Classifying Algebraic Varieties I

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Outline of the talk

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1. Introduction
2. 2 equations in 2 variables
3. 1 equation in 2 variables
4. Irreducible subsets

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Polynomial equations

Algebraic Geometry is concerned with the study of solutions of polynomial equations say n-equations in m-variables

$$\begin{cases} P_1(x_1, \ldots, x_m) = 0 \\ \qquad \ldots \\ \qquad \ldots \\ P_n(x_1, \ldots, x_m) = 0. \end{cases}$$

1 equation in 1 variable

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$$P(x) = a_d x^d + a_{d-1} x^{d-1} + \ldots + a_1 x + a_0 \qquad a_d \neq 0$$

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- Therefore a polynomial of degree d always has exactly d solutions or roots (when counted with multiplicity).
- If one is interested in solutions that belong to $\mathbb{R}$ (or $\mathbb{Q}$, or $\mathbb{Z}$ etc.), then the problem is much more complicated, but at least we know that there are at most d solutions.

Complex solutions

Throughout this talk I will always look for complex solutions $(z_1, \ldots, z_m) \in \mathbb{C}^m$ to polynomial equations

$$P_1(x_1, \ldots, x_m) = 0 \qquad \ldots \qquad P_n(x_1, \ldots, x_m) = 0$$

where $P_i \in \mathbb{C}[x_1, \ldots, x_m]$.

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Lines in the plane

Consider now 2 equations in 2 variables.
For example 2 lines in the plane i.e.

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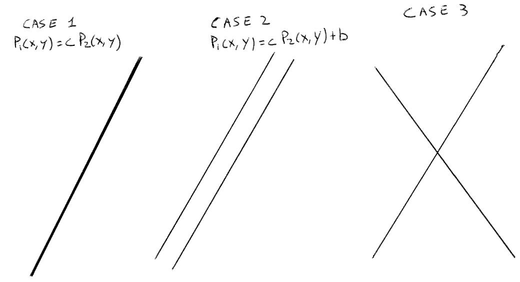
2. If the lines are distinct but parallel then there are no solutions.
3. If the lines are not parallel, there is a unique solution.

Lines in the plane



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The "line at infinity" is given by $\mathbb{C} \cup \{\infty\}$. It corresponds to all possible slopes of a line.

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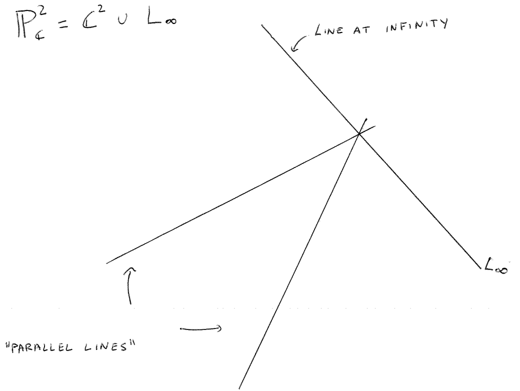
Theorem

Let L_1 and L_2 be two distinct lines in $\mathbb{P}^2_{\mathbb{C}}$.
Then L_1 and L_2 meet in exactly one point.

Lines in the projective plane



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- The points of $\mathbb{P}^2_{\mathbb{C}} - \mathbb{C}^2$ (the line at infinity) are of the form $[b_1 : b_2 : 0]$.

Lines in the projective plane

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- For any line parallel to $2x + 3y - 5 = 0$ we will obtain the same point at infinity.
- More generally one can show the following.

Bezout's Theorem

Theorem (Bezout's Theorem)

Let $P(x, y, z)$ and $Q(x, y, z)$ be homogeneous polynomials of degrees d and d'. If the intersection of the curves

$$\{P(x, y, z) = 0\} \cap \{Q(x, y, z) = 0\} \subset \mathbb{P}^2_{\mathbb{C}}$$

is finite, then it consist of exactly $d \cdot d'$ points (counted with multiplicity).

