THE MINIMAL MODEL PROGRAM FOR VARIETIES OF LOG GENERAL TYPE

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Contents

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1. Preliminaries

1.1. **Resolution of singularities.** We will need the following result of Hironaka on the resolution of singularities.

Theorem 1.1. *Let X be an irreducible complex projective variety and D be an effective Cartier divisor on X. Then there is a birational morphism $\mu : X' \to X$ from a smooth variety X' given by a finite sequence of blow ups along smooth centers supported over the singularities of D and X such that*

$$\mu^* D + \mathrm{Exc}(\mu)$$

is a divisor with simple normal crossings support.

The above statement is taken from [11, §4]. For a particularly clear exposition of the proof of this result as well as references to the literature, we refer to [7].

1.2. **Divisors.** Let X be a normal complex variety.

Definition 1.2. *A* **prime divisor** *is an irreducible and reduced codimension 1 subvariety of X. The group of* **Weil divisors** $\mathrm{WDiv}(X)$ *is the set of all finite formal linear combinations $D = \sum d_i D_i$ where*

$d_i \in \mathbb{Z}$ and D_i are prime divisors with addition defined component by component

$$\sum d_i D_i + \sum d_i' D_i = \sum (d_i + d_i') D_i.$$

A divisor $D \in \mathrm{WDiv}(X)$ is **effective** *(denoted by $D \geq 0$) if $D = \sum d_i D_i$, with $d_i \geq 0$ and D_i prime divisors.*

Definition 1.3. *For any divisor $D \in \mathrm{WDiv}(X)$, one may define the* **divisorial sheaf** *$\mathcal{O}_X(D)$ by setting*

$$\Gamma(U, \mathcal{O}_X(D)) = \{f \in \mathbb{C}(X) | ((f) + D)|_U \geq 0\}.$$

Remark 1.4. *Note that $\mathcal{O}_X(D)$ is a reflexive sheaf of rank one so that $\mathcal{O}_X(D)^{\vee\vee} \cong \mathcal{O}_X(D)$. Conversly, for any torsion free reflexive sheaf of rank one $\mathcal{F}$ there is a Weil divisor D such that $\mathcal{F} \cong \mathcal{O}_X(D)$. Notice moreover that if $U = X - X_{\mathrm{sing}}$ and $i : U \to X$ is the inclusion, then $i_* \mathcal{O}_U(D|_U) = \mathcal{O}_X(D)$.*

Definition 1.5. *For any rational function $0 \neq f \in \mathbb{C}(X)$, we let $(f) \in \mathrm{Div}(X)$ be the* **principal** *divisor corresponding to the zeroes and poles of f. We say that two divisors $D, D' \in \mathrm{Div}(X)$ are* **linearly equivalent** *if $D - D' = (f)$ where $f \in \mathbb{C}(X)$. The* **complete linear series** *corresponding to a divisor $D \in \mathrm{Div}(X)$ is given by*

$$|D| = \{D' \geq 0 | D' \sim D\}.$$

Definition 1.6. *For any divisor $D \in \mathrm{Div}(X)$, the* **base locus** *of D is given by*

$$\mathbf{Bs}(D) = \cap_{D' \in |D|} \mathrm{Supp}(D).$$

(Here $\mathrm{Supp}(D)$ is the support of D i.e. the subset of X given by the points of D.)

Definition 1.7. *If $|D| \neq \emptyset$, then $|D| \cong \mathbb{P}^k \cong \mathbb{P}H^0(\mathcal{O}_X(D))$ for some $k > 0$. We let k be the* **dimension** *of $|D|$ and*

$$\phi_{|D|} : X \dashrightarrow \mathbb{P}^k$$

be the corresponding rational map. Note that if $U = X - \mathrm{Bs}(D)$, then $(\phi_{|D|})|_U$ is a morphism. More explicitely, if $\{s_0, \ldots, s_k\}$ is a basis of $H^0(\mathcal{O}_X(D))$, then

$$(\phi_{|D|})|_U(x) = [s_0(x) : \ldots : s_k(x)].$$

Definition 1.8. *A k-***cycle** *on X is a $\mathbb{Z}$-linear combination of irreducible subvarieties of dimension k. The set of all k-cycles on X is denoted by $Z_k(X)$ and it is an abelian group with respect to addition. Note that $Z_{\dim(X)-1}(X) = \mathrm{WDiv}(X)$.*

Definition 1.9. *A* **Cartier divisor** *is a Weil divisor D which is locally defined by the zeroes and poles of a rational function $f \in \mathbb{C}(X)$. The group of Cartier divisors* $\mathrm{Div}(X)$ *is a subgroup of* $\mathrm{WDiv}(X)$ *and it may be identified with $\Gamma(X, \mathbb{C}(X)^*/\mathcal{O}_X^*)$ (here $\mathbb{C}(X)$ denotes the sheaf of rational functions). Note that a Weil divisor D is Cartier if and only if the sheaf $\mathcal{O}_X(D)$ is invertible.*

If $K \in \{\mathbb{Q}, \mathbb{R}, \ldots\}$, then we let $\mathrm{WDiv}_K(X) = \mathrm{WDiv}(X) \otimes_{\mathbb{Z}} K$ and $\mathrm{Div}_K(X) = \mathrm{Div}(X) \otimes_{\mathbb{Z}} K$. If $D, D' \in \mathrm{WDiv}(X)$, then $D \sim_K D'$ if and only if $D - D' = \sum d_i(f_i)$ with $f_i \in \mathbb{C}(X)$ and $d_i \in K$.

Definition 1.10. *If D is a Cartier divisor on X and $f : Y \to X$ is a dominant morphism, then we define the* **pullback** *f^*D of D as follows: Let U_i be an open covering of X and $g_i \in \mathbb{C}(X)^*$ such that $D \cap U_i = (g_i) \cap U_i$, then f^*D is defined by $g_i \circ f$ on $f^{-1}(U_i)$.*

Definition 1.11. *If D is Cartier divisor on a proper normal variety X and $C \subset X$ is a curve contained in X, then the* **intersection** *of D and C is given by $D \cdot C = \deg(i^*D)$ where $i : C' \to X$ is the induced map from the normalization of C to X. Two Cartier divisors D and D' (or more generally two elements $D, D' \in \mathrm{WDiv}(X)$ such that $D - D' \in \mathrm{Div}_{\mathbb{R}}(X)$) are* **numerically equivalent** *(denoted by $D \equiv D'$) if $(D - D') \cdot C = 0$ for any curve $C \subset X$.*

Numerical equivalence generates an equivalence relation in $\mathrm{Div}(X)$ *and in $Z_1(X)$. We let*

$$N^1(X) = \mathrm{Div}_{\mathbb{R}}(X)/\equiv \qquad \textit{and} \qquad N_1(X) = (Z_1(X) \otimes_{\mathbb{Z}} \mathbb{R})/\equiv .$$

Note that $N^1(X)$ and $N_1(X)$ are dual vector spaces over $\mathbb{R}$. Their dimension $\rho(X)$ is the **Picard number** *of X.*

Definition 1.12. *The* **cone of effective 1-cycles** *is the cone*

$$NE(X) \subset N_1(X)$$

generated by $\{\sum n_i C_i$ s. t. $n_i \geq 0\}$.

Given a proper morphism of normal varieties $f : X \to Y$ and an irreducible curve $C \subset X$, we let $f_*(C) = df(C)$ where $d = \deg(C \to f(C))$. If $f(C)$ is a point, then we set $f_*C = 0$. One sees that

$$f^*D \cdot C = D \cdot f_*C \qquad \forall D \in \mathrm{Div}(Y).$$

Extending by linearity, we get an injective homomorphism

$$f^* : N^1(Y) \to N^1(X)$$

and surjective homomorphisms

$$f_* : N_1(X) \to N_1(Y), \qquad NE(X) \to NE(Y).$$

Definition 1.13. *If $D = \sum d_i D_i \in \mathrm{WDiv}_{\mathbb{R}}(X)$, where D_i are distinct prime divisors, then we define the* **round down**, *the* **round up** *and the* **fractional part** *of D by the formulas*

$$\ulcorner D \urcorner = \sum \ulcorner d_i \urcorner D_i, \qquad \llcorner D \lrcorner = \sum \llcorner d_i \lrcorner D_i, \qquad \{D\} = \sum \{d_i\} D_i$$

where $\llcorner d_i \lrcorner$ is the biggest integer $\leq d_i$, $\ulcorner d_i \urcorner$ is the smallest integer $\geq d_i$ and $\{d_i\} = d_i - \llcorner d_i \lrcorner$.

Remark 1.14. *Note that if $D \sim_{\mathbb{Q}} D'$ it is not the case that $\ulcorner D \urcorner = \ulcorner D' \urcorner$. If $D \in \mathrm{Div}_{\mathbb{Q}}(X)$, and $Y \subset X$ is a subvariety, it is also not the case that $\llcorner D|_Y \lrcorner = \llcorner D \lrcorner|_Y$. We have $\ulcorner -D \urcorner = -\llcorner D \lrcorner$.*

Definition 1.15. *If $D \in \mathrm{WDiv}(X)$ and $|D| \neq \emptyset$, we let $\mathrm{Fix}(D) = \sum f_i F_i$ where f_i is the minimum of the multiplicities of any divisor $D' \in |D|$ along the prime divisor F_i. We let $\mathrm{Mob}(D) = D - \mathrm{Fix}(D)$. Note that $\mathrm{Bs}(\mathrm{Mob}(D))$ has codimension at least 2.*

1.3. **Ample divisors.**

Definition 1.16. *A Cartier divisor $D \in \mathrm{Div}(X)$ is* **very ample** *if it is base point free and $\phi_{|D|} : X \to \mathbb{P}^N$ is an embedding. A $\mathbb{Q}$-Cartier divisor $D \in \mathrm{Div}_{\mathbb{Q}}(X)$ is* **ample** *if mD is very ample for some $m > 0$.*

Recall the following:

Definition 1.17. *A coherent sheaf $\mathcal{F}$ on a variety X is* **globally generated** *if the homomorphism*

$$H^0(X, \mathcal{F}) \otimes \mathcal{O}_X \to \mathcal{F}$$

is surjective.

Theorem 1.18 (Serre)**.** *Let $D \in \mathrm{Div}(X)$ be an ample divisor on a projective scheme and $\mathcal{F}$ a coherent sheaf on X. Then there is an integer $n_0 > 0$ such that for all $n \geq n_0$, $\mathcal{F} \otimes \mathcal{O}_X(nD)$ is globally generated.*

Proof. [5] II §5. The idea is that we may assume that $X = \mathbb{P}^N$. Since $\mathcal{F}$ is coherent, it is locally generated by finitely many sections of $\mathcal{O}_X$. Each local section is the restriction of some global section of $\mathcal{O}_{\mathbb{P}^N}(n)$ for $n \gg 0$. By compactness, we only need finitely many such sections. □

Theorem 1.19. *[Serre Vanishing] Let X be a projective scheme and $D \in \mathrm{Div}(X)$ be a very ample line bundle and $\mathcal{F}$ a coherent sheaf. Then there is an integer $n_0 > 0$ such that for all $n \geq n_0$ and all $i > 0$,*

$$H^i(X, \mathcal{F} \otimes \mathcal{O}_X(nD)) = 0.$$

Proof. (cf. [5] III.5.2) We may assume that $X = \mathbb{P}^N$ (replace $\mathcal{F}$ by $\phi_{D*}\mathcal{F}$). The Theorem is clear if $\mathcal{F}$ is a finite direct sum of sheaves of the form $\mathcal{O}_{\mathbb{P}^N}(q)$. We can find a short exact sequence of coherent sheaves

$$0 \longrightarrow \mathcal{K} \longrightarrow \oplus\mathcal{O}_{\mathbb{P}^N}(q_i) \longrightarrow \mathcal{F} \longrightarrow 0$$

(eg. use (1.18)). We now consider the exact sequence

$$\cdots \oplus H^i(\mathcal{O}_{\mathbb{P}^N}(q_i+n)) \to H^i(\mathcal{F}\otimes\mathcal{O}_{\mathbb{P}^N}(n)) \to H^{i+1}(\mathcal{K}\otimes\mathcal{O}_{\mathbb{P}^N}(n))\cdots$$

and proceed by descending induction on i so that we may assume that $h^{i+1}(\mathcal{K}\otimes\mathcal{O}_{\mathbb{P}^N}(n)) = 0$ for $n \gg 0$ and $i \geq 0$. Since $h^i(\mathcal{O}_{\mathbb{P}^N}(q_i+n)) = 0$ for $n \gg 0$ and $i \geq 0$, we have $h^i(\mathcal{F}\otimes\mathcal{O}_{\mathbb{P}^N}(n)) = 0$ as required for $n \gg 0$ and $i \geq 0$. □

Recall that we also have the following

Proposition 1.20. *Let X be a projective scheme and $D \in \mathrm{Div}(X)$. The following are equivalent*

1. *D is ample;*
2. *mD is ample for some $m > 0$;*
3. *mD is very ample for some $m > 0$;*
4. *there exists an integer $m_1 > 0$ such that mD is very ample for all $m \geq m_1$;*
5. *for any coherent sheaf $\mathcal{F}$, there exists an integer $m_2 = m_2(\mathcal{F}) > 0$ such that $\mathcal{F}\otimes\mathcal{O}_X(mD)$ is globally generated for all $m \geq m_2$;*
6. *for any coherent sheaf $\mathcal{F}$, there exists an integer $m_3 = m_3(\mathcal{F}) > 0$ such that*

$$H^i(X, \mathcal{F}\otimes\mathcal{O}_X(mD)) = 0 \qquad \textit{for all } i > 0,\ m \geq m_3(\mathcal{F}).$$

Proof. Exercise. (See [5] II §7). □

Proposition 1.21. *Let $f : X \to Y$ be a finite map of projective varieties, L an ample line bundle on Y, then f^*L is ample on X.*

Proof. For any coherent sheaf $\mathcal{F}$ on X, one has that $R^i f_*\mathcal{F} = 0$ for all $i > 0$, and so by the projection formula

$$H^i(Y, \mathcal{F}\otimes f^*L^m) = H^i(X, f_*\mathcal{F}\otimes L^m) = 0$$

for all $i > 0$ (as L is ample on Y). The proposition now follows from (1.20). □

For a very ample divisor D on a projective variety X, and a subvariety $V \subset X$ of dimension i, we let

$$D^i \cdot V$$

be the degree of V viewed as a subvariety of $\mathbb{P}H^0(\mathcal{O}_X(D))$. More generally, given $D \in \operatorname{Div}(X)$, we may pick $H \in \operatorname{Div}(X)$ such H and $D+H$ are very ample. For all $j > 0$, we view $H \cdot W$ and $(D+H) \cdot W$ as subvarieties of X of dimension $i-1$ so that proceeding by induction on the dimension of W, we may assume that $D^{i-1} \cdot H \cdot W$ and $D^{i-1} \cdot (D+H) \cdot W$ are defined. We then let

$$D^i \cdot W = D^{i-1} \cdot (D+H) \cdot W - D^{i-1} \cdot H \cdot W.$$

By linearity, we may define $D^i \cdot W$ for any $D \in \operatorname{Div}_{\mathbb{R}}(X)$ and $W \in Z_i(X) \otimes_{\mathbb{Z}} \mathbb{R}$.

We have the following important result:

Theorem 1.22. *[Nakai-Moishezon criterion] Let $D \in \operatorname{Div}(X)$ be a Cartier divisor on a proper scheme X, then D is ample if and only if for any $0 \le i \le n-1$ and any subvariety W of dimension i, one has $D^i \cdot W > 0$.*

Proof. (cf. [10] 1.37) We will assume that X is projective. We may also assume that X is irreducible. Clearly, if D is ample, then $D^i \cdot W > 0$. For the converse implication, we proceed by induction on $n = \dim X$. When $\dim X = 1$, the Theorem is obvious. So we may assume that $D|_Z$ is ample for all for all proper closed sub-schemes $Z \subsetneq X$.

Claim 1. $h^0(X, \mathcal{O}_X(kD)) > 0$ for some $k > 0$ (actually $\kappa(D) = n$).

We choose a very ample divisor $B \in \operatorname{Div}(X)$ such that $D+B$ is very ample. Let $A \in |D+B|$ be a general member. Consider the short exact sequence

$$0 \to \mathcal{O}_X(kD - B) \to \mathcal{O}_X(kD) \to \mathcal{O}_B(kD) \to 0.$$

Since $\mathcal{O}_B(kD)$ is ample, for all $k \gg 0$ we have that $h^i(B, \mathcal{O}_B(kD)) = 0$ for all $i > 0$, and so

$$h^i(\mathcal{O}_X(kD - B)) = h^i(\mathcal{O}_X(kD)) \qquad \textit{for } i \ge 2 \textit{ and } k \gg 0.$$

The same argument applied to the short exact sequence

$$0 \to \mathcal{O}_X(kD - B) \to \mathcal{O}_X((k+1)D) \to \mathcal{O}_A((k+1)D) \to 0$$

shows that

$$h^i(\mathcal{O}_X(kD - B)) = h^i(\mathcal{O}_X((k+1)D)) \qquad \textit{for } i \ge 2 \textit{ and } k \gg 0.$$

Putting this together, we see that for $i \ge 2$ and $k \gg 0$, one has $h^i(\mathcal{O}_X(kD)) = h^i(\mathcal{O}_X((k+1)D))$ and so the number $h^i(\mathcal{O}_X(kD))$ is constant. But then for $k \gg 0$

$$h^0(X, \mathcal{O}_X(kD)) \ge h^0(X, \mathcal{O}_X(kD)) - h^1(X, \mathcal{O}_X(kD))$$
$$= \chi(X, \mathcal{O}_X(kD)) + (\textit{constant}) = D^n/n! \cdot k^n + O(k-1).$$

Claim 2. $\mathcal{O}_X(kD)$ is generated by global sections for some $k > 0$.

Fix a non-zero section $s \in H^0(X, \mathcal{O}_X(mD))$. Let S be the divisor defined by s. Consider the short exact sequence

$$0 \to \mathcal{O}_X((k-1)mD) \to \mathcal{O}_X(kmD) \to \mathcal{O}_S(kmD) \to 0.$$

By induction $\mathcal{O}_S(kmD)$ is generated by global sections and so it suffices to show that for $k \gg 0$, the homomorphism $H^0(X, \mathcal{O}_X(kmD)) \to H^0(S, \mathcal{O}_S(kmD))$ is surjective. Arguing as in Claim 1, one sees that $h^1(X, \mathcal{O}_X(kmD))$ is a decreasing sequence and is hence eventually constant as required.

Conclusion of the proof. $\phi_{kD} : X \to \mathbb{P}^N$ is a finite morphism for $k \gg 0$. If in fact C is a curve contracted by ϕ_{kD}, then $kD \cdot C = \mathcal{O}_{\mathbb{P}^N}(1) \cdot \phi_{kD}(C) = 0$ which contradicts $C \cdot D > 0$. The Theorem now follows as $kD = \phi^*_{kD}\mathcal{O}_{\mathbb{P}^N}(1)$ is ample (as it is the pull-back of an ample line bundle via a finite map). □

Exercise 1.23. *Let X be a projective variety, H an ample divisor on X. A divisor D on X is ample if and only if there exists an $\epsilon > 0$ such that*

$$\frac{D \cdot C}{H \cdot C} \geq \epsilon$$

for all irreducible curves $C \subset X$.

Theorem 1.24. *[Nakai's Criterion] Let D be a divisor on a projective scheme X. D is ample if and only if $D \cdot Z > 0$ for any $Z \in \overline{NE}(X) - \{0\}$.*

Proof. Exercise. □

Theorem 1.25. *[Seshadri's Criterion] Let D be a divisor on a projective scheme X. D is ample if and only if there exists an $\epsilon > 0$ such that*

$$C \cdot D \geq \epsilon \, \mathrm{mult}_x(C)$$

for all $x \in C \subset X$.

Proof. Assume that D is ample, then there exists an integer $n > 0$ such that nD is very ample, but then

$$nD \cdot C \geq \mathrm{mult}_x(C)$$

for all $x \in C \subset X$.

For the reverse implication, we proceed by induction. Therefore, we may assume that for any irreducible subvariety $Z \subsetneq X$, the divisor $D|_Z$ is ample and so $D^{\dim Z} \cdot Z > 0$. By (1.22), it is enough to show that $D^n > 0$. Let

$$\mu : X' \to X$$

be the blow up of X at a smooth point. Then $\mu^*D-\epsilon E$ is nef. In fact, for any curve $C'\subset X'$ we either have $C=\mu(C')$ is a curve and then

$$(\mu^*D-\epsilon E)\cdot C' = D\cdot C-\epsilon\,\mathrm{mult}_x C\geq 0,$$

or $\mu(C')=x$ and then since $C'\subset E\cong\mathbb{P}^{n-1}$ and $\mathcal{O}_E(E)=\mathcal{O}_{\mathbb{P}^{n-1}}(-1)$

$$(\mu^*D-\epsilon E)\cdot C' = \epsilon\deg C'>0.$$

But then, by (1.35)

$$(\mu^*D-\epsilon E)^n = D^n-\epsilon^n\geq 0$$

and this completes the proof. □

1.4. **Positivity of divisors.** Given a divisor on a variety X there are several notions of positivity that will be essential in what follows. We begin with the following:

Definition 1.26. *If $D\in\mathrm{Div}_{\mathbb{R}}(X)$ is a $\mathbb{R}$-Cartier divisor on a proper variety X, then D is* **nef** *if $D\cdot C\geq 0$ for any $C\in Z_1(X)$.*

The terminology *nef* was introduced by M. Reid. It stands for numerically eventually free. The point is that if D is eventually free (i.e. if it is semiample see (1.28)) then it is easy to see that D is nef.

Exercise 1.27. *Let X be a projective variety, H an ample divisor on X. A divisor D on X is nef if and only if $D+\epsilon H$ is ample for all rational numbers $\epsilon>0$.*

Definition 1.28. *A divisor $D\in\mathrm{Div}(X)$ (or more precisely the complete linear series $|D|$) is* **base point free** *if for any point $x\in X$ there is a divisor $D'\sim D$ such that $x\notin\mathrm{Supp}(D')$. This is equivalent to requiring that the sheaf $\mathcal{O}_X(D)$ is generated by global sections.*

A divisor $D\in\mathrm{Div}_{\mathbb{Q}}(X)$ is **semiample** *if there is an integer $m>0$ such that $|mD|$ is base point free.*

A divisor $D\in\mathrm{Div}_{\mathbb{R}}(X)$ is **semiample** *if we may write $D=\sum_{i=1}^k r_iD_i$ where $r_i\in\mathbb{R}$ and $D_i\in\mathrm{Div}_{\mathbb{Q}}(X)$ are semiample.*

Note that if $D\in\mathrm{Div}(X)$ is base point free, then there is a morphism

$$\phi_{|D|}:X\to\mathbb{P}H^0(\mathcal{O}_X(D)).$$

Definition 1.29. *For any divisor D (or line bundle L), one can define the* **Kodaira dimension**

$$\kappa(D):=\max_{m>0}\{\dim\phi_{mD}(X)\}$$

here we set $\dim\phi_{mD}(X)=-\infty$ if $h^0(\mathcal{O}_X(mD))=0$.

Remark 1.30. *Frequently, one lets* $\dim \phi_{mD}(X) = -1$ *if* $h^0(\mathcal{O}_X(mD)) = 0$*. If we adopt this convention, then*

$$\kappa(D) = \text{tr. deg.}_{\mathbb{C}} R(D) - 1.$$

We have that $\kappa(D) < 0$ *if and only if* $h^0(\mathcal{O}_X(mD)) = 0$ *for all* $m > 0$*.*

If $\kappa(D) = 0$ *then there exists an integer* m_0 *such that* $h^0(\mathcal{O}_X(mD)) = 1$ *if and only if* $m > 0$ *is divisible by* m_0*. (It is an instructive excercise to prove this.)*

It is known (cf. [11, §2]*) that if* $\kappa(D) > 0$*, then there exist constants* $A, B > 0$ *such that for all* m *sufficiently divisible, we have*

$$Am^{\kappa(X)} \leq h^0(\mathcal{O}_X(mD)) \leq Bm^{\kappa(X)},$$

so that $\kappa(D) = \kappa$ *if and only if* $\limsup h^0(\mathcal{O}_X(mD))/m^\kappa \neq 0$*.*

Definition 1.31. *A divisor* $D \in \mathrm{WDiv}_{\mathbb{Q}}(X)$ *is* **big** *if* $\kappa(D) = \dim(X)$*.*

Remark 1.32. *If* D *is big, then we define the* **volume** *of* D *by*

$$\mathrm{vol}(D) = \limsup \frac{h^0(\mathcal{O}_X(mD))}{m^n/n!}$$

where $n = \dim(X)$*. It is known that in this case*

$$\limsup h^0(\mathcal{O}_X(mD))/m^n = \lim h^0(\mathcal{O}_X(mD))/m^n$$

(cf. [11]*) and that* $D \sim_{\mathbb{Q}} A + B$ *where* A *is an ample* $\mathbb{Q}$*-divisor and* B *is effective cf.* [11, §2]*. Notice moreover that if* $D \equiv D'$ *then* D *is big if and only if* D' *is big cf.* [11, §2]*.*

Remark 1.33. *If* $0 < \kappa = \kappa(D) < \dim(X)$*, then it is not known if*

$$\lim h^0(\mathcal{O}_X(mD))/m^\kappa$$

always exists.

Definition 1.34. *A divisor* $D \in \mathrm{WDiv}_{\mathbb{Q}}(X)$ *is* **pseudo-effective** *if and only if for any ample divisor* A *and any rational number* $\epsilon > 0$*, the divisor* $D + \epsilon A$ *is big. (Equivalently* D *is pseudo-effective if and only if* D *is in the closure of the big cone. This property is also determined by the numerical equivalence class of* D*.)*

Theorem 1.35. *[Kleiman's Theorem] Let* X *be a proper variety,* D *a nef divisor. Then* $D^{\dim Z} \cdot Z \geq 0$ *for all irreducible subvarieties* $Z \subset X$*.*

Proof. (cf. [11] 1.4.9) We assume that X is projective (Chow's Lemma) and irreducible. When $\dim X = 1$, the Theorem is clear. By induction on $n = \dim X$, we may assume that

$$D^{\dim Z} \cdot Z \geq 0 \qquad \forall Z \subset X \textit{ irreducible of } \dim Z < n,$$

and we must show that $D^n \geq 0$. Fix H an ample divisor and consider the polynomial

$$P(t) := (D + tH)^n \in \mathbb{Q}[t].$$

We must show that $P(0) \geq 0$. For $1 \leq k \leq n$, the coefficient of t^k is

$$D^{n-k} H^k \geq 0.$$

Assume that $P(0) < 0$, then one sees that $P(t)$ has a unique real root $t_0 > 0$.

For any rational number $t > t_0$, one sees that

$$(D + tH)^{\dim Z} \cdot Z > 0 \qquad \forall Z \subset X \textit{ irreducible of } \dim Z \leq n,$$

and so by (1.22), $D + tH$ is ample. We write

$$P(t) = Q(t) + R(t) = D \cdot (D + tH)^{n-1} + tH \cdot (D + tH)^{n-1}.$$

As $D+tH$ is ample for $t > t_0$, one has that $(D+tH)^{n-1}$ is an effective 1-cycle, so $Q(t) \geq 0$ for all rational numbers $t > t_0$ and so $Q(t_0) \geq 0$. One sees that all the coefficients of $R(t)$ are non-negative and the coefficient of t^n is $H^n > 0$. It follows that $R(t_0) > 0$ and so $P(t_0) > 0$ which is the required contradiction. □

2. The Singularities of the Minimal Model Program

Definition 2.1. *If X is a normal variety and $i : U \to X$ is the inclusion of the nonsingular locus. Then U is a big open subset and we let ω_U be the canonical line bundle of U. ω_U is an invertible sheaf whose sections may be locally written as $f \cdot dz_1 \wedge \ldots \wedge dz_n$ where $z_1, \ldots, z_n$ are local coordinates and f is a regular function. We define the* **canonical sheaf** *as the divisorial sheaf $\omega_X = i_* \omega_U$. A* **canonical divisor** *on X is a divisor K_X such that $\mathcal{O}_X(K_X) \cong \omega_X$. Note that, despite the fact that it is usaually referred to as "the canonical divisor", K_X is not uniquely defined and may be non-effective.*

Definition 2.2. *A* **log pair** *(X, D) consits of a normal variety X and a divisor $D \in \mathrm{WDiv}_{\mathbb{R}}(X)$ such that $K_X + D \in \mathrm{Div}_{\mathbb{R}}(X)$.*

Definition 2.3. *A* **log resolution** *of a pair (X, D) is a proper birational morphism $f : Y \to X$ from a smooth variety such that $\mathrm{Exc}(f)$ is a divisor and $f^{-1}(D) \cup \mathrm{Exc}(f)$ has simple normal crossings support (i.e. each component is a smooth divisor and all components meet transversely).*

Exercise 2.4. *Compute a log resolution for 3 lines meeting at a point and for the cusp $y^2 = x^3$.*

Definition 2.5. *Given a log pair* (X, D) *and a log resolution* $f : Y \to X$*, we write*

$$K_Y = f^*(K_X + D) + A_Y(X, D)$$

where $f_* K_Y = K_X$ *and* $f_* A_Y(X, D) = -D$*. The divisor* $A_Y(X, D)$ *is the* **discrepancy divisor** *of* (X, D)*. We will also write* $A_Y(X, D) = \sum a_P(X, D)P$ *where* P *are prime divisors on* Y*. The numbers* $a_P(X, D)$ *are the* **discrepancies** *of* (X, D) *along* P*. We will also write*

$$A_Y(X, D) = E_Y(X, D) - \Gamma_Y(X, D)$$

where $E_Y(X, D)$ *and* $\Gamma_Y(X, D)$ *are effective with no common components. The* **total discrepancy** *of* (X, D) *is given by*

$$\text{total discrepancy}(X, D) = \inf\{a_P(X, D) | P \text{ is a prime divisor over } X\},$$

and the **discrepancy** *of* (X, D) *is given by*

$$\text{discrepancy}(X, D) = \inf\{a_P(X, D) | P \text{ is an exceptional prime divisor over } X\}.$$

Remark 2.6. *Note that* $A_Y(X, D)$ *is uniquely defined. To prove this, use the Negativity Lemma given below.*

Lemma 2.7 (Negativity Lemma)**.** *Let* $f : Y \to X$ *be a proper birational morphism of normal varieties. If* $-B \in \mathrm{Div}_{\mathbb{Q}}(Y)$ *is* f*-nef, then* B *is effective if and only if* $f_* B$ *is effective. Moreover, if* B *is effective, then for any* $x \in X$*, either* $f^{-1}(x) \subset \mathrm{Supp}(B)$ *or* $f^{-1}(x) \cap \mathrm{Supp}(B) = \emptyset$*.*

Proof. See [10, Lemma 3.39]. □

Exercise 2.8. *Let* $f : Y \to X$ *be a proper birational morphism and set* $D_Y = -A_Y(X, D)$*. Show that* total discrepancy$(X, D) =$ total discrepancy(Y, D_Y) *and give an example where* discrepancy$(X, D) =$ discrepancy(Y, D_Y).

Exercise 2.9. *If* (X, D) *and* (X, D') *are two log pairs such that* $D \leq D'$*, then show that for any log resolution* $f : Y \to X$ *of* (X, D) *and* (X, D')*, we have* $A_Y(X, D) \geq A_Y(X, D')$*.*

Exercise 2.10. *Let* X *be a smooth variety* $D = \sum a_i D_i$ *a sum of distinct prime divisors,* $Z \subset X$ *a smooth subvariety of codimension* k*. Let* $p : B_Z(X) \to X$ *be the blow up of* X *along* Z *and* E *be the exceptional divisor dominating* Z*. Show that* $a_E(X, D) = k - 1 - \sum a_i \cdot \mathrm{mult}_Z D_i$.

The numbers $a_P(X, D)$ will allow us to define several important classes of singularities that are essential for the Minimal Model Program. The idea is that the bigger the discrepancy or total discrepancy of (X, D) is, then the less singular the pair (X, D) is. It is important to notice the following:

Lemma 2.11. *If the total discrepancy of (X, D) is < -1, then the total discrepancy of (X, D) is $-\infty$.*

Proof. Exercise. □

Definition 2.12. *A pair (X, D) is* **log canonical** *(respectively* **kawamata log terminal***) if $a_P(X, D) \geq -1$ (resp. $a_P(X, D) > -1$) for all prime divisors P over X. A pair (X, D) is* **canonical** *(respectively* **terminal***) if $a_P(X, D) \geq 0$ (resp. $a_P(X, D) > 0$) for all prime divisors P exceptional over X.*

Remark 2.13. *The condition that (X, D) is log canonical or kawamata log terminal can be checked on any log resolution of (X, D). It is known that kawamata log terminal singularities are rational (i.e. for any resolution $f : Y \to X$, we have $R^i f_* \mathcal{O}_Y = 0$ for $i > 0$) and Cohen-Macaulay cf.* [10, §5].

Remark 2.14. *If $\dim X = 2$ and (X, D) is a terminal pair, then X is smooth. If $\dim X = 2$ and (X, D) is a canonical pair, then X has at most rational double point singularities which are not contained in* Supp(D).

Remark 2.15. *If $\dim X = 2$ then $(X, 0)$ is a terminal (resp. canonical, kawamata log terminal, log canonical) pair if and only if X is smooth (resp. $\mathbb{C}^2$/*finite subgroup of *$SL(2, \mathbb{C})$, $\mathbb{C}^2$/*finite subgroup of *$GL(2, \mathbb{C})$), simple elliptic, cusp, smooth, or a quotient of these by a finite group).*

Exercise 2.16. *If X is the cone over a curve of genus g, and E is the exceptional divisor corresponding to the blow up of the vertex. Show that $a_E(X, 0) = -1$ (resp. $-1 + 2/n$, < -1) iff $g = 1$ (resp. $g = 0$ is a rational curve of degree $n > 0$, $g \geq 2$).*

Remark 2.17. *As observed above, if $x \in X$ is a rational double point, then $(X, 0)$ is canonical but not terminal. If $x \in X$ is the vertex of a cone over a rational curve, then $(X, 0)$ is Kawamata log terminal, but not canonical. If $x \in X$ is the vertex of a cone over an elliptic curve, then $(X, 0)$ is log canonical but not Kawamata log terminal.*

Exercise 2.18. *Given a log pair (X, D) and two log resolutions $f : Y \to X$ and $f' : Y' \to X$ such that $f' = f \circ \nu$ for some morphism $\nu : X' \to X$, show that $\nu_* A_{Y'}(X, D) = A_Y(X, D)$.*

Definition 2.19. *We say that a pair (X, D) is* **purely log terminal** *if the discrepancy of any exceptional divisor is greater than -1.*

Remark 2.20. *The notion of a purely log terminal pair (X, D) is particularly useful when $S = \llcorner D \lrcorner$ is irreducible. In this case S is normal* *and the pair (S, Θ) defined by adjunction $(K_X + D)|_S = K_S + \Theta$ is Kawamata log terminal.*

Definition 2.21. *We say that a pair (X, D) is* **divisorially log terminal** *if there is a log resolution $f : Y \to X$ such that all f-exceptional divisors $E \subset Y$ have discrepancy greater than -1.*

Remark 2.22. *If $X = \mathbb{P}^2$ and D is a curve with a node, then (X, D) is log canonical but not divisorially log terminal. If $X = \mathbb{C}^2$ and D is the union of the x and y axis, then (X, D) is divisorially log terminal but not purely log terminal.*

Proposition 2.23. *Given a divisorially log terminal pair (X, D), there is a resolution $f : Y \to X$ which is an isomorphism at the general point of each component of the strata of $\llcorner D \lrcorner$.*

Proof. [14] □

Remark 2.24. *Using the above proposition, one can show that a pair (X, D) is divisorially log terminal if and only if there is a closed subset $Z \subset X$ such that $(X - Z, D|_{X-Z})$ is log smooth (cf. 2.29) and if E is a divisor over X with center contained in Z, then $a_E(X, D) > -1$.*

Exercise 2.25. *A divisorially log terminal pair (X, D) is Kawamata log terminal if and only if $\llcorner D \lrcorner = 0$.*

Definition 2.26. *Given a log pair (X, D), a* **place of non Kawamata log terminal singularities** *of (X, D) is a divisor E over X such that $a_E(X, D) \leq -1$. A* **center of non Kawamata log terminal singularities** *of (X, D) is the image of a place of non Kawamata log terminal singularities of (X, D). We let the* **non Kawamata log terminal locus of** *(X, D) denoted by $\mathrm{Nklt}(X, D)$ be the subset of X defined by the union of all centers of non Kawamata log terminal singularities of (X, D).*

Remark 2.27. *Traditionally non Kawamata log terminal places or centers are called log canonical places or centers. This is meaningful for log canonical pairs, but otherwise confusing.*

Remark 2.28. *One can similarly define places and centers of non log canonical, non canonical and non terminal singularities. In the case of non canonical and non terminal singularities, one should only consider divisors E exceptional over X.*

Definition 2.29. *A pair (X, D)* **is log smooth** *if X is smooth and D has simple normal crossings.*

Remark 2.30. *If* (X, D) *is log smooth, then* $\mathrm{Nklt}(X, D) = \mathrm{Supp}(\llcorner D \lrcorner)$. *Each component of the strata of* $\llcorner D \lrcorner$ *is a non Kawamata log terminal center of* (X, D). *If* X *is smooth and* P *is a prime divisor on* X *and* $Z \subset P$ *is a subvariety of codimension* $c \leq \mathrm{mult}_P(D)$ *in* X, *then* Z *is a non Kawamata log terminal centers of* (X, D).

Exercise 2.31. *If* (X, D) *is a Kawamata log terminal pair, then there is a log resolution* $f : Y \to X$ *such that* $\Gamma_Y(X, D)$ *is smooth.*

Exercise 2.32. *If* (X, D) *is a Kawamata log terminal pair, then* (X, D) *has finitely many places of discrepancy* $a_E(X, D) > 0$.

Definition 2.33. *If* (X, D) *is a log canonical pair,* $Z \subset X$ *is a closed subscheme and* $G \in \mathrm{Div}_{\mathbb{R}}(X)$ *is an effective* $\mathbb{R}$*-divisor. Then the* **log canonical threshold of** G *along* Z *with respect to* (X, D) *is given by*

$$c_Z(X, D; G) := \sup\{c > 0 | (X, D + cG) \text{ is LC near } Z\}.$$

Note that in order to compute $c_Z(X, D; G)$ it suffices to pick a log resolution $f : Y \to X$ of $(X, D + G)$, then $c_Z(X, D; G)$ is given by the supremum of $c \in \mathbb{R}$ such that $\mathrm{mult}_E(-A_Y(X, D) + f^*G) \leq 1$ for all divisors E on Y whose image intersects Z. Equivalently $c = \min\{\frac{1+a_E(X,D)}{\mathrm{mult}_E(f^*G)}\}$ for all divisors E on Y whose image intersects Z. If $Z = X$ we let $c_Z(X, D; G) =: c(X, D; G)$.

Exercise 2.34. *Let* $X = \mathbb{C}^2$ *and* G *be the cusp defined by* $y^2 = x^3$. *Show that* $c(X, 0; G) = 5/6$.

Exercise 2.35. *Let* X *be the cone over a rational curve of degree* n *and* G *be a line through the vertex* $v \in X$. *Show that* $c(X, 0; G) = 1$ *and* (X, G) *is PLT.*

2.1. **Vanishing theorems.**

Theorem 2.36. *[Kodaira Vanishing Theorem] Let* X *be a smooth projective variety and* $D \in \mathrm{Div}(X)$ *an ample divisor, then*

$$H^i(X, \mathcal{O}_X(K_X + D)) = 0 \qquad \text{for all } i > 0.$$

Remark 2.37. *By Serre duality, this is equivalent to the condition that* $H^i(X, \mathcal{O}_X(-D)) = 0$ *for all* $i < \dim X$.

In applications, it is usually necessary to have a more flexible version of (2.36). The following theorem is often sufficient.

Theorem 2.38. *[Kawamata-Viehweg Vanishing] Let* X *be a smooth projective variety and* $D \in \mathrm{Div}(X)$. *If* $D \equiv M + F$ *where* $M \in \mathrm{Div}_{\mathbb{Q}}(X)$ *is nef and big and* $F \in \mathrm{Div}_{\mathbb{Q}}(X)$ *has simple normal crossings and* $\llcorner F \lrcorner = 0$.

Then $H^i(X, \mathcal{O}_X(K_X + D)) = 0$ *for all* $i > 0$.

Exercise 2.39. *Use* (2.38) *to deduce that if* X *is a smooth projective variety,* $f : X \to Y$ *is a projective morphism and* $D \in \mathrm{Div}(X)$*,* $D \equiv M + F$ *where* $M \in \mathrm{Div}_{\mathbb{Q}}(X)$ *is relatively nef and big and* $F \in \mathrm{Div}_{\mathbb{Q}}(X)$ *has simple normal crossings and* $\llcorner F \lrcorner = 0$*, then* $R^i f_*(X, \mathcal{O}_X(K_X + D)) = 0$ *for all* $i > 0$*.*

The above theorem generalizes to the following:

Theorem 2.40. *[General Kawamata-Viehweg Vanishing] Let* (X, Δ) *be a Kawamata log terminal pair and* $D \in \mathrm{WDiv}(X)$*. If* $D \equiv \Delta + M$ *where* $M \in \mathrm{Div}_{\mathbb{Q}}(X)$ *is nef and big.*

Then $H^i(X, \mathcal{O}_X(K_X + D)) = 0$ *for all* $i > 0$*.*

Proof. Assume for simplicity that $D \in \mathrm{Div}(X)$. Let $f : Y \to X$ be a log resolution. Then since $-E_Y + \ulcorner E_Y \urcorner = -E_Y - \llcorner -E_Y \lrcorner = \{-E_Y\}$, we have

$$f^*(K_X + D) + \ulcorner E_Y \urcorner \equiv K_Y + \Gamma_Y + \{-E_Y\} + f^*M$$

where f^*M is nef and big. It follows that $R^i f_* \mathcal{O}_Y(f^*(K_X + D) + \ulcorner E_Y \urcorner) = 0$ for $i > 0$ and that $H^i(\mathcal{O}_Y(f^*(K_X + D) + \ulcorner E_Y \urcorner)) = 0$ for $i > 0$ (cf. the log smooth case of (2.39)). But then, as

$$f_* \mathcal{O}_Y(f^*(K_X + D) + \ulcorner E_Y \urcorner) = \mathcal{O}_X(K_X + D),$$

we have

$$H^i(\mathcal{O}_X(K_X + D)) \cong H^i(\mathcal{O}_Y(f^*(K_X + D) + \ulcorner E_Y \urcorner))$$

and the theorem follows. □

The above theorem is a special case of the following

Theorem 2.41. *[Relative Kawamata-Viehweg Vanishing] Let* (X, Δ) *be a Kawamata log terminal pair and* $D \in \mathrm{WDiv}(X)$*. If* $f : X \to Y$ *is a projective morphism and* $D \equiv \Delta + M$ *where* $M \in \mathrm{Div}_{\mathbb{Q}}(X)$ *is nef and big over* Y *(i.e.* $M \cdot C \geq 0$ *for any curve* $C \subset X$ *contracted by* f *and* $M|_{X_\eta}$ *is big where* X_η *is the general fiber of* f*).*

Then $R^i f_* \mathcal{O}_X(K_X + D) = 0$ *for all* $i > 0$*.*

Proof. Let H be a sufficiently ample divisor on Y, then $M + f^*H$ is nef and big, $R^i f_* \mathcal{O}_X(K_X + D) \otimes \mathcal{O}_Y(H)$ is generated by global sections and $H^j(X, R^i f_* \mathcal{O}_X(K_X + D) \otimes \mathcal{O}_Y(H)) = 0$ for all $j > 0$. By the projection formula and a spectral sequence argument we have that $H^i(X, \mathcal{O}_X(K_X + D + f^*H)) \cong H^0(Y, R^i f_* \mathcal{O}_X(K_X + D) \otimes \mathcal{O}_Y(H))$. By (2.40) the group on the left vanishes and since $R^i f_* \mathcal{O}_X(K_X + D) \otimes \mathcal{O}_Y(H)$ is generated by global sections, then $R^i f_* \mathcal{O}_X(K_X + D) \otimes \mathcal{O}_Y(H) = 0$. Since $\mathcal{O}_Y(H) = 0$ is locally free, the claim follows. □

Exercise 2.42. *Let M and H be as above. Show that $M + f^*H$ is nef and big.*

We will need a slightly more general version that applies to divisorially log terminal pairs.

Theorem 2.43. *Let (X, D) be a log smooth log canonical pair and $f : X \to Z$ be a projective morphism. Let $N \in \mathrm{Div}(X)$ be a divisor such that $N - D$ is nef over Z and big over Z and the restriction of N to any non-Kawamata log terminal center of (X, D) is big over Z.*

Then

$$R^i f_* \mathcal{O}_X(K_X + N) = 0, \qquad \text{for } i > 0.$$

Proof. We proceed by induction on the dimension of X. If $\dim X = 1$, then $\deg(N) > 0$ and the claim follows by (2.41). If $\dim X \geq 2$, then we proceed by induction on the number of components of $\llcorner D \lrcorner$. If $\llcorner D \lrcorner = 0$, the claim follows by (2.41). Otherwise, let $S \in \llcorner D \lrcorner$ be any prime divisor and consider the short exact sequence

$$0 \to \mathcal{O}_X(K_X + N - S) \to \mathcal{O}_X(K_X + N) \to \mathcal{O}_S(K_S + (N - S)|_S) \to 0.$$

By induction on the number of components of $\llcorner D \lrcorner$, we have that $R^i f_* \mathcal{O}_X(K_X + N - S) = 0$ for all $i > 0$ and by induction on the dimension, we obtain that $R^i f_*(K_S + (N - S)|_S) = 0$. The assertion now follows immediately. □

2.2. Calculus of non Kawamata log terminal centers.

Theorem 2.44. *[The connectedness lemma of Kollár and Shokurov] Let $f : X \to Z$ be a proper morphism of normal varieties with connected fibers and $D \in \mathrm{WDiv}_{\mathbb{Q}}(X)$ such that $-(K_X + D) \in \mathrm{Div}_{\mathbb{Q}}(X)$ is f-nef and f-big. Write $D = D^+ - D^-$ where D^+ and D^- are effective with no common components. If D^- is f-exceptional (i.e. all of its components have image of codimension at least 2), then*

$$\mathrm{Nklt}(X, D) \cap f^{-1}(z)$$

is connected for any $z \in Z$.

Proof. Let $\mu : Y \to X$ be a log resolution of (X, D) and $D_Y = -A_Y(X, D)$. Then

$$\mathrm{Nklt}(X, D) = \mu(\mathrm{Nklt}(Y, D_Y)).$$

Replacing X by Y, we may assume that X is smooth and D has simple normal crossings support. We write $D = D^{\geq 1} + D^{<1}$ for the decomposition of D in to components of multiplicity ≥ 1 and < 1 respectively.

In particular $\mathrm{Nklt}(X, D) = \mathrm{Supp}(D^{\geq 1})$. Consider the short exact sequence

$$0 \to \mathcal{O}_X(\ulcorner -D \urcorner) \to \mathcal{O}_X(\ulcorner -D^{<1} \urcorner) \to \mathcal{O}_S(\ulcorner -D^{<1} \urcorner) \to 0$$

where $S = \llcorner D^{\geq 1} \lrcorner$. Since

$$\ulcorner -D \urcorner = K_X + \ulcorner -(K_X + D) \urcorner \equiv K_X - (K_X + D) + \{K_X + D\},$$

by Kawamata Viehweg vanishing, we have that $R^1 f_* \mathcal{O}_X(\ulcorner -D \urcorner) = 0$ so that the homomorphism

$$f_* \mathcal{O}_X(\ulcorner -D^{<1} \urcorner) \to f_* \mathcal{O}_S(\ulcorner -D^{<1} \urcorner)$$

is surjective. Now, $\ulcorner -D^{<1} \urcorner = -\llcorner D^{<1} \lrcorner \geq 0$ is effective and exceptional and so $f_* \mathcal{O}_X(\ulcorner -D^{<1} \urcorner) = \mathcal{O}_Z$. It follows that $\mathcal{O}_Z \to f_* \mathcal{O}_S(\ulcorner -D^{<1} \urcorner)$ is surjective. As $\ulcorner -D^{<1} \urcorner$ is effective, we have an inclusion $f_* \mathcal{O}_S \subset f_* \mathcal{O}_S(\ulcorner -D^{<1} \urcorner)$ and hence a surjection $\mathcal{O}_Z \to \mathcal{O}_{f(S)} \to f_* \mathcal{O}_S$. Therefore $S \to f(S)$ has connected fibers. □

Remark 2.45. *There are two main cases of interest in the above Theorem. If $Z = \mathrm{Spec}(\mathbb{C})$ so that (X, D) is a weak log Fano, then $\mathrm{Nklt}(X, D)$ is connected. If $f : X \to Z$ is birational, (Z, B) is a log pair $D = -A_X(Z, B)$ and (X, D) is log smooth, then this says that the fibers of the log canonical places of (Z, B) on any log resolution are connected.*

Theorem 2.46. *Let (X, D) be a log canonical pair such that (X, D_0) is Kawamata log terminal for some $D_0 \in \mathrm{WDiv}_{\mathbb{Q}}(X)$. If W_1 and W_2 are non Kawamata log terminal centers of (X, D), then so is any irreducible component W of $W_1 \cap W_2$. Therefore, for any point $x \in X$ such that (X, D) is not Kawamata log terminal near x, there is a unique minimal center of not Kawamata log terminal singularities for (X, D) containing x.*

Proof. The question is local, so we may assume that X is affine and $W = W_1 \cap W_2$. Pick D_i general divisors containing W_i and $\mu : Y \to X$ a log resolution of $(X, D + D_0 + D_1 + D_2)$ such that there are divisors $E_i \subset Y$ which are non Kawamata log terminal places of (X, D) with centers W_i. Therefore, we have that $\mathrm{mult}_{E_i} \Gamma_Y(X, D) = 1$. Let $e_i = \mathrm{mult}_{E_i} \mu^* D$ and $e'_i = \mathrm{mult}_{E_i} \mu^* D_i$. By our assumptions $e_i, e'_i > 0$. Let $a_i = \frac{e_i}{e'_i}$, then E_1 and E_2 are non Kawamata log terminal places of $(X, (1-\epsilon)D + \epsilon(a_1 D_1 + a_2 D_2))$ for $0 < \epsilon \ll 1$ and $\mathrm{NKLT}(X, (1-\epsilon)D + \epsilon(a_1 D_1 + a_2 D_2)) = W_1 \cup W_2$. By the Connectedness Theorem (2.44), for any $\epsilon > 0$ there are non Kawamata log terminal places $F_i(\epsilon) \subset Y$ of $(X, (1-\epsilon)D + \epsilon(a_1 D_1 + a_2 D_2))$ with centers contained in W_i such that $F_1(\epsilon) \cap F_2(\epsilon) \neq \emptyset$. We may assume that $F_i(\epsilon) = F_i$ are independent of ϵ (by finiteness of the number of exceptional divisors). By continuity, these are also non Kawamata log terminal places of (X, D). But then $W = f(F_1) \cap f(F_2)$ is a non Kawamata log terminal center of (X, D). □

Theorem 2.47. *Let (X, D) be a log canonical pair and W a minimal non Kawamata log terminal center of (X, D). Assume that (X, D_0) is Kawamata log terminal for some $D_0 \in \mathrm{WDiv}_{\mathbb{Q}}(X)$. If W is a prime divisor, then there exists a divisor $D_W \in \mathrm{WDiv}_{\mathbb{Q}}(W)$ such that (W, D_W) is Kawamata log terminal and*

$$(K_X + D)|_W = K_W + D_W.$$

Let $f : Y \to X$ be a log resolution of (X, D), $W' = (f^{-1})_* W$ and set $K_W + D_W = (f|_{W'})_*((K_Y - A_Y(X, D))|_{W'})$.

Proof. See [9, §16]. □

Remark 2.48. *It is conjectured that (2.47) holds regardless of the codimension of W. By a result of Kawamata, it is known that if H is ample and $\epsilon > 0$ is a rational number, then there exists a divisor $D_W \in \mathrm{WDiv}_{\mathbb{Q}}$ such that*

$$(K_X + D + \epsilon H)|_W \sim_{\mathbb{Q}} K_W + D_W,$$

and (W, D_W) is Kawamata log terminal.

2.3. **Rational Singularities.**

Definition 2.49. *A variety Y has rational singularities if there is a resolution $f : X \to Y$ such that $f_*\mathcal{O}_X = \mathcal{O}_Y$ and $R^i f_* \mathcal{O}_X = 0$ for all $i > 0$*

Remark 2.50. *Y has rational singularities if and only for any resolution $f : X \to Y$, we have $f_*\mathcal{O}_X = \mathcal{O}_Y$ and $R^i f_*\mathcal{O}_X = 0$ for all $i > 0$. It is also known that Y has rational singularities if and only if Y is Cohen-Macaulay and for some resolution $f : X \to Y$, we have $f_*\omega_Y = \omega_X$.*

Definition 2.51. *A coherent sheaf F on a scheme X is S_d at a point $x \in X$, if so is its stalk F_x as a module over the local ring $\mathcal{O}_{x,X}$. This means that there is a F_x regular sequence $x_1, \ldots, x_r \in \mathfrak{m}_x$ of length $r = \min\{d, \dim \mathcal{O}_{x,X}\}$ i.e. x_i is not a zero divisor for $F_x/(x_1, \ldots, x_{i-1})F_x$. F is S_d on X if it is S_d at every point $x \in X$. X is S_d if $\mathcal{O}_X$ is S_d.*

A coherent sheaf F on a scheme X is **Cohen-Macaulay** *if for any point $x \in X$ it is S_d for $d = \dim \mathrm{Supp} F_x$ (i.e. if it admits a regular sequence of length equal to the dimension of its support). In other words F is Cohen-Macaulay at x if there are elements $x_1, \ldots, x_r \in \mathfrak{m}_x$*

with $r = \dim \operatorname{Supp}(F_x)$ *and the image of* x_i *in* $F_x/(x_1, \ldots, x_{i-1})F_x$ *is not a zero divisor for* $1 \le i \le r$*. Equivalently* F *is Cohen-Macaulay at* x *if there is an element* $y \in \mathfrak{m}_x$ *such that its image in* F_x/yF_x *is not a zero divisor and* F_x/yF_x *is Cohen-Macaulay or if there are elements* $x_1, \ldots, x_r \in \mathfrak{m}_x$ *with* $r = \dim \operatorname{Supp}(F_x)$ *and* $\dim \operatorname{Supp} F_x/(x_1, \ldots, x_r)F_x = 0$*.*

Remark 2.52 (Serre's Criterion)**.** *If* $\dim X = 2$ *and* $x \in X$ *is an isolated singularity, then* X *is Cohen-Macaulay at* x *if and only if it is normal.*

Theorem 2.53. *If* X *is a normal projective variety and* $A \in \operatorname{Div}(X)$ *is ample on* X*, then a coherent sheaf* F *on* X *with* $\dim \operatorname{Supp}(F) = n$ *then* F *is Cohen-Macaulay if and only if* $H^i(X, F \otimes \mathcal{O}_X(-rA)) = 0$ *for* $i < n$ *and* $r \gg 0$*.*

Proof. [10, 5.72]. ☐

Proposition 2.54. *If* X *is a normal projective variety and* $f : Y \to X$ *a resolution, then* f *is a rational resolution if and only if* X *is Cohen-Macaulay and* $f_*\omega_Y = \omega_X$*.*

Proof. (see [10, 5.12]) Let $A \in \operatorname{Div}(X)$ be ample. Since f^*A is nef and big, by (2.38), one sees that $H^i(Y, \omega_Y(rf^*A)) = 0$ for all $i > 0$ and $r > 0$. and by Serre duality we have

$$H^{n-i}(Y, \mathcal{O}_Y(-rf^*A)) = 0,$$

where $n = \dim X$.

If f is a rational resolution, then $R^i f_*\mathcal{O}_Y = 0$ for $i > 0$ and so by an easy spectral sequence argument,

$$H^j(X, f_*\mathcal{O}_Y \otimes \mathcal{O}_X(-rA)) = H^j(Y, \mathcal{O}_Y(-rf^*A)) = 0$$

for any $r > 0$ and $j < n$. Since X is normal $f_*\mathcal{O}_Y = \mathcal{O}_X$ and so by (2.53), X is Cohen-Macaulay. Notice that we also have

$$h^0(X, \omega_X(rA)) = h^n(X, \mathcal{O}_X(-rA)) = h^n(Y, \mathcal{O}_Y(-rf^*A))$$
$$= h^0(Y, \omega_Y(rf^*A)) = h^0(X, f_*\omega_Y(rA))$$

where the first equality holds by [10, 5.71]. But since $\omega_X(rA)$ and $f_*\omega_Y(rA)$ are generated for $r \gg 0$, it follows that the inclusion $f_*\omega_Y \to \omega_X$ is an isomorphism.

Suppose now that X is Cohen-Macaulay and $f_*\omega_Y = \omega_X$. Proceeding by induction on the dimension and cutting down by hyperplanes we may assume that $R^i f_*\mathcal{O}_Y$ is supported on points for any $i > 0$. Thus $H^j(X, R^i f_*\mathcal{O}_Y(-rA)) = 0$ for $i, j > 0$ and since X is Cohen-Macaulay

$H^j(X, f_*\mathcal{O}_Y(-rA)) = 0$ for $j < n$ and $r \gg 0$. By an easy spectral sequence argument,

$$H^0(X, R^i f_* \mathcal{O}_Y \otimes \mathcal{O}_X(-rA)) = H^i(Y, \mathcal{O}_Y(-rf^*A)) = 0$$

for any $r > 0$ and $0 \leq i \leq n-2$. Thus $R^i f_* \mathcal{O}_Y = 0$ for $1 \leq i \leq n-2$. Also there is a short exat sequence

$$0 \to H^0(R^{n-1} f_* \mathcal{O}_Y(-rA)) \to H^n(\mathcal{O}_X(-rA)) \to H^n(\mathcal{O}_Y(-rf^*A)) \to 0.$$

We claim that the last map is an isomorphism and hence $R^{n-1} f_* \mathcal{O}_Y = 0$.

To see the claim note that as $f_*\omega_Y = \omega_X$, then

$$H^0(Y, \omega_Y(rf^*A)) \cong H^0(X, f_*\omega_Y(rA)) \cong H^0(X, \omega_X(rA))$$

and the claim follows by Serre duality. □

Theorem 2.55. *If (X, D) is a divisorially log terminal pair, then X has rational singularities.*

Proof. We will assume that X is projective, the pair (X, D) is Kawamata log terminal. Consider a log resolution $f : Y \to X$. We may write

$$\ulcorner E_Y(X, D) \urcorner \equiv K_Y - f^*(K_X + D) + \Gamma_Y(X, D) + \{-E_Y(X, D)\}.$$

As $\Gamma_Y(X, D) + \{-E_Y(X, D)\}$ has simple normal crossings and $\llcorner \Gamma_Y(X, D) + \{-E_Y(X, D)\} \lrcorner = 0$, then

$$R^j f_* \mathcal{O}_Y(\ulcorner E_Y(X, D) \urcorner) = 0 \qquad \forall i > 0.$$

Note that we also have $f_* \mathcal{O}_Y(\ulcorner E_Y(X, D) \urcorner) = \mathcal{O}_X$. Let $A \in \mathrm{Div}(X)$ be ample. We have a diagramm

$$\begin{array}{ccc} H^i(\mathcal{O}_X(-rA)) & \xrightarrow{=} & H^i(\mathcal{O}_X(-rA)) \\ \downarrow \beta & & \downarrow \alpha \\ H^i(\mathcal{O}_Y(-rf^*A)) & \longrightarrow & H^i(\mathcal{O}_Y(\ulcorner E_Y(X, D) \urcorner - rf^*A)). \end{array}$$

The existence of the vertical map β follows since there is a map of complexes $\mathcal{O}_X(-rA) = f_*\mathcal{O}_Y(-rf^*A) \to R^{\cdot} f_* \mathcal{O}_Y(-rf^*A)$ and hence of cohomology groups

$$H^i(X, \mathcal{O}_X(-rA)) \to \mathbb{H}^i(X, R^{\cdot} f_* \mathcal{O}_Y(-rf^*A)) = H^i(Y, \mathcal{O}_Y(-rf^*A)).$$

From the Leray Spectral sequence, we also get that α is an isomorphism. But as $h^i(\mathcal{O}_Y(-rf^*A)) = h^{n-i}(\omega_Y(rf^*A)) = 0$ for $i < n$, we have that $H^i(\mathcal{O}_X(-rA)) = 0$ for $i < n$ and $r > 0$. It is easy to see that the diagram commutes. Therefore X is CM. When $i = n$, we get an injection

$$H^n(\mathcal{O}_X(-rA)) \to H^n(\mathcal{O}_Y(-rf^*A))$$

i.e. a surjection

$$H^0(\omega_Y(rf^*A)) = H^0(f_*\omega_Y \otimes \mathcal{O}_X(rA)) \to H^0(\omega_X \otimes \mathcal{O}_X(rA))$$

so that $f_*\omega_Y \to \omega_X$ is surjective. □

3. Multiplier ideal sheaves

Definition 3.1. *Let $D \geq 0$ be a $\mathbb{R}$-divisor on a smooth variety X and $f : Y \to X$ be a log resolution of (X, D). Then the* **multiplier ideal sheaf of** (X, D) *is defined as*

$$\mathcal{J}(X, D) = \mathcal{J}(D) = \mathcal{J}_D := f_*\mathcal{O}_Y(K_{Y/X} - \llcorner f^*D \lrcorner).$$

Notice that $\mathcal{J}(X, D) \subset f_*\mathcal{O}_Y = \mathcal{O}_X$. In order to show that multiplier ideal sheaves are well defined, one needs the following.

Proposition 3.2. *The definition in* (3.1) *does not depend on the log resolution.*

Proof. Let $f : Y \to X$ and $f' : Y' \to X$ be two log resolutions of (X, D). We may assume that $f' = f \circ g$ for some morphism $g : Y' \to Y$. We have

$$f'_*\mathcal{O}_{Y'}(K_{Y'/X} - \llcorner (f')^*D \lrcorner) = f_*(\mathcal{O}_Y(K_{Y/X}) \otimes g_*\mathcal{O}_{Y'}(K_{Y'/Y} - \llcorner g^*f^*D \lrcorner))$$

and so it suffices to prove that $g_*\mathcal{O}_{Y'}(K_{Y'/Y} - \llcorner g^*f^*D \lrcorner) = \mathcal{O}_{Y'}(-\llcorner f^*D \lrcorner)$. This follows from Lemma (3.3) below (cf. [12, Lemma 9.2.19]). □

Lemma 3.3. *Let X be a smooth variety and D be a divisor with simple normal crossings support and $f : Y \to X$ be a log resolution of (X, D), then*

$$f_*\mathcal{O}_Y(K_{Y/X} - \llcorner f^*D \lrcorner) = \mathcal{O}_X(-\llcorner D \lrcorner).$$

Proof. Using the projection formula, it is easy to see that we may assume that $D = \{D\}$. We must then show that $K_{Y/X} - \llcorner f^*D \lrcorner \geq 0$. This can be done by a local computation. Let E be any divisor in Y with center Z on X. We may work locally around a general point of Z and assume that $D = \sum d_i D_i$ where $Z \subset \mathrm{Supp}D$. Let x_i be local coordinates on X with $D_i = \{x_i = 0\}$ and y_i be local coordinates on Y with $E = \{y_1 = 0\}$. We let $c_i = \mathrm{mult}_E(f^*D_i)$ so that $\mathrm{mult}_E(f^*D) = \sum d_i c_i < \sum c_i$. We have $x_i = y_1^{c_i} \cdot b_i$ for some regular functions b_i on Y. It follows that $dx_i = y_1^{c_i - 1} c_i b_i dy_1 + y_1^{c_i} db_i$ and hence

$$dx_1 \wedge \ldots \wedge dx_n = y_1^{\gamma - 1} g dy_1 \wedge \ldots \wedge dy_n$$

for some regular function g on Y and $\gamma = \sum c_i$. Therefore

$$\text{mult}_E(K_{Y/X}) \geq \sum c_i - 1 > \text{mult}_E(f^*D) - 1.$$

□

Remark 3.4. *One could also define the multiplier ideal sheaf of a pair (X, Δ) with respect to a divisor $D \in \text{Div}_{\mathbb{R}}(X)$ by*

$$\mathcal{J}((X,\Delta); D) = f_*\mathcal{O}_Y(K_Y - \llcorner f^*(K_X + \Delta + D)\lrcorner)$$

where $f : Y \to X$ is a log resolution of $(X, \Delta + D)$. Note that $\mathcal{J}((X,\Delta); D) = \mathcal{J}((X,\Delta+D); 0)$ and if X is smooth, then $\mathcal{J}((X,\Delta); D) = \mathcal{J}(X, \Delta + D)$.

One should view the multiplier ideal sheaf $\mathcal{J}(X, D)$ as a measure of the singularities of (X, D). Notice for example that a pair (X, Δ) is kawamata log terminal if and only if $\mathcal{J}((X,\Delta); 0) = \mathcal{O}_X$. We have the following basic properties:

Proposition 3.5. *Let $D \geq 0$ be an $\mathbb{R}$-divisor on a smooth n-dimensional variety X.*

1. *If $G \in \text{Div}(X)$, then $\mathcal{J}(D + G) = \mathcal{J}(D) \otimes \mathcal{O}_X(-G)$.*
2. *If D has simple normal crossings support, then $\mathcal{J}(D) = \mathcal{O}_X(-\llcorner D \lrcorner)$.*
3. *If $D_1 \leq D_2$ with $0 \leq D_i \in \text{Div}_{\mathbb{R}}(X)$, then $\mathcal{J}(D_2) \subset \mathcal{J}(D_1)$.*
4. *If $f : Y \to X$ is a proper birational morphism of smooth varieties, then*
$$\mathcal{J}(X, D) = f_*(\mathcal{J}(Y, f^*D) \otimes \omega_{Y/X}).$$
5. *If $\text{mult}_x D \geq n$, then $\mathcal{J}(D) \subset \mathcal{I}_x$ where $x \in X$ is any point and $\mathcal{I}_x$ is the corresponding maximal ideal.*
6. *If $\text{mult}_x D < 1$ then $\mathcal{J}(D)_x = \mathcal{O}_{x,X}$.*

Proof. Properties (1-4) are easy excercises. To see property (5), let $\mu : X' \to X$ be the blow up of X at x and E be the exceptional divisor. By (4) and (1), we have

$$\mathcal{J}(D) = \mu_*(\mathcal{J}(\mu^*D) \otimes \omega_{X'/X}) = \mu_*(\mathcal{J}(\mu^*D - nE) \otimes \omega_{X'/X}(-nE))$$
$$\subset \mu_*(\omega_{X'/X}(-nE)) = \mu_*\mathcal{O}_{X'}(-E) \subset \mathcal{I}_x.$$

Property (6) is (3.20). □

Remark 3.6. *The same proof shows that if Z is an irreducible subvariety of dimension k and $\text{mult}_Z D \geq n - k + p - 1$, then $\mathcal{J}(D) \subset \mathcal{I}_Z^{<p>}$ where $\mathcal{I}_Z^{<p>}$ is the p-th symbolic power of $\mathcal{I}_Z$ i.e. the ideal of regular functions vanishing along a general point of Z to order at least p.*

Theorem 3.7. *[Nadel Vanishing] Let X be a smooth variety, $0 \leq D \in \operatorname{Div}_{\mathbb{R}}(X)$ and $f : X \to Z$ be a projective morphism. If $N \in \operatorname{Div}(X)$ is such that $N - D$ is f-nef and f-big, then*

$$R^i f_*(\mathcal{O}_X(K_X + N) \otimes \mathcal{J}(D)) = 0 \qquad \forall i > 0.$$

Proof. Let $g : Y \to X$ be a log resolution of (X, D), then $g^*(N - D)$ is h-nef and h-big where $h = f \circ g$. As $g^*(N - D)$ is also g-nef and g-big and it has simple normal crossings support, by (2.41), we have

$$R^j g_* \mathcal{O}_Y(K_Y + \ulcorner f^*(N - D) \urcorner) = 0 \qquad \forall j > 0.$$

Similarly, we have

$$R^i h_*(\mathcal{O}_Y(K_Y + \ulcorner f^*(N - D) \urcorner)) = 0 \qquad \forall i > 0.$$

Since

$$\begin{aligned} g_* \mathcal{O}_Y(K_Y + \ulcorner f^*(N - D) \urcorner) &= \mathcal{O}_X(K_X + N) \otimes g_* \mathcal{O}_Y(K_{Y/X} - \llcorner f^*(D) \lrcorner) \\ &= \mathcal{O}_X(K_X + N) \otimes \mathcal{J}(D), \end{aligned}$$

we have that

$$R^i f_*(\mathcal{O}_X(K_X + N) \otimes \mathcal{J}(D)) \cong R^i h_* \mathcal{O}_Y(K_Y + \ulcorner f^*(N - D) \urcorner) = 0 \qquad \forall i > 0.$$

□

Corollary 3.8. *Let X be a smooth projective variety, $0 \leq D \in \operatorname{Div}_{\mathbb{R}}(X)$, $N, B \in \operatorname{Div}(X)$ are such that $N - D$ is nef and big and $|B|$ is very ample, then*

$$\mathcal{O}_X(K_X + nB + N) \otimes \mathcal{J}(D)$$

is generated by global sections for all $n \geq \dim X$.

Proof. Let $X_i = B_1 \cap \ldots \cap B_i$ be the intersection of i general elements $B_i \in |B|$ that contain x. Consider the short exact sequences

$$\begin{aligned} 0 \to \mathcal{O}_{X_i}(K_X + jB + N) \otimes \mathcal{J}(D) &\to \mathcal{O}_{X_i}(K_X + (j+1)B + N) \otimes \mathcal{J}(D) \\ &\to \mathcal{O}_{X_{i+1}}(K_X + (j+1)B + N) \otimes \mathcal{J}(D) \to 0. \end{aligned}$$

By induction on i, one shows that $H^k(\mathcal{O}_{X_i}(K_X + jB + N)) = 0$ for all $k > 0$ and $j \geq i$. It follows that if $n \geq \dim X$, then $\mathcal{O}_X(K_X + nB + N) \otimes \mathcal{J}(D) \to \mathcal{O}_{X_{\dim X}}(K_X + nB + N) \otimes \mathcal{J}(D)$ is surjective and hence $\mathcal{O}_X(K_X + nB + N) \otimes \mathcal{J}(D)$ is generated at x. □

Remark 3.9. *More generally, if F is a coherent sheaf on a projective variety of dimension n and B is very ample such that $H^p(F \otimes \mathcal{O}_X(jB)) = 0$ for any $p > 0$ and $j \geq 0$, then $F \otimes \mathcal{O}_X(nB)$ is generated by global sections. To see this, consider the short exact sequence*

$$0 \to F'' \to F \to F' \to 0$$

where $\dim \operatorname{Supp}(F'') = 0$ *and* F' *contains no subsheaves supported at points. Then* F'' *is generated by global sections and we may apply the argument of* (3.8) *to* F'*.*

Definition 3.10. *Let* X *be a smooth variety and* V *a nonempty linear series on* X *(resp.* $0 \neq a \subset \mathcal{O}_X$ *an ideal). Pick* $f : Y \to X$ *a log resolution of* (X, V) *(resp. of* (X, a)*) i.e. a proper birational morphism* $f : Y \to X$ *such that* Y *is smooth,* $f^*V = V' + F$ *where* V' *is a free linear series and* $F = \operatorname{Fix}(f^*V)$*, and* $F + \operatorname{Exc}(f)$ *has simple normal crossings support (resp.* $a \cdot \mathcal{O}_Y = \mathcal{O}_Y(-F)$ *and* $F + \operatorname{Exc}(f)$ *has simple normal crossings support). For any* $0 < c \in \mathbb{R}$*, we define the corresponding multiplier ideal sheaf*

$$\mathcal{J}(c \cdot V) = f_*\mathcal{O}_Y(K_{Y/X} - \llcorner cF \lrcorner) \qquad (\text{resp. } \mathcal{J}(c \cdot a) = f_*\mathcal{O}_Y(K_{Y/X} - \llcorner cF \lrcorner).$$

Proposition 3.11. *Let* X *be a smooth variety and* $V_1 \subset V_2$ *be nonempty linear series on* X*,* $0 \neq a_1 \subset a_2 \subset \mathcal{O}_X$ *ideals, then*

$$\mathcal{J}(V_1) \subset \mathcal{J}(V_2) \qquad \text{and} \qquad \mathcal{J}(a_1) \subset \mathcal{J}(a_2).$$

If b *is the base ideal of a nonempty linear series* V *and* $D \in V$ *is a general member and* $0 < c < 1$ *is a real number, then*

$$\mathcal{J}(c \cdot V) = \mathcal{J}(c \cdot b) = \mathcal{J}(c \cdot D).$$

Proof. Exercise. □

3.1. **First geometric applications of multiplier ideals.** In this section we will discuss two geometric applications of multiplier ideals.

Theorem 3.12. *Let* S *be a finite set of points on* $\mathbb{P}^n$ *and* $D \subset \mathbb{P}^n$ *be an hypersurface of degree* d *such that* $\operatorname{mult}_x D \geq k$ *for all* $x \in S$*. Then there is a hypersurface of degree* $\leq \llcorner \frac{dn}{k} \lrcorner$ *containing* S*.*

Proof. Set $G = \frac{n}{k}A$ so that $\operatorname{mult}_x G \geq n$ for all $x \in S$. Therefore $\mathcal{J}(G) \subset \mathcal{I}_S$. Since $\mathcal{O}_{\mathbb{P}^n}(K_{\mathbb{P}^n}) = \mathcal{O}_{\mathbb{P}^n}(-n-1)$, we have that

$$H^i(\omega_{\mathbb{P}^n} \otimes \mathcal{O}_{\mathbb{P}^n}(l) \otimes \mathcal{J}(G)) = 0 \qquad \forall i > 0,\ l > \frac{dn}{k}.$$

Therefore, for $l \geq \llcorner \frac{dn}{k} \lrcorner + 1$, we have

$$H^0(\omega_{\mathbb{P}^n} \otimes \mathcal{O}_{\mathbb{P}^n}(l) \otimes \mathcal{J}(G)) = \chi(\omega_{\mathbb{P}^n} \otimes \mathcal{O}_{\mathbb{P}^n}(l) \otimes \mathcal{J}(G)) = P(l)$$

is a polynomial of degree n. So $P(l)$ has at most n zeroes and hence $H^0(\omega_{\mathbb{P}^n} \otimes \mathcal{O}_{\mathbb{P}^n}(l) \otimes \mathcal{J}(G)) \neq 0$ for some $l \leq \llcorner \frac{dn}{k} \lrcorner + n + 1$. □

Remark 3.13. *Conjecturally, one should be able to produce a hypersurface of degree* $\leq \llcorner \frac{dn}{k} \lrcorner - n$ *vanishing along* S*. It is hard to find interesting examples for a hypersurface* D *as above. It is maybe more* *interesting to think of the above theorem as giving necessary conditions for such a hypersurface to exist.*

The next application is due to J. Kollár.

Theorem 3.14. *Let (A, Θ) be a principally polarized abelian variety (in particular A is a complex torus, $\Theta \in \mathrm{Div}(X)$ is ample and $h^0(\mathcal{O}_A(\Theta)) = 1$). Then (A, Θ) is log canonical (i.e. $\mathcal{J}((1-\epsilon)\Theta) = \mathcal{O}_A$ for any $0 < \epsilon \ll 1$).*

In particular $\mathrm{mult}_x \Theta \leq \dim A$ *for any point $x \in A$.*

Proof. Consider the short exact sequence

$$0 \to \mathcal{O}_A(\Theta) \otimes \mathcal{J}((1-\epsilon)\Theta) \to \mathcal{O}_A(\Theta) \to \mathcal{O}_Z(\Theta) \to 0$$

where $Z = Z(\mathcal{J}((1-\epsilon)\Theta))$. By (3.7), we have $H^1(\mathcal{O}_A(\Theta) \otimes \mathcal{J}((1-\epsilon)\Theta)) = 0$ so that $H^0(\mathcal{O}_A(\Theta)) \to H^0(\mathcal{O}_Z(\Theta))$ is surjective. By (3) and (1) of (3.5), we have

$$\mathcal{I}_Z = \mathcal{J}((1-\epsilon)\Theta) \subset \mathcal{J}(\Theta) = \mathcal{I}_\Theta.$$

It then follows that $H^0(\mathcal{O}_Z(\Theta)) = 0$. By semicontinuity, for general $x \in A$, we have $H^0(\mathcal{O}_Z(t_x^*\Theta)) = 0$ where t_x denotes translation by $x \in A$. But then a general translate of Θ vanishes along Z so that $Z = \emptyset$. □

Remark 3.15. *The same proof shows that if $D \in |m\Theta|$, then $(A, \frac{1}{m}D)$ is log canonical and hence* $\mathrm{mult}_Z(D) \leq m(n-k)$ *where $Z \subset X$ is a subvariety of dimension k. It is also known that equality holds if and only if $(A, \Theta) \cong (A', \Theta') \times (A'', \Theta'')$ where (A', Θ') is the product of $n-k$ principally polarized elliptic curves.*

3.2. **Further properties of multiplier ideal sheaves.**

Theorem 3.16. *Let X be a smooth quasi-projective variety and $0 \leq D \in \mathrm{Div}_{\mathbb{R}}(X)$. If H is a smooth irreducible divisor on X not contained in the support of D, then*

$$\mathcal{J}(H, D|_H) \subset \mathcal{J}(X, D) \cdot \mathcal{O}_H$$

where $\mathcal{J}(X, D) \cdot \mathcal{O}_H := \mathrm{Im}(\mathcal{J}(X, D) \hookrightarrow \mathcal{O}_X \to \mathcal{O}_H) \subset \mathcal{O}_H$.

Moreover, if $0 < s < 1$, then for all $0 < t \ll 1$ we have

$$\mathcal{J}(X, D + (1-t)H) \cdot \mathcal{O}_H \subset \mathcal{J}(H, (1-s)D|_H).$$

Proof. Let $f : Y \to X$ be a log resolution of $(X, D + H)$ and write $f^*H = H' + \sum a_j E_j$ where $H' = (f^{-1})_* H$, $a_j \geq 0$ and E_j are exceptional. We may assume that $g = f|_{H'} : H' \to H$ is a log resolution of $(H, D|_H)$. By adjunction $K_{H'} = (K_Y + H')|_{H'}$ and $K_H = (K_X + H)|_H$, so that $K_{H'/H} = (K_{Y/X} - \sum a_j E_j)|_{H'}$. Consider the short exact sequence

$$0 \to \mathcal{O}_Y(K_{Y/X} - \llcorner f^*D \lrcorner - f^*H) \to \mathcal{O}_Y(K_{Y/X} - \llcorner f^*D \lrcorner - \sum a_j E_j)$$
$$\to \mathcal{O}_{H'}(K_{H'/H} + \llcorner f^*D|_H \lrcorner) \to 0.$$

Since $-\llcorner f^*D \lrcorner - f^*H \sim_{\mathbb{Q},f} \{f^*D\}$, we have that $R^1 f_*\mathcal{O}_Y(K_{Y/X} - \llcorner f^*D \lrcorner - f^*H) = 0$ cf. (2.41). Therefore there is a surjection

$$f_*\mathcal{O}_Y(K_{Y/X} - \llcorner f^*D \lrcorner - \sum a_j E_j) \to g_*\mathcal{O}_{H'}(K_{H'/H} + \llcorner f^*D|_{H'} \lrcorner) = \mathcal{J}(H, D|_H).$$

The first assertion now follows as

$$\mathcal{J}(X, D) = f_*\mathcal{O}_Y(K_{Y/X} - \llcorner f^*D \lrcorner) \supset f_*\mathcal{O}_Y(K_{Y/X} - \llcorner f^*D \lrcorner - \sum a_j E_j).$$

To see the second assertion, note that

$$\mathcal{J}(X, D + (1-t)H) = f_*\mathcal{O}_Y(K_{Y/X} - \llcorner f^*((1-t)H + D) \lrcorner)$$

and

$$\mathcal{J}(H, (1-s)D|_H) = g_*\mathcal{O}_{H'}(K_{H'/H} - \llcorner g^*((1-s)D|_H) \lrcorner).$$

Therefore, if $E \subset Y$ is any divisor on Y with $\mathrm{mult}_{E\cap H'}(K_{H'/H} - f^*((1-s)D|_H)) \leq -1$, then we must show that

$$\mathrm{mult}_E(K_{Y/X} - \llcorner f^*((1-t)H + D) \lrcorner) \leq \mathrm{mult}_{E\cap H'}(K_{H'/H} - \llcorner f^*((1-s)D|_H) \lrcorner).$$

Let $k = \mathrm{mult}_E(K_{Y/X})$, $a = \mathrm{mult}_E(f^*H)$ and $d = \mathrm{mult}_E(f^*D)$, then we must show that

$$k - \llcorner (1-t)a + d \lrcorner \leq k - a - \llcorner (1-s)d \lrcorner.$$

But for $0 < t \leq \frac{sd}{a}$, the equation is easily seen to hold. $\square$

Corollary 3.17. *[Inversion of adjunction] If $\mathcal{J}(H, D|_H) = \mathcal{O}_H$ near a point $x \in H$, then $\mathcal{J}(X, D) = \mathcal{O}_X$ near $x \in X$. In other words $(H, D|_H)$ is kawamata log terminal near x then (X, D) is kawamata log terminal near x.*

Corollary 3.18. *[Inversion of adjunction II] If $\mathcal{J}(H, (1-s)D|_H) \subset m_x$ for a point $x \in H$ and any number $0 < s < 1$, then $\mathcal{J}(X, D + (1-t)H) \subset m_x$ for any $0 < t \ll 1$. In other words, if $(H, (1-s)D|_H)$ is not kawamata log terminal near x then $(X, D + (1-t)H)$ is not kawamata log terminal near x.*

Remark 3.19. *A more general version of inversion of adjunction is the following. Let $(X, S + B)$ be a pair such that S is a prime divisor not contained in the support of B, let $\nu : S' \to S$ be the normalization of S and (S', B') be the log pair defined by the adjunction formula $\nu^*(K_X + S + B) = K_{S'} + B'$. Then*

(1) *$(X, S+B)$ is purely log terminal if and only if (S', B') is kawamata log terminal, and*
(2) *$(X, S+B)$ is log canonical if and only if (S', B') is log canonical.*

The implications $(X, S+B)$ is purely log terminal (resp. log canonical) implies that (S', B') is kawamata log terminal (resp. log canonical) is easy to see. The implication (S', B') is kawamata log terminal implies $(X, S+B)$ is purely log terminal follows from the Connectedness Lemma. The implication (S', B') is log canonical implies $(X, S+B)$ is log canonical is a deep result due to Kawakita.

Corollary 3.20. *If X is a smooth quasi-projective variety, $0 \leq D \in \mathrm{Div}_{\mathbb{Q}}(X)$ and $\mathrm{mult}_x(D) < 1$, then*

$$\mathcal{J}(X, D)_x = \mathcal{O}_{x,X}.$$

Proof. We proceed by induction on $n = \dim X$. The case $n = 1$ is clear. Assume $n > 1$ and fix $x \in H \subset X$ a smooth divisor not contained in the support of D. For a general choice of H, we have $\mathrm{mult}_x(D|_H) = \mathrm{mult}_x(D) < 1$. Therefore, $\mathcal{J}(H, D|_H)_x = \mathcal{O}_{x,H}$ and by (3.17), it follows that $\mathcal{J}(X, D)_x = \mathcal{O}_{x,X}$. □

Proposition 3.21. *Let X be a smooth variety, $0 \leq D \in \mathrm{Div}_{\mathbb{R}}(X)$ and $Z \subset X$ an irreducible subvariety of dimension d such that (X, D) is log canonical at the general point z of Z and Z is a non Kawamata log terminal center for (X, D). If B is an effective divisor whose support does not contain Z and such that*

$$\mathrm{mult}_z(B|_Z) > d,$$

then for any $0 < \epsilon \ll 1$, we have

$$\mathcal{J}(X, (1-\epsilon)D + B) \subset m_z.$$

Proof. Let $f : Y \to X$ be a log resolution of (X, D), then there is a divisor $E \subset Y$ with center Z such that $a_E(X, D) = -1$. We let $k = \mathrm{mult}_E(K_{Y/X})$ so that $\mathrm{mult}_E(f^*D) = k + 1$. Since $z \in Z$ is general, we may assume that $f|_E$ is smooth over z and we let E_z be the fiber over z. We have

$$\mathcal{J}(X, (1-\epsilon)D + B) = f_*\mathcal{J}(Y, f^*((1-\epsilon)D + B) - K_{Y/X}).$$

Since $\mathrm{mult}_E(B|_Z) > d$, it follows that

$$\mathrm{mult}_{E_z}(f^*((1-\epsilon)D + B) - K_{Y/X}) \geq d + 1 = \mathrm{codim}_Y E_z.$$

By (3.5), we have $\mathcal{J}(Y, f^*((1-\epsilon)D + B) - K_{Y/X}) \subset \mathcal{I}_{E_z}$ and the proposition follows easily. □

Theorem 3.22. *[Subadditivity for multiplier ideal sheaves] Let X be a smooth variety, $0 \leq D_i \in \mathrm{Div}_{\mathbb{R}}(X)$. Then*

$$\mathcal{J}(D_1 + D_2) \subset \mathcal{J}(D_1) \cdot \mathcal{J}(D_2).$$

Proof. See [12, 9.5.20]. □

Theorem 3.23. *Let $\pi : X \to T$ be a surjective morphism of smooth varieties. Then*

(1) *for general $t \in T$ we have $\mathcal{J}(X_t, D|_{X_t}) = \mathcal{J}(X, D) \cdot \mathcal{O}_{X_t}$, and*
(2) *if $\dim T = 1$, X_0 is a divisor contained in the fiber over $0 \in T$ and there is a section $g : T \to X$ such that $g(0) \in X_0$ and*

$$\mathcal{J}(X_t, D|_{X_t}) \subset m_g(t) \qquad \text{for } t \in T - 0$$

then $\mathcal{J}(X_0, D|_{X_0}) \subset m_{g(0)}$.

Proof. Since the assertion is local, we may assume that X is affine. Let $f : Y \to X$ be a log resolution of (X, D), then $\pi \circ f$ is smooth over an open subset U of T and the simple normal crossings divisor given by the support of f^*D and the exceptional locus of f, meets each fiber Y_t transversely for any $t \in U$. Then

$$(f^*D)|_{Y_t} = (f|_{Y_t})^*(D|_{X_t}) \qquad \text{and} \qquad K_{Y/X}|_{Y_t} = K_{Y_t/X_t}.$$

Consider now the short exact sequence

$$0 \to \mathcal{O}_Y(K_{Y/X} - \llcorner f^*D \lrcorner) \otimes \mathcal{I}_{Y_t} \to \mathcal{O}_Y(K_{Y/X} - \llcorner f^*D \lrcorner) \to \mathcal{O}_{Y_t}(K_{Y/X} - \llcorner f^*D \lrcorner) \to 0.$$

Since Y_t is obtained by intersecting the pull-backs of $\dim T$ general hypersurfaces of T containing t, one can show that $R^1 f_*(\mathcal{O}_Y(K_{Y/X} - \llcorner f^*D \lrcorner) \otimes \mathcal{I}_{Y_t}) = 0$ and hence the homomorphism

$$\mathcal{J}(X, D) \to \mathcal{J}(X_t, D|_{X_t})$$

is surjective and (1) follows.

For (2), notice that by (1), there is an open subset U of T such that over U we have an inclusion $\mathcal{J}(X, D) \subset \mathcal{I}_{g(T)}$. Since the zero set of $\mathcal{J}(X, D)$ is closed, the above inclusion holds over T. By (3.16), we have

$$\mathcal{J}(X_0, D|_{X_0}) \subset \mathcal{J}(X, D) \cdot \mathcal{O}_{X_0} \subset \mathcal{I}_{g(0)}$$

as required. □

3.3. **The theorem of Anhern and Siu.** Recall the following.

Conjecture 3.24 (Fujita's conjecture)**.** *Let X be a smooth projective variety of dimension n and A be an ample line bundle, then $K_X + (n+1)A$ is generated.*

Remark 3.25. *By a result of Kawamata, the conjecture is true in dimension ≤ 4. It is also conjectured that that $K_X + (n+2)A$ is very ample and that if $A^2 \geq 2$, then $K_X + nA$ is generated.*

While this appears to be a very bold conjecture, there is the following important result that works in all dimensions.

Theorem 3.26. *Let $x \in X$ be a point on a smooth projective variety of dimension n and A be an ample line bundle such that for any subvariety $x \in Z \subset X$, we have*

$$A^{\dim Z} \cdot Z > (\frac{n^2+n}{2})^{\dim Z}.$$

Then $K_X + A$ is generated at x.

Proof. It suffices to show that there is a divisor $D \sim_{\mathbb{Q}} cA$ such that $c < 1$ and x is an isolated component of $Z(\mathcal{J}(D))$. By (3.7), it then follows that the map

$$H^0(\mathcal{O}_X(K_X + A)) \to H^0(\mathcal{O}_X(K_X + A)/\mathcal{O}_X(K_X + A) \otimes \mathcal{J}(D))$$

is surjective. The theorem then follows as $H^0(\mathcal{O}_X(K_X + A)/\mathcal{O}_X(K_X + A) \otimes \mathcal{J}(D))$ surjects on to $H^0(\mathcal{O}_X(K_X + A)/\mathcal{O}_X(K_X + A) \otimes m_x)$.

In order to construct such a divisor, we will need several intermediate results.

Lemma 3.27. *[Constructing singular divisors] Let $x \in V$ be a smooth point on an irreducible projective variety of dimension d, $0 < a \in \mathbb{Q}$ and A an ample Cartier divisor on V such that $A^d > a^d$. Then, for any $k \gg 0$, there exists a divisor $A_k \in |kA|$ such that* $\operatorname{mult}_x(A) > ka$.

Proof. This follows easily as by (1.19), for $k \gg 0$ we have

$$h^0(\mathcal{O}_V(kA)) = \frac{k^d A^d}{d!} + O(k^{d-1})$$

and the number of conditions required to vanish to order $\geq m$ at the smooth point $x \in V$ is

$$\binom{d+m-1}{d} = \frac{m^d}{d!} + O(m^{d-1}).$$

□

Therefore, we may find a divisor $D_1 \sim_{\mathbb{Q}} c_1 A$ with $\operatorname{mult}_x(D_1) \geq n$ and $c_1 < \frac{n}{M}$ where $M = \frac{n^2+n}{2}$. Therefore (X, D_1) is not kawamata log terminal at x i.e. $\mathcal{J}(X, D_1)_x \neq \mathcal{O}_{x,X}$. Replacing D_1 by λD_1 (where $\lambda = c_x(X, 0; D_1)$ is the log canonical threshold) we may assume that (X, D_1) is log canonical but not kawamata log terminal at x. Perturbing D_1

by a general element of A, we may assume that (X, D_1) has a unique center of non kawamata log terminal singularities Z_1 at x.

We now proceed to show by induction on $\dim Z$ that for any $k > 0$ there exists a $\mathbb{Q}$-divisor $D_k \sim_{\mathbb{Q}} c_k A$ such that

(1) (X, D_k) is log canonical but not kawamata log terminal at x,
(2) (X, D_k) has a unique center of non kawamata log terminal singularities $Z_k \neq \emptyset$ at x with $\dim Z_k \leq n - k$, and
(3) $c_k < \frac{1}{M} \sum_{i=1}^{k} (n - i + 1)$.

The case $k = 1$ has already been established. Assume now that we have constructed D_k as above. Let $g : T \to Z_k$ be the normalization of a general curve containing x. For general $t \in T$, the point $z = g(t) \in Z_k$ is a general point. By (3.27), there is a divisor $G_t \sim_{\mathbb{Q}} gA|_{Z_k}$ such that $\operatorname{mult}_{g(t)} G_t > \dim Z_k$ and $g < \frac{k}{M}$. As A is ample, by (1.19) $H^1(\mathcal{O}_X(mA) \otimes \mathcal{I}_{Z_k}) = 0$ for all $m \gg 0$ so that $H^0(\mathcal{O}_X(mA)) \to H^0(\mathcal{O}_{Z_k}(mA))$ is surjective. Therefore, there is a divisor $G'_t \sim_{\mathbb{Q}} gA$ such that $G'_t|_{Z_k} = G_t$. We may assume that there exists $0 < m \in \mathbb{Z}$ such that $mG'_t \sim mA$. After replacing T by a finite cover, we may assume that there is a divisor $mG' \sim p_X^* mA \in \operatorname{Div}(X \times T)$ such that $G'_t = G'|_{X \times t}$ for any $t \in T - 0$ and a section $\gamma : T \to X \times T$ such that $\gamma(0) = (x, 0)$, for general $t \in T$, $p_X(\gamma(t))$ is a general point of Z_k and

$$\operatorname{mult}_{p_X(\gamma(t))}(G'_t|_{Z_k}) > \dim Z_k.$$

By (3.22)

$$\mathcal{J}(X, (1-\epsilon)D_k + G'_t) \subset m_{p_X(\gamma(t))}$$

for $0 < \epsilon \ll 1$ and general $t \in T$. By (3.23), we have that $\mathcal{J}(X, (1-\epsilon)D_k + G'_0) \subset m_x$. Since $\mathcal{O}_X(mA) \otimes \mathcal{I}_{Z_k}$ is generated, we may assume that the zeroes of

$$\mathcal{J}(X, D_k + G'_0)$$

are contained in Z_k. It follows that

$$\mathcal{J}((1-\epsilon)D_k + G'_0) \subset m_x$$

and that the zeroes of $\mathcal{J}((1-\epsilon)D_k + G'_0)$ are strictly contained in Z_k. After peturbing $(1-\epsilon)D_k + G'_0$ by a general ample divisor and multiplying it by the log canonical threshold, we obtain a divisor D_{k+1} with the required properties. □

3.4. Asymptotic multiplier ideal sheaves.

Definition 3.28. *Let X be a smooth projective variety, $D \in \operatorname{Div}(X)$ be a divisor such that $\kappa(D) \geq 0$. Then there exists an integers $e = e(D) > 0$ and $m_0 = m_0(D)$ such that if $m \geq m_0$, then*

$$H^0(\mathcal{O}_X(mD)) \neq 0 \qquad \text{if and only if} \qquad e \text{ divides } m.$$

The integer $e(D)$ is the **exponent** *of D and the integer $m_0(D)$ is the Iitaka threshold of D.*

Exercise 3.29. *If D is big, then $e(D) = 1$.*

Lemma 3.30. *Let X be a smooth projective variety, $D \in \mathrm{Div}(X)$ be a divisor such that $h^0(\mathcal{O}_X(mD)) \geq 0$ for an integer $m > 0$. Then for any numbers $0 < c \in \mathbb{R}$ and $0 < k \in \mathbb{Z}$, we have*

$$\mathcal{J}(\frac{c}{m} \cdot |mD|) \subset \mathcal{J}(\frac{c}{mk} \cdot |mkD|).$$

Proof. This is an easy excercise that follows from the inclusion of linear series $k|mD| \subset |kmD|$. $\square$

Definition 3.31. *Let $D \in \mathrm{Div}(X)$ be a divisor such that $\kappa(D) \geq 0$ and $0 < c \in \mathbb{R}$, then we define the* **asymptotic multiplier ideal sheaf** *of D by*

$$\mathcal{J}(c \cdot ||D||) = \cup_{m \in I} \mathcal{J}(\frac{c}{m} \cdot |mD|)$$

where $I = \{m \geq m_0(D) | e(D) \text{ divides } m\}$. Notice that as X is Noetherian, we have $\mathcal{J}(c \cdot ||D||) = \mathcal{J}(\frac{c}{m} \cdot |mD|)$ for any $m > 0$ sufficiently divisible.

One of the reasons for introducing asymptotic multiplier ideal sheaves is that they satisfy many useful formal properties.

Proposition 3.32. *Let $L \in \mathrm{Div}(X)$ be a divisor such that $\kappa(L) \geq 0$, $0 < m, l \in \mathbb{Z}$ and $0 < c \in \mathbb{R}$, then*

1. $\mathcal{J}(c \cdot ||mL||) = \mathcal{J}(cm \cdot ||L||)$,
2. $\mathcal{J}(c \cdot ||(m+1)L||) \subset \mathcal{J}(c \cdot ||mL||)$,
3. $b_l \cdot \mathcal{J}(||mL||) \subset \mathcal{J}(||(m+l)L||)$,
4. $H^0(\mathcal{O}_X(mL) \otimes \mathcal{J}(||mL||)) \cong H^0(\mathcal{O}_X(mL))$, *and*
5. $\mathcal{J}(c \cdot ||mL||) = \mathcal{J}(\frac{c}{mp} \cdot |D|)$ *where $D \in |mpL|$ is general.*

Proof. We may pick $p > 0$ such that $\mathcal{J}(c \cdot ||mL||) = \mathcal{J}(\frac{cp}{p} \cdot |mpL|)$ and $\mathcal{J}(cm \cdot ||L||) = \mathcal{J}(\frac{cmp}{pm} \cdot |pmL|)$. (1) follows immediately.

By (1) and (3.5), we have that

$$\mathcal{J}(c \cdot ||mL||) = \mathcal{J}(cm \cdot ||L||) \supset \mathcal{J}(c(m+1) \cdot ||L||) = \mathcal{J}(c \cdot ||(m+1)L||).$$

Hence (2).

To see (3), consider $f : Y \to X$ a log resolution of $(X, |tL|)$ for $t \in \{l, pm, pl, p(m+l)\}$. We write $f^*|tL| = V_t + F_t$ where V_t is free and $F_t = \mathrm{Fix}(f^*|tL|)$. If $p > 0$ is sufficiently divisible, we have

$$pF_l + F_{mp} \geq F_{pl} + F_{mp} \geq F_{p(m+l)},$$

so that

$$-F_l - \frac{1}{p}\llcorner F_{mp}\lrcorner \leq -\llcorner \frac{1}{p} F_{p(m+l)}\lrcorner.$$

It follows that

$$b_l \cdot \mathcal{J}(||mL||) \subset f_*\mathcal{O}_Y(K_{Y/X} - F_l - \frac{1}{p}\llcorner F_{mp}\lrcorner)$$

$$\subset f_*\mathcal{O}_Y(K_{Y/X} - \llcorner \frac{1}{p} F_{p(m+l)}\lrcorner) = \mathcal{J}(||(m+l)L||).$$

(4) follows as by (3), we have $b_m \subset \mathcal{J}(||mL||)$.

To see (5), consider a log resolution $f : Y \to X$ of $(X, |mpL|)$ for any $p > 0$ sufficiently divisible. Then $f^*|mpL| = V_{mp} + F_{mp}$ where $F_{mp} = \mathrm{Fix}(f^*|mpL|)$. But then $f^*D = D' + F_{mp}$ where D' is a smooth divisor, and so $\llcorner \frac{1}{mp}(D' + F_{mp})\lrcorner = \llcorner \frac{1}{mp} F_{mp}\lrcorner$ and the assertion follows easily. □

Theorem 3.33 (Subadditivity for asymptotic multiplier ideal sheaves.). *Let $L \in \mathrm{Div}(X)$ be a divisor such that $\kappa(L) \geq 0$, $0 < m, l \in \mathbb{Z}$ and $0 < c \in \mathbb{R}$, then*

$$\mathcal{J}(c \cdot ||(m+l)L||) \subset \mathcal{J}(c \cdot ||mL||) \cdot \mathcal{J}(c \cdot ||lL||).$$

In particular $\mathcal{J}(c \cdot ||mL||) \subset \mathcal{J}(c \cdot ||L||)^m$.

Proof. Let $p > 0$ be sufficiently divisible and $D \in |ml(m+l)pL|$ be general. Then

$$\mathcal{J}(c \cdot ||(m+l)L||) = \mathcal{J}(\frac{c}{mpl} D) = \mathcal{J}(\frac{c(m+l)}{mlp(m+l)} D) \subset$$

$$\mathcal{J}(\frac{cm}{mlp(m+l)} D) \cdot \mathcal{J}(\frac{cl}{mlp(m+l)} D) = \mathcal{J}(c \cdot ||mL||) \cdot \mathcal{J}(c \cdot ||lL||).$$

□

Remark 3.34. *Note that we have used* (3.22) *which states that if* $0 \leq D_i \in \mathrm{Div}(X)$*, then $\mathcal{J}(D_1 + D_2) \subset \mathcal{J}(D_1) \cdot \mathcal{J}(D_2)$. However, it is not the case that $\mathcal{J}(|D_1 + D_2|) \subset \mathcal{J}(|D_1|) \cdot \mathcal{J}(|D_2|)$ (eg. let D_i be general points on an elliptic curve E, then $\mathcal{J}(|D_i|) = \mathcal{J}(D_i) = \mathcal{O}_E(-D_i)$ but $|D_1 + D_2|$ is free so that $\mathcal{J}(|D_1 + D_2|) = \mathcal{O}_E$).*

The following result is due to Wilson.

Proposition 3.35. *Let $D \in \mathrm{Div}(X)$ be nef and big divisor on a smooth projective variety. Then there exists an integer $m_0 > 0$ and a divisor $N \in \mathrm{Div}(X)$ such that $|mD - N|$ is free for all $m \geq m_0$. In particular if $x \in X$ and $G \in |mD|$ is general, then $\mathrm{mult}_x G$ is bounded (by $\mathrm{mult}_x(N)$).*

Proof. Let $H \in \mathrm{Div}(X)$ be sufficiently ample. Since D is big, there is an integer $m_0 > 0$ such that $(n+1)H + N \sim m_0 D$ for some $0 \leq N \in \mathrm{Div}(X)$. For any $m_0 \leq m \in \mathbb{Z}$, we have

$$mD - N = (m - m_0)D + (n+1)H.$$

By (3.8), $mD - N$ is free. □

Proposition 3.36. *Let $D \in \mathrm{Div}(X)$ be a big divisor on a smooth projective variety. Then D is nef if and only if $\mathcal{J}(||mD||) = \mathcal{O}_X$ for all $m \geq 1$.*

Proof. Assume that D is nef. Fix any $m \geq 1$. We have

$$\mathcal{J}(||mD||) = \mathcal{J}(\frac{1}{k} \cdot |mkD|) = \mathcal{J}(\frac{1}{k} D_k)$$

where $k > 0$ is sufficiently divisible and D_k is general in $|mkD|$. By (3.35), we may assume that $\mathrm{mult}_x(\frac{1}{k}D_k) < 1$ for all $x \in X$ so that $\mathcal{J}(\frac{1}{k}D_k) = \mathcal{O}_X$.

Assume now that $\mathcal{J}(||mD||) = \mathcal{O}_X$ for all $m \geq 1$. Fix B a very ample divisor. By (3.8)

$$\mathcal{O}_X(K_X + (n+1)B + mD) \otimes \mathcal{J}(||mD||) = \mathcal{O}_X(K_X + (n+1)B + mD)$$

is generated by global sections. If $C \subset X$ is any curve, then $D \cdot C \geq -\frac{1}{m}(K_X + (n+1)B) \cdot C$. As

$$\lim_{m \to +\infty} -\frac{1}{m}(K_X + (n+1)B) \cdot C = 0,$$

it follows that $D \cdot C \geq 0$ and so D is nef. □

The next application concerns the diminished stable base locus of a pseudo-effective divisor $D \in \mathrm{Div}(X)$ which is defined by

$$\mathbf{B}_-(D) = \cup_{0<\epsilon\in\mathbb{Q}} \mathrm{SBs}(D + \epsilon A)$$

where A is any fixed divisor.

Exercise 3.37. *Show that the above definition is independent of A.*

Exercise 3.38. *Show that if $0 < \epsilon_1 \leq \epsilon_2 \in \mathbb{Q}$ then $\mathrm{SBs}(D + \epsilon_2 A) \subset \mathrm{SBs}(D + \epsilon_1 A)$ so that $\mathbf{B}_-(D)$ is a countable union of subvarieties of X.*

Proposition 3.39. *Let $D \in \mathrm{Div}(X)$ be a pseudoeffective divisor on a smooth projective variety and $Z \subset X$ be a subvariety. If*

$$\lim \frac{1}{m!} \mathrm{mult}_Z(|m!D|) = 0,$$

then Z is not contained in $\mathbf{B}_-(D)$.

Proof. If $A \in \mathrm{Div}(X)$ is ample and m is sufficiently large, then by (3.8), $\mathcal{O}_X(m(D+A)) \otimes \mathcal{J}(||mD||)$ is generated by global sections. Notice that $\mathcal{J}(||mD||) = \mathcal{J}(\frac{1}{p}D_p)$ where $D_p \in |mpD|$ is general and $p > 0$ is sufficiently divisible. Since p is sufficiently divisible, then $\frac{1}{p}\operatorname{mult}_Z D_p < 1$ and so $I_Z \not\subset \mathcal{J}(||mD||)$. □

3.5. **Adjoint ideal sheaves.**

Definition 3.40. *Let (X, D) be a log smooth pair where D is a reduced divisor, $0 < c \in \mathbb{R}$ and let V be a linear system whose base locus contains no log canonical centers of (X, D). For any log resolution $f : Y \to X$ of $(X, D+|V|)$, we write $f^*D = M+F$ where $F = \mathrm{Fix}(f^*V)$ and M is free.*

We define the **multiplier ideal sheaf**

$$\mathcal{J}_{D,c\cdot V} := f_*\mathcal{O}_Y(E_Y(X, D) - \llcorner cF \lrcorner).$$

If $B = cG$ where $0 < G \in \mathrm{Div}(X)$, then we define $\mathcal{J}_{D,B} := \mathcal{J}_{D,c\cdot V}$ where $V = \{G\}$.

Lemma 3.41. *The above definition is independent of the log resolution $f : Y \to X$.*

Proof. Given two log resolutions of $(X, D + |V|)$, $f : Y \to X$ and $f' : Y' \to X$ we may assume that $f' = f \circ \nu$ where $\nu : Y' \to Y$. Then ν^*M is free and $\nu^*F = \mathrm{Fix}(\nu^*f^*V)$. We let $E_{Y'} = E_{Y'}(X, D)$ and similarly for Y and Γ. We have

$$= E_{Y'} - c\nu^*F = K_{Y'} + \Gamma_{Y'} - f'^*(K_X + D) - c\nu^*F$$
$$= K_{Y'} + \Gamma_{Y'} - \nu^*(K_Y + \Gamma_Y - E_Y + cF)$$
$$= \nu^*(E - \llcorner cF \lrcorner) + K_{Y'} + \Gamma_{Y'} - \nu^*(K_Y + \Gamma_Y + \{cF\})$$

One sees that $(Y, \Gamma_Y + \{cF\})$ is log canonical and its log canonical places coincide with those of (Y, Γ_Y) and hence with those of (X, D). It follows that the divisor

$$G = \ulcorner K_{Y'} + \Gamma_{Y'} - \nu^*(K_Y + \Gamma_Y + \{cF\})\urcorner$$

is effective and ν exceptional. Therefore

$$f'_*\mathcal{O}_{Y'}(\ulcorner E_{Y'} - c\nu^*F\urcorner) = f_*(\nu_*\mathcal{O}_{Y'}(\ulcorner E_{Y'} - c\nu^*F\urcorner)$$
$$= f_*(\nu_*\mathcal{O}_{Y'}(\nu^*(\ulcorner E_Y - cF\urcorner) + G) = f_*(\mathcal{O}_Y(\ulcorner E_Y - cF\urcorner)).$$

□

Lemma 3.42. *Let (X, D) be a log smooth pair where D is a reduced divisor, $0 < c \in \mathbb{R}$ and let V (resp. $0 \le G, H \in \mathrm{Div}_{\mathbb{R}}(X)$) be a linear system whose base locus (resp. a divisors whose support) contains no log canonical centers of (X, D). Then*

(1) $\mathcal{J}_{D,G} = \mathcal{O}_X$ *if and only if* $(X, D+G)$ *is divisorially log terminal and* $\llcorner D + G \lrcorner = \llcorner D \lrcorner$.
(2) *If* $0 \leq D' \leq D$ *and* D' *is a reduced divisor, then* $\mathcal{J}_{D,c\cdot V} \subset \mathcal{J}_{D',c\cdot V}$.
(3) *If* $0 \leq \Sigma \in \operatorname{Div}(X)$, $H \leq G + \Sigma$ *and* $\mathcal{J}_{D,G} = \mathcal{O}_X$ *then* $\mathcal{I}_\Sigma \subset \mathcal{J}_{D,H}$.

Proof. (1) and (2) follow easily from the definitions.

To see (3), let $f : Y \to X$ be a log resolution of $(X, D+G+H+\Sigma)$. As $\Sigma \in \operatorname{Div}(X)$, we have

$$\llcorner f^*H \lrcorner \leq \llcorner f^*G \lrcorner + f^*\Sigma.$$

As $\mathcal{J}_{D,G} = \mathcal{O}_X$, it follows that

$$E_Y(X, D) - \llcorner f^*G \lrcorner \geq 0.$$

Therefore, we have that

$$-f^*\Sigma \leq \llcorner f^*G \lrcorner - \llcorner f^*H \lrcorner = (\llcorner f^*G \lrcorner - E_Y) + (E_Y - \llcorner f^*H \lrcorner) \leq E_Y - \llcorner f^*H \lrcorner.$$

Therefore

$$\mathcal{I}_\Sigma = f_*\mathcal{O}_Y(-f^*\Sigma) \subset f_*\mathcal{O}_Y(E_Y - \lfloor f^*H \rfloor) = \mathcal{J}_{D,H}.$$

□

Theorem 3.43. *[Nadel Vanishing for adjoint ideals] Let* $\pi : X \to Z$ *be a projective morphism to a normal affine variety. Assume that* (X, D) *is a log smooth pair where* D *is reduced and* $0 \leq G \in \operatorname{Div}_{\mathbb{R}}(X)$ *is a divisor whose support contains no centers of* $\operatorname{NKLT}(X, D)$.

If $N \in \operatorname{Div}(X)$ *and* $N - G$ *is ample, then*

$$R^i\pi_*\mathcal{J}_{D,G}(K_X + D + N) = 0 \qquad \forall\ i > 0.$$

Proof. Let $f : Y \to X$ be a log resolution of $(X, D + G)$. By [14] we may assume that f is an isomorphism at a general point of each log canonical center of (X, D). We have

$$f^*(K_X + D + N) + E - \llcorner f^*G \lrcorner = K_Y + \Gamma + \{f^*G\} + f^*(N - G)$$

where $E = E_Y(X, D)$ and $\Gamma = \Gamma_Y(X, D)$. By (2.43), we have that $R^if_*\mathcal{O}_Y(K_Y + \Gamma + \{f^*G\} + f^*(N - G)) = 0$ for $i > 0$ and $R^i(\pi \circ f)_*\mathcal{O}_Y(K_Y + \Gamma + \{f^*G\} + f^*(N - G)) = 0$ for $i > 0$. Since $f_*\mathcal{O}_Y(K_Y + \Gamma + \{f^*G\} + f^*(N - G)) = \mathcal{J}_{D,G}(K_X + D + N)$, the claim follows from an easy spectral sequence argument. □

Lemma 3.44. *Let* $\pi : X \to Z$ *be a projective morphism to a normal affine variety. Assume that* (X, D) *is a log smooth pair where* D *is reduced,* S *is a component of* D *and* $G \in \operatorname{Div}_{\mathbb{R}}(X)$ *is effective and its* *support contains no centers of* NKLT(X,D). *Then there is a short exact sequence*

$$0 \to \mathcal{J}_{D-S,G+S} \to \mathcal{J}_{D,G} \to \mathcal{J}_{(D-S)|_S,G|_S} \to 0.$$

If moreover, $N \in \mathrm{Div}(X)$ *and* $N - G - (K_X + D)$ *is ample, then the homomorphism*

$$\pi_*\mathcal{J}_{D,G}(N) \to \pi_*\mathcal{J}_{(D-S)|_S,G|_S}(N)$$

is surjective.

Proof. Let $f : Y \to X$ be a log resolution of $(X, D+G)$. By [14] we may assume that f is an isomorphism at a general point of each log canonical center of (X,D). Let $T = (f^{-1})_*S$ and consider the short exact sequence

$$0 \to \mathcal{O}_Y(E - \llcorner f^*G \lrcorner - T) \to \mathcal{O}_Y(E - \llcorner f^*G \lrcorner) \to \mathcal{O}_T(E - \llcorner f^*G \lrcorner) \to 0$$

where $E = E_Y(X,D)$. Let $\Gamma = \Gamma_Y(X,D)$. Since

$$E - f^*G - T = (K_Y + \Gamma - T) - f^*(K_X + D - S + (G+S)),$$

we have

$$E - f^*G - T = E_Y(X, D-S) - f^*(G+S)$$

and

$$\begin{aligned}(E - f^*G)|_T &= K_T + (\Gamma - T)|_T - f^*(K_S + (D - S + G)|_S)\\ &= E_T(S, (D-S)|_S) - f^*(G|_S).\end{aligned}$$

Since $\ulcorner E - f^*G - T \urcorner \sim_f K_Y + \Gamma - T + \{f^*G\}$, by (2.43)

$$R^1 f_*\mathcal{O}_Y(\ulcorner E - f^*G - T \urcorner) = 0.$$

Therefore, pushing forward the above exact sequence via f, we obtain the required short exact sequence. The surjection $\pi_*\mathcal{J}_{D,G} \to \pi_*\mathcal{J}_{(D-S)|_S,G|_S}$ follows by (2.43) as

$$f^*N + \ulcorner E - f^*G - T \urcorner \sim$$
$$(K_Y + \Gamma - T) + f^*(N - K_X - D - G) + \{f^*G\}.$$

□

Exercise 3.45. *Use* (3.44) *to reprove* (3.41).

The next result is sometimes referred to as the process of "squeezing out the extra positivity". Roughly speaking it says that under appropriate hypothesis, if N is the multiple of an adjoint bundle, $S \subset X$ is a divisor and H is an ample line bundle, such that sections of $(mN+H)|_S$ extend for $m \gg 0$, then sections of $N|_S$ also extend.

Theorem 3.46. *Let $\pi : X \to Z$ be a projective morphism to a normal affine variety. Assume that $(X, D = S + A + B)$ is a log pair where $\llcorner D \lrcorner = S$, X and S are smooth and $0 \le A, B \in \mathrm{Div}_{\mathbb{Q}}(X)$. Let $0 < k \in \mathbb{Z}$, $C = \frac{1}{k}A$ and $0 \le \Phi \le \Omega = (D - S)|_S$ be a $\mathbb{Q}$-divisor such that $k(K_S + \Phi)$ and $k(K_X + D)$ are integral.*

If there is an integer $m > 1$ divisible by k and a divisor $0 \le P \in \mathrm{Div}(X)$ such that $C - \frac{k-1}{m}P$ is ample, $mC \in \mathrm{Div}(X)$, the pair $(X, D + \frac{k-1}{m}P)$ is purely log terminal and

$$\frac{m}{k}|k(K_S + \Phi)| + m(\Omega - \Phi + C|_S) + P|_S \subset |m(K_X + D + C) + P|_S,$$

then

$$|k(K_S + \Phi)| + k(\Omega - \Phi) \subset |k(K_X + D)|_S.$$

Proof. Pick any divisor $\Sigma \in |k(K_S + \Phi)|$, then there exists a divisor

$$G \in |m(K_X + D + C) + P| \qquad \text{with} \qquad G|_S = \frac{m}{k}\Sigma + m(\Omega - \Phi + C|_S) + P|_S.$$

We define

$$\Lambda = \frac{k-1}{m}G + B \qquad \text{and} \qquad N = k(K_X + D) - K_X - S.$$

Since $\Lambda \ge 0$, $S \not\subset \mathrm{Supp}(\Lambda)$ and $N - \Lambda \sim_{\mathbb{Q}} C - \frac{k-1}{m}P$ is ample, then by (3.7), the homomorphism

$$H^0(X, \mathcal{O}_X(k(K_X + D))) \to H^0(S, \mathcal{O}_S(k(K_S + \Omega)) \otimes \mathcal{J}_{\Lambda|_S})$$

is surjective. It suffices then to check that $\Sigma + k(\Omega + \Phi)$ vanishes along the ideal $\mathcal{J}_{\Lambda|_S}$. This follows by (3.42) since $(S, \Omega + \frac{k-1}{m}P|_S)$ is klt (as $(X, D + \frac{k-1}{m}P)$ is purely log terminal) and

$$\begin{aligned} &\Lambda|_S - (\Sigma + k(\Omega - \Phi)) \\ &= \frac{k-1}{m}\left(\frac{m}{k}\Sigma + m(\Omega - \Phi + C|_S) + P|_S\right) + B|_S - (\Sigma + k(\Omega - \Phi)) \\ &\le \Omega + \frac{k-1}{m}P|_S. \end{aligned}$$

□

3.6. Asymptotic multiplier ideal sheaves II.

Definition 3.47. *Let X be a normal variety and $D \in \mathrm{Div}(X)$. An* **additive sequence of linear systems associated to** *D is a sequence of sublinear series $V_i \subset |iD|$ such that*

$$V_i + V_j \subset V_{i+j}.$$

Definition 3.48. *Suppose that (X, D) is a log smooth pair and D is reduced. If $V_\bullet$ is an additive sequence of linear systems (associated to a divisor G) such that for some $0 < m \in \mathbb{Z}$ no non kawamata log terminal center of (X, D) is contained in $\mathrm{Bs}(V_m)$ then we define the* **asymtotic multiplier ideal sheaf** *of $c \cdot V_\bullet$ with respect to (X, D) by*

$$\mathcal{J}_{D,c\cdot V_\bullet} = \cup_{p>0} \mathcal{J}_{D,\frac{c}{p}\cdot V_p}.$$

If $V_m = |mG|$, we let

$$\mathcal{J}_{D,c\cdot||G||} = \mathcal{J}_{D,c\cdot V_\bullet}$$

and if S is a component of D and $W_m = |mD|_S$ we let

$$\mathcal{J}_{(D-S)|_S,c\cdot||G||_S} = \mathcal{J}_{(D-S)|_S,c\cdot W_\bullet}.$$

Exercise 3.49. *Show that if q divides p and m divides q, then*

$$\mathcal{J}_{D,\frac{c}{p}\cdot V_p} \supset \mathcal{J}_{D,\frac{c}{q}\cdot V_q}$$

and therefore,

$$\mathcal{J}_{D,c\cdot V_\bullet} = \mathcal{J}_{D,\frac{c}{p}\cdot V_p}$$

for any $p > 0$ sufficiently divisible.

Exercise 3.50. *Show that $\mathcal{J}_{(D-S)|_S,c\cdot||G||_S} \subset \mathcal{J}_{(D-S)|_S,c\cdot||G|_S||}$.*

We will need the following preliminary results.

Lemma 3.51. *Let $\pi : X \to Z$ be a projective morphism to a normal affine variety and $G \in \mathrm{Div}_{\mathbb{Q}}(X)$. If (X, D) is a log smooth pair, D is reduced and $\mathrm{SBs}(G)$ does not contain any non kawamata log terminal center of (X, D), then*

(1) *for any $0 < c_1 \leq c_2 \in \mathbb{R}$, we have*

$$\mathcal{J}_{D,c_2\cdot||G||} \subset \mathcal{J}_{D,c_1\cdot||G||}$$

(2) *if $G \in \mathrm{Div}(X)$ and S is a component of D, then*

$$\mathrm{Im}\,(\pi_*\mathcal{O}_X(G) \to \mathcal{O}_S(G)) \subset \pi_*\mathcal{J}_{(D-S)|_S,||G||_S}(G).$$

Proof. (1) follows easