Curves III

- The **canonical ring** $R(X) = \bigoplus_{k\geq 0} H^0(X, \omega_X^{\otimes k})$ is a fundamental invariant of X.
- If $g = 0$, then $\omega_X = \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(-2)$ so that $H^0(X, \omega_X^{\otimes k}) = 0$ for all $k > 0$ i.e. $R(X) \cong \mathbb{C}$.
- If $g = 1$, then $\omega_X = \mathcal{O}_X$ so that $H^0(X, \omega_X^{\otimes k}) = \mathbb{C}$ for all $k > 0$ i.e. $R(X) \cong \mathbb{C}[t]$.

Curves III

- The **canonical ring** $R(X) = \bigoplus_{k\geq 0} H^0(X, \omega_X^{\otimes k})$ is a fundamental invariant of X.
- If $g = 0$, then $\omega_X = \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(-2)$ so that $H^0(X, \omega_X^{\otimes k}) = 0$ for all $k > 0$ i.e. $R(X) \cong \mathbb{C}$.
- If $g = 1$, then $\omega_X = \mathcal{O}_X$ so that $H^0(X, \omega_X^{\otimes k}) = \mathbb{C}$ for all $k > 0$ i.e. $R(X) \cong \mathbb{C}[t]$.
- If $g \geq 2$ then $\omega_X^{\otimes 3}$ defines an embedding $\phi : X \to \mathbb{P}^N_{\mathbb{C}}$.

Curves III

- The **canonical ring** $R(X) = \bigoplus_{k\geq 0} H^0(X, \omega_X^{\otimes k})$ is a fundamental invariant of X.
- If $g = 0$, then $\omega_X = \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(-2)$ so that $H^0(X, \omega_X^{\otimes k}) = 0$ for all $k > 0$ i.e. $R(X) \cong \mathbb{C}$.
- If $g = 1$, then $\omega_X = \mathcal{O}_X$ so that $H^0(X, \omega_X^{\otimes k}) = \mathbb{C}$ for all $k > 0$ i.e. $R(X) \cong \mathbb{C}[t]$.
- If $g \geq 2$ then $\omega_X^{\otimes 3}$ defines an embedding $\phi : X \to \mathbb{P}^N_{\mathbb{C}}$.
- It follows that $\omega_X^{\otimes 3} = \phi^* \mathcal{O}_{P^N_{\mathbb{C}}}(1)$. It is then easy to see that $R(X)$ is finitely generated and $X = \mathrm{Proj} R(X)$.

Outline of the talk

Surfaces

- One would naturally like to generalize these results to higher dimensions.

Surfaces

- One would naturally like to generalize these results to higher dimensions.
- The next case of interest are **surfaces** i.e. $\dim X = 2$. (Note that $\dim_{\mathbb{R}} X = 4$!)

Surfaces

- One would naturally like to generalize these results to higher dimensions.
- The next case of interest are **surfaces** i.e. $\dim X = 2$. (Note that $\dim_{\mathbb{R}} X = 4$!)
- The first difficulty is that given any point on a surface $x \in X$, one can produce a new surface

$$\nu : \tilde{X} = \mathrm{Bl}_x(X) \to X$$

known as the blow up of X at x.

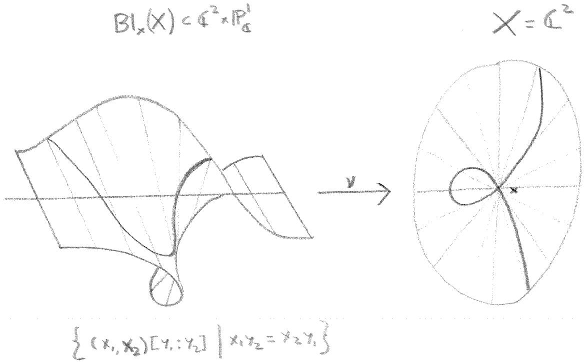
Surfaces

- One would naturally like to generalize these results to higher dimensions.
- The next case of interest are **surfaces** i.e. $\dim X = 2$. (Note that $\dim_{\mathbb{R}} X = 4!$)
- The first difficulty is that given any point on a surface $x \in X$, one can produce a new surface

$$\nu : \tilde{X} = \mathrm{Bl}_x(X) \to X$$

known as the blow up of X at x.
- The morphism ν may be viewed as a surgery that replaces the point x by a rational curve $E \cong \mathbb{P}^1 = \mathbb{P}(T_x X)$.

Blowing up a point



Birational Surfaces

- The exceptional curve E is called a -1 curve since

$$\omega_{\tilde{X}} \cdot E = \deg(\omega_{\tilde{X}}|_E) = -1.$$

Birational Surfaces

- The exceptional curve E is called a -1 curve since
$$\omega_{\tilde{X}} \cdot E = \deg(\omega_{\tilde{X}}|_E) = -1.$$
- One says that X, X' are **birational** if they have isomorphic open subsets $U \cong U'$.

Birational Surfaces

- The exceptional curve E is called a -1 curve since

$$\omega_{\tilde{X}} \cdot E = \deg(\omega_{\tilde{X}}|_E) = -1.$$

- One says that X, X' are **birational** if they have isomorphic open subsets $U \cong U'$.
- For example X is birational to $\mathrm{Bl}_x(X)$.

Birational Surfaces

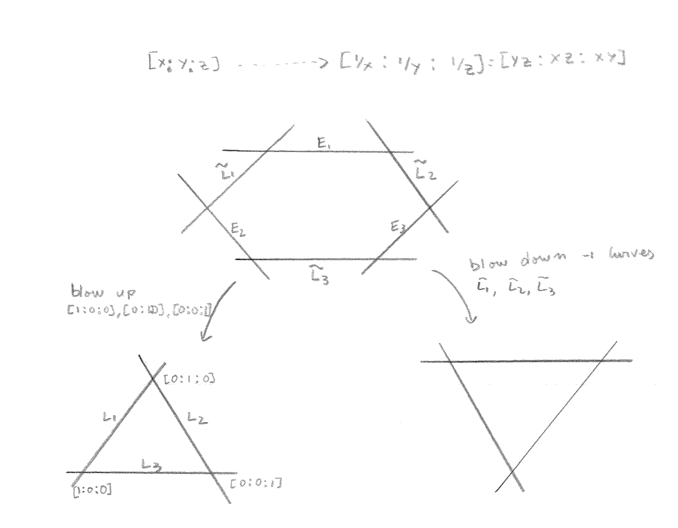
- The exceptional curve E is called a -1 curve since

$$\omega_{\tilde{X}} \cdot E = \deg(\omega_{\tilde{X}}|_E) = -1.$$

- One says that X, X' are **birational** if they have isomorphic open subsets $U \cong U'$.
- For example X is birational to $\mathrm{Bl}_x(X)$.
- It turns out that two surfaces are birational if and only if there is a finite sequence of blow ups such that

$$\mathrm{Bl}_{x_t}\mathrm{Bl}_{x_{t-1}}\cdots\mathrm{Bl}_{x_1}(X) = \mathrm{Bl}_{x'_s}\mathrm{Bl}_{x'_{s-1}}\cdots\mathrm{Bl}_{x'_1}(X').$$

Example of a birational map



Birational Surfaces II

- Since the space $\mathbb{C}(X)$ of **rational functions** on X are determined by any open subset $U \subset X$, two surfaces X, X' are birational if and only if $\mathbb{C}(X) \cong \mathbb{C}(X')$.

Birational Surfaces II

- Since the space $\mathbb{C}(X)$ of **rational functions** on X are determined by any open subset $U \subset X$, two surfaces X, X' are birational if and only if $\mathbb{C}(X) \cong \mathbb{C}(X')$.
- For example a surface X is **rational** (i.e. birational to $\mathbb{P}^2_{\mathbb{C}}$ if and only if $\mathbb{C}(X) \cong \mathbb{C}(x, y)$.

Birational Surfaces II

- Since the space $\mathbb{C}(X)$ of **rational functions** on X are determined by any open subset $U \subset X$, two surfaces X, X' are birational if and only if $\mathbb{C}(X) \cong \mathbb{C}(X')$.
- For example a surface X is **rational** (i.e. birational to $\mathbb{P}^2_{\mathbb{C}}$ if and only if $\mathbb{C}(X) \cong \mathbb{C}(x, y)$.
- It is then natural to study surfaces modulo birational equivalence.

Birational Surfaces II

- Since the space $\mathbb{C}(X)$ of **rational functions** on X are determined by any open subset $U \subset X$, two surfaces X, X' are birational if and only if $\mathbb{C}(X) \cong \mathbb{C}(X')$.
- For example a surface X is **rational** (i.e. birational to $\mathbb{P}^2_{\mathbb{C}}$ if and only if $\mathbb{C}(X) \cong \mathbb{C}(x, y)$.
- It is then natural to study surfaces modulo birational equivalence.
- Given X one would like to identify a unique representative X' birational to X. Ideally this representative X' has nice properties that are useful in understanding the geometry of X' (and hence of X).

Castelnuovo's Theorem

- The answer is given by a result of Castelnuovo. It is easiest to describe this in terms of the canonical bundle.

Castelnuovo's Theorem

- The answer is given by a result of Castelnuovo. It is easiest to describe this in terms of the canonical bundle.
- Let $\omega_X = \Lambda^2 T_X^\vee$ be the canonical line bundle.

Castelnuovo's Theorem

- The answer is given by a result of Castelnuovo. It is easiest to describe this in terms of the canonical bundle.
- Let $\omega_X = \Lambda^2 T_X^\vee$ be the canonical line bundle.
- If $\nu : \tilde{X} = \mathrm{Bl}_x(X) \to X$ is the blow up of a point, and $E = \nu^{-1}(x)$ is the exceptional curve, then E is a -1 curve i.e. $E \cong \mathbb{P}^1_{\mathbb{C}}$ and $E \cdot \omega_X = \deg \omega_X|_E = -1$.

Castelnuovo's Theorem

- The answer is given by a result of Castelnuovo. It is easiest to describe this in terms of the canonical bundle.
- Let $\omega_X = \Lambda^2 T_X^\vee$ be the canonical line bundle.
- If $\nu : \tilde{X} = \mathrm{Bl}_x(X) \to X$ is the blow up of a point, and $E = \nu^{-1}(x)$ is the exceptional curve, then E is a -1 curve i.e. $E \cong \mathbb{P}^1_{\mathbb{C}}$ and $E \cdot \omega_X = \deg \omega_X|_E = -1$.
- Castelnuovo's Theorem says that one may reverse this procedure i.e. given any surface X and any -1 curve $E \subset X$, there exists a morphism $\nu : X \to X_1$ such that $X = \mathrm{Bl}_x(X_1)$ for some point $x \in X_1$.

Minimal Surfaces

- A **minimal surface** is a surface that contains no -1 curves.

Minimal Surfaces

- A **minimal surface** is a surface that contains no -1 curves.
- Note that $\rho(\mathrm{Bl}_x(X)) = \rho(X) + 1$ so that one can only blow down finitely many curves.

Minimal Surfaces

- A **minimal surface** is a surface that contains no -1 curves.
- Note that $\rho(\mathrm{Bl}_x(X)) = \rho(X) + 1$ so that one can only blow down finitely many curves.
- So, given a surface X, after blowing down finitely many -1 curves, one obtains a morphism $X \to X_{min}$ where X_{min} is a minimal surface.

Minimal Surfaces

- A **minimal surface** is a surface that contains no -1 curves.
- Note that $\rho(\mathrm{Bl}_x(X)) = \rho(X) + 1$ so that one can only blow down finitely many curves.
- So, given a surface X, after blowing down finitely many -1 curves, one obtains a morphism $X \to X_{min}$ where X_{min} is a minimal surface.
- The natural question is then: Is X_{min} unique? Does X_{min} have any interesting special properties?

Canonical Ring

- The answer is given in terms of the canonical ring

$$R(X)=\bigoplus_{k\geq 0} H^0(X,\omega_X^{\otimes k}).$$

Canonical Ring

- The answer is given in terms of the canonical ring

$$R(X) = \bigoplus_{k \geq 0} H^0(X, \omega_X^{\otimes k}).$$

- The **Kodaira dimension** of X is

$$\kappa(X) := \mathrm{tr.deg.}_{\mathbb{C}} R(X) - 1.$$

Canonical Ring

- The answer is given in terms of the canonical ring

$$R(X) = \bigoplus_{k\geq 0} H^0(X, \omega_X^{\otimes k}).$$

- The **Kodaira dimension** of X is

$$\kappa(X) := \mathrm{tr.deg.}_{\mathbb{C}} R(X) - 1.$$

- We have $\kappa(X) := \max_{k>0}\{\dim \phi_{\omega_X^{\otimes k}}(X)\}$.

Canonical Ring

- The answer is given in terms of the canonical ring
$$R(X) = \bigoplus_{k\geq 0} H^0(X, \omega_X^{\otimes k}).$$
- The **Kodaira dimension** of X is
$$\kappa(X) := \mathrm{tr.deg.}_{\mathbb{C}} R(X) - 1.$$
- We have $\kappa(X) := \max_{k>0}\{\dim \phi_{\omega_X^{\otimes k}}(X)\}$.
- If $\kappa(X) = -1$, then the minimal model $X \to X_{min}$ is not unique (but it is well understood how two different minimal models are related).

Canonical Ring

- The answer is given in terms of the canonical ring

$$R(X) = \bigoplus_{k \geq 0} H^0(X, \omega_X^{\otimes k}).$$

- The **Kodaira dimension** of X is

$$\kappa(X) := \mathrm{tr.deg.}_{\mathbb{C}} R(X) - 1.$$

- We have $\kappa(X) := \max_{k>0}\{\dim \phi_{\omega_X^{\otimes k}}(X)\}$.
- If $\kappa(X) = -1$, then the minimal model $X \to X_{min}$ is not unique (but it is well understood how two different minimal models are related).
- If $\kappa(X) \geq 0$, then the minimal model $X \to X_{min}$ is unique.

Birational geometry of Surfaces

- However, if $\kappa(X) = -1$, then X_{min} is either $\mathbb{P}^2_{\mathbb{C}}$ or a ruled surface so that X is covered by rational curves (i.e. by $\mathbb{P}^1$'s). The sections of $H^0(\omega_X^{\otimes m})$ must vanish along these curves for $m > 0$. Hence $R(X) \cong \mathbb{C}$.

Birational geometry of Surfaces

- However, if $\kappa(X) = -1$, then X_{min} is either $\mathbb{P}^2_{\mathbb{C}}$ or a ruled surface so that X is covered by rational curves (i.e. by $\mathbb{P}^1$'s). The sections of $H^0(\omega_X^{\otimes m})$ must vanish along these curves for $m > 0$. Hence $R(X) \cong \mathbb{C}$.
- If $\kappa(X) \geq 0$, then the minimal model $X \to X_{min}$ is unique and $\omega_{X_{min}}$ is **nef**. This means that for any curve $C \subset X_{min}$ we have

$$\omega_{X_{min}} \cdot C \geq 0.$$

Birational geometry of Surfaces

- However, if $\kappa(X)=-1$, then X_{min} is either $\mathbb{P}^2_{\mathbb{C}}$ or a ruled surface so that X is covered by rational curves (i.e. by $\mathbb{P}^1$'s). The sections of $H^0(\omega_X^{\otimes m})$ must vanish along these curves for $m>0$. Hence $R(X)\cong\mathbb{C}$.
- If $\kappa(X)\geq 0$, then the minimal model $X\to X_{min}$ is unique and $\omega_{X_{min}}$ is **nef**. This means that for any curve $C\subset X_{min}$ we have

$$\omega_{X_{min}}\cdot C\geq 0.$$

- In fact one can show that $\omega_{X_{min}}$ is **semiample** i.e. there is an integer $m>0$ such that the sections of $H^0(\omega_{X_{min}}^{\otimes m})$ have no common zeroes.

Birational geometry of surfaces II

- Therefore, $\varphi_{\omega_{X_{min}}^{\otimes m}} : X_{min} \to \mathbb{P}^N$ is a morphism, $\omega_{X_{min}}^{\otimes m} \cong \varphi^* \mathcal{O}_{\mathbb{P}^N}(1)$.

Birational geometry of surfaces II

- Therefore, $\varphi_{\omega^{\otimes m}_{X_{min}}} : X_{min} \to \mathbb{P}^N$ is a morphism, $\omega^{\otimes m}_{X_{min}} \cong \varphi^* \mathcal{O}_{\mathbb{P}^N}(1)$.
- It is then easy to show that $R(X)$ is finitely generated.

Birational geometry of surfaces II

- Therefore, $\varphi_{\omega^{\otimes m}_{X_{min}}} : X_{min} \to \mathbb{P}^N$ is a morphism, $\omega^{\otimes m}_{X_{min}} \cong \varphi^* \mathcal{O}_{\mathbb{P}^N}(1)$.
- It is then easy to show that $R(X)$ is finitely generated.
- Conclusion: If X is not covered by rational curves, then after performing a finite sequence of geometrically meaningful operations (contracting -1 curves) we obtain a birational model on which ω is positive (nef and semiample).

Birational geometry of surfaces II

- Therefore, $\varphi_{\omega_{X_{min}}^{\otimes m}} : X_{min} \to \mathbb{P}^N$ is a morphism, $\omega_{X_{min}}^{\otimes m} \cong \varphi^* \mathcal{O}_{\mathbb{P}^N}(1)$.
- It is then easy to show that $R(X)$ is finitely generated.
- Conclusion: If X is not covered by rational curves, then after performing a finite sequence of geometrically meaningful operations (contracting -1 curves) we obtain a birational model on which ω is positive (nef and semiample).
- Finite generation of $R(X)$ then follows.

Birational geometry of surfaces III

More precisely we have the following:

- If $\kappa(X) = 0$, then X_{min} is either an abelian, hyperellyptic, K3 or Enriques surface. $\omega_{X_{min}}^{\otimes 12} = \mathcal{O}_X$ so that $R(X)^{(12)} = \oplus_{m \geq 0} H^0(\omega_X^{\otimes 12m}) \cong \mathbb{C}[t]$.

Birational geometry of surfaces III

More precisely we have the following:

- If $\kappa(X)=0$, then X_{min} is either an abelian, hyperellyptic, K3 or Enriques surface. $\omega_{X_{min}}^{\otimes 12}=\mathcal{O}_X$ so that $R(X)^{(12)}=\oplus_{m\geq 0}H^0(\omega_X^{\otimes 12m})\cong\mathbb{C}[t]$.
- If $\kappa(X)=1$, then $\omega_{X_{min}}^{\otimes 12}$ defines a morphism $f:X_{min}\to C$ whose general fiber is an elliptic curve. We have $\omega_{X_{min}}^{\otimes 12}=f^*H$ where H is an ample line bundle on C.

Birational geometry of surfaces III

More precisely we have the following:

- If $\kappa(X) = 0$, then X_{min} is either an abelian, hyperellyptic, K3 or Enriques surface. $\omega_{X_{min}}^{\otimes 12} = \mathcal{O}_X$ so that $R(X)^{(12)} = \oplus_{m \geq 0} H^0(\omega_X^{\otimes 12m}) \cong \mathbb{C}[t]$.
- If $\kappa(X) = 1$, then $\omega_{X_{min}}^{\otimes 12}$ defines a morphism $f : X_{min} \to C$ whose general fiber is an elliptic curve. We have $\omega_{X_{min}}^{\otimes 12} = f^* H$ where H is an ample line bundle on C.
- If $\kappa(X) = 2$, then $\omega_{X_{min}}^{\otimes 5}$ defines a morphism $\varphi : X_{min} \to X_{can} \subset \mathbb{P}^N$ where $X_{can} = \mathrm{Proj}(R(X))$ is the canonical model of X. The map $X \to X_{can}$ is birational and X_{can} has finitely many mildly singular points (rational double points).

Birational geometry of surfaces IV

If $\kappa(X) = 2$, it is natural to ask if one can understand the geometry of X_{min} more precisely. We have the following.

Birational geometry of surfaces IV

If $\kappa(X) = 2$, it is natural to ask if one can understand the geometry of X_{min} more precisely. We have the following.

Theorem (Castelnuovo)

If X is a surface with $H^0(X, \omega_X^{\otimes 2}) = 0$ and $H^0(X, \Omega^1_X) = 0$, then X is birational to $\mathbb{P}^2_{\mathbb{C}}$.

Birational geometry of surfaces IV

If $\kappa(X) = 2$, it is natural to ask if one can understand the geometry of X_{min} more precisely. We have the following.

Theorem (Castelnuovo)

If X is a surface with $H^0(X, \omega_X^{\otimes 2}) = 0$ and $H^0(X, \Omega_X^1) = 0$, then X is birational to $\mathbb{P}^2_{\mathbb{C}}$.

Corollary (Lüroth problem)

Let $\mathbb{C} \subset L \subset \mathbb{C}(x, y)$ be any field then $L \cong \mathbb{C}$ or $L \cong \mathbb{C}(x)$ or $L \cong \mathbb{C}(x, y)$. (Note that we may have a strict inclusion even if $L \cong \mathbb{C}(x, y)$.)

Birational geometry of surfaces V

Idea of proof. The inclusion $L \subset \mathbb{C}(x, y)$ corresponds to a morphism $X' \to X$ where $\mathbb{C}(X') = L$ and $\mathbb{C}(X) = \mathbb{C}(x, y)$.

Birational geometry of surfaces V

Idea of proof. The inclusion $L \subset \mathbb{C}(x, y)$ corresponds to a morphism $X' \to X$ where $\mathbb{C}(X') = L$ and $\mathbb{C}(X) = \mathbb{C}(x, y)$. If $\dim X' = 0$, then $L \cong \mathbb{C}$.

Birational geometry of surfaces V

Idea of proof. The inclusion $L \subset \mathbb{C}(x, y)$ corresponds to a morphism $X' \to X$ where $\mathbb{C}(X') = L$ and $\mathbb{C}(X) = \mathbb{C}(x, y)$.
If $\dim X' = 0$, then $L \cong \mathbb{C}$.
If $\dim X' = 1$, then $H^0(\omega_{X'}) = H^0(\Omega^1_{X'}) = 0$ as $H^0(\Omega^1_X) = 0$.
Therefore $X' = \mathbb{P}^1_{\mathbb{C}}$ and $L = \mathbb{C}(X') = \mathbb{C}(x)$.

Birational geometry of surfaces V

Idea of proof. The inclusion $L \subset \mathbb{C}(x,y)$ corresponds to a morphism $X' \to X$ where $\mathbb{C}(X') = L$ and $\mathbb{C}(X) = \mathbb{C}(x,y)$.
If $\dim X' = 0$, then $L \cong \mathbb{C}$.
If $\dim X' = 1$, then $H^0(\omega_{X'}) = H^0(\Omega^1_{X'}) = 0$ as $H^0(\Omega^1_X) = 0$.
Therefore $X' = \mathbb{P}^1_{\mathbb{C}}$ and $L = \mathbb{C}(X') = \mathbb{C}(x)$.
If $\dim X' = 2$, then $H^0(X', \omega_{X'}^{\otimes 2}) = 0$ and $H^0(X, \Omega^1_{X'}) = 0$ so that X' is birational to $\mathbb{P}^2_{\mathbb{C}}$ and hence $L \cong \mathbb{C}(x,y)$.
Remark. The above result fails in higher dimension.

Birational geometry of surfaces VI

- In general it is impossible to classify surfaces with $\kappa(X) = 2$.

Birational geometry of surfaces VI

- In general it is impossible to classify surfaces with $\kappa(X) = 2$.
- It is not even known if there exists a surface X homeomorphic to $\mathbb{P}^2_{\mathbb{C}}$ with $\kappa(X) = 2$.

Birational geometry of surfaces VI

- In general it is impossible to classify surfaces with $\kappa(X) = 2$.
- It is not even known if there exists a surface X homeomorphic to $\mathbb{P}^2_{\mathbb{C}}$ with $\kappa(X) = 2$.
- Never-the-less we have a good qualitative understanding of the birational geometry of X.

Birational geometry of surfaces VI

- In general it is impossible to classify surfaces with $\kappa(X) = 2$.
- It is not even known if there exists a surface X homeomorphic to $\mathbb{P}^2_{\mathbb{C}}$ with $\kappa(X) = 2$.
- Never-the-less we have a good qualitative understanding of the birational geometry of X.
- In the next lecture, I will illustrate how to extend the Kodaira-Enriques surface classification to arbitrary dimension.

Outline of the talk

Higher dimensional varieties

Theorem (Birkar-Cascini-Hacon-M^{c}Kernan - Siu)

Let X be a smooth complex projective variety of general type (i.e. $\kappa(X) = \dim(X)$) then X has a canonical model

$$X_{can} = \mathrm{Proj} R(X).$$

Higher dimensional varieties

Theorem (Birkar-Cascini-Hacon-M^{c}Kernan - Siu)

Let X be a smooth complex projective variety of general type (i.e. $\kappa(X) = \dim(X)$) then X has a canonical model

$$X_{can} = \mathrm{Proj} R(X).$$

One can also show that X has a minimal model X_{min}.

Higher dimensional varieties

- There are several new feaures X_{can} and X_{min} may have mild singularities.

Higher dimensional varieties

- There are several new feaures X_{can} and X_{min} may have mild singularities.
- X_{min} may not be unique.

Higher dimensional varieties

- There are several new feaures X_{can} and X_{min} may have mild singularities.
- X_{min} may not be unique.
- The map $X \dashrightarrow X_{min}$ is given by a carefully chosen finite sequence of divisorial contractions (the analog of contracting a -1 curve) and flips (a new operation).

Higher dimensional varieties

- There are several new feaures X_{can} and X_{min} may have mild singularities.
- X_{min} may not be unique.
- The map $X \dashrightarrow X_{min}$ is given by a carefully chosen finite sequence of divisorial contractions (the analog of contracting a -1 curve) and flips (a new operation).
- In dimension 3, flips were constructed by S. Mori, in dimension 4 by V. Shokurov, and in dimension ≥ 5 by Birkar-Cascini-Hacon-M^{c}Kernan using techniques of Shokurov and Siu.