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In fact, Bezout's Theorem works in any number of variables.

Bezout's Theorem II

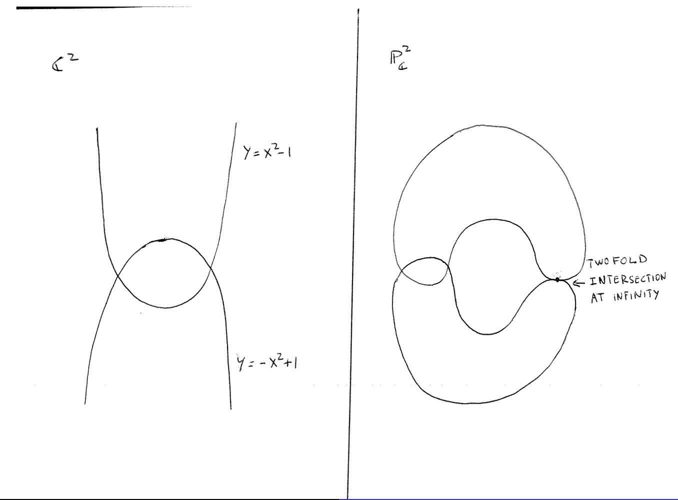
Theorem (Bezout's Theorem)

Let $P_1(x_0, \ldots, x_n), \ldots, P_n(x_0, \ldots, x_n)$ be n homogeneous polynomials of degrees $d_1, \ldots, d_n$. If the set

$$\bigcap_{i=1}^{n} \{P_i(x_0, \ldots, x_n) = 0\} \subset \mathbb{P}^n_{\mathbb{C}}$$

is finite, then it consist of exactly $d_1 \cdots d_n$ points (counted with multiplicity).

Bezout's Theorem III



Outline of the talk

Infinite number of solutions

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Consider the case of 1 equation in 2 variables $P(x, y) = 0$. For example

$$y^2 = x(x-1)(x-2)\cdots(x-d+1).$$

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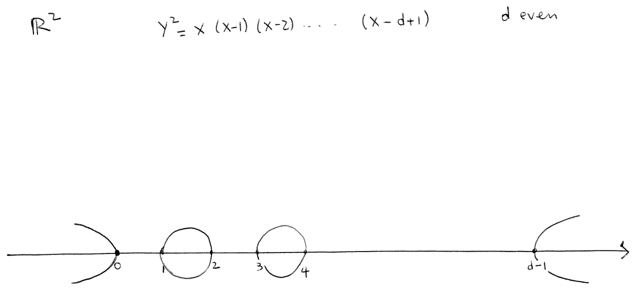
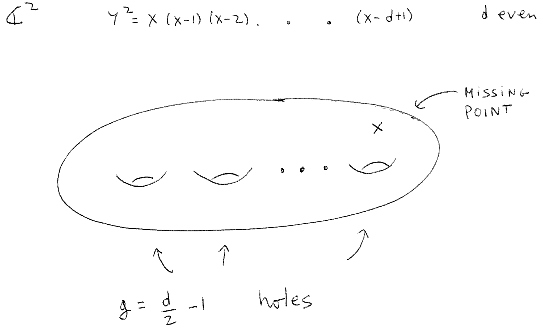
The solutions in $\mathbb{R}^2$ and in $\mathbb{C}^2$ look like this.

Hyperelliptic curve I



Hyperelliptic curve II



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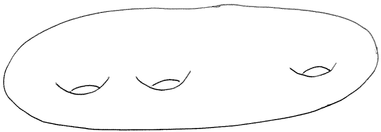
then the missing point is just the point at infinity $[0:1:0]$. The picture is now much more natural. It corresponds to a Riemann Surface.

Hyperelliptic curve IV

$\mathbb{P}^2_{\mathbb{C}}$ $\qquad y^2 z^{d-2} = x(x-z)(x-2z)\ .\ .\ .\ (x-(d-1)z)$

d even

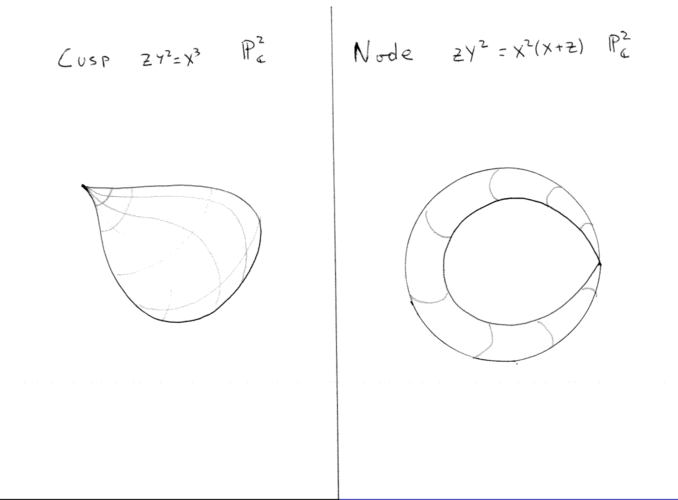


$g = \frac{d}{2} - 1$ holes

Singular curves

In special cases, it is possible for the set of solutions to be singular. For example $y^2 - x^3 = 0$ and $y^2 - x^2(x+1) = 0$.

Singular curves II



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  we obtain smooth Riemann Surfaces.
- Moreover, there is a (natural) way to resolve (=remove) singularities.

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Irreducible subsets

- In general a **Zariski closed subset** of $\mathbb{P}^m_{\mathbb{C}}$ is a set of the form

$$X = \bigcap_{i=1}^{n} \{P_i(x_1, \ldots, x_m) = 0\} \subset \mathbb{P}^m_{\mathbb{C}}.$$

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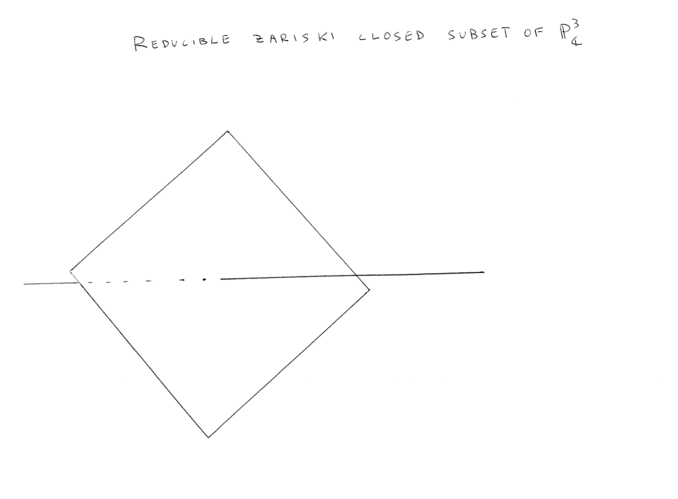
- Any Zariski closed subset is a finite union of irreducible Zariski closed subsets.
- If $P(x, y, z) \in \mathbb{C}[x, y, z]$ is homogeneous of degree $r > 0$, then
$$X = \{P(x, y, z) = 0\}$$
is irreducible if and only if $P(x, y, z)$ does not factor.

Reducible set



Irreducible subsets II

If X is an infinite irreducible Zariski closed subset, then there is an integer $d > 0$ such that for most points $x \in X$, one can find an open neighborhood $x \in U_x \subset X$ where U_x is analytically isomorphic to a ball in $\mathbb{C}^d$.

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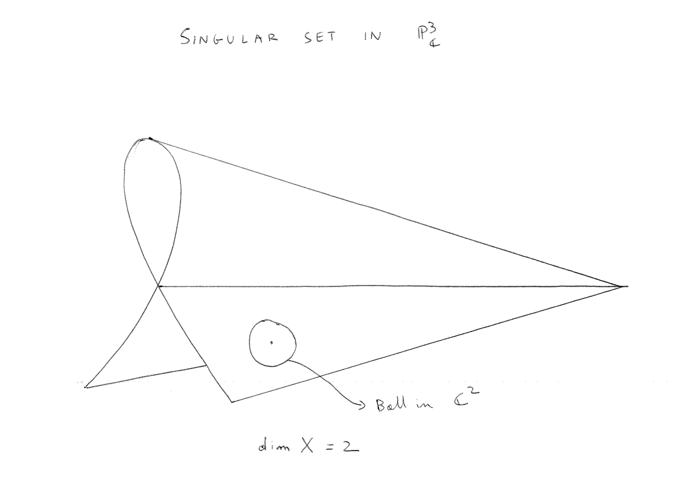
We say that X has **dimension** d.

Irreducible set of dimension 2



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Question

How many curves of genus g are there?

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- For example

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- To see that $X \cong \mathbb{P}^1_{\mathbb{C}}$ notice that X may be "parameterized" by $[s : t] \to [s^3 : s^2t : st^2 : t^3]$.

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- Any other elliptic curve is equivalent to X_t for some t.

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Number Theory. By a theorem of Faltings, they have at most finitely many rational solutions (over $\mathbb{Q}$), whereas rational curves always have infinitely many solutions. (eg. $x^n + y^n = z^n$!)

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Its abelianization is $\mathbb{Z}^{2g}$.

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Complex analysis. Consider the space of global holomorphic 1-forms $H^0(X, \omega_X)$ (i.e. objects that may locally be written as $f(x)dx$). Then $H^0(X, \omega_X) \cong \mathbb{C}^g$.

Curve embeddings

- Note that (as mentioned above) for any curve X there are (infinitely) many different descriptions

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- It is then more interesting to consider meromorphic functions on X.

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- The set $H^0(X, \mathcal{O}_X(D))$ is a finite dimensional complex vector space.
- E.g. $H^0(\mathbb{P}^1_{\mathbb{C}}, \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(dP_\infty))$ corresponds to homogeneous $p(x)$ of degree $\leq d$.

Rational functions II

- Given a basis $s_0, \ldots, s_m$ of $H^0(X, \mathcal{O}_X(D))$, we obtain a map $X \dashrightarrow \mathbb{P}^m_{\mathbb{C}}$ given by

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(undefined at the common zeroes of the s_i).
- Alternatively one can think of $\mathcal{O}_X(D)$ as a line bundle and s_i as sections of this line bundle.
- To find a natural embedding $X \subset \mathbb{P}^m_{\mathbb{C}}$, it suffices to find a "natural" line bundle $\mathcal{O}_X(D)$ such that the sections $s_i \in H^0(X, \mathcal{O}_X(D))$ separate the points and tangent directions of X.

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Theorem

Let X be a curve of genus $g \geq 2$, then for any $k \geq 3$ the sections of $H^0(X, (\omega_X)^{\otimes k})$ define an embedding

$$X \hookrightarrow \mathbb{P}_{\mathbb{C}}^{(2k-1)(g-1)-1}.$$

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- In coordinates this may be described as follows. If $r_0, \cdots, r_m$ are generators of $R(X)$ and

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then $X \subset \mathbb{P}^m_{\mathbb{C}}$ is defined by the equations in K.
- There are many interesting open problems on the structure of $R(X)$ (even if $\dim X = 1$).

Outline of the talk

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- The **Canonical Ring** of X given by

$$R(X) = \bigoplus_{k \geq 0} H^0(X, \omega_X^{\otimes k}).$$

Surfaces and 3-folds

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- In particular, if $\dim X = 3$, then $R(X)$ is finitely generated.

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It follows that there is a natural map $\phi : X \dashrightarrow \mathrm{Proj} R(X) \cong \mathbb{P}^m_{\mathbb{C}}$. In most cases, ϕ carries a lot of information about X and it can be used to study the geometry of X in a coordinate free manner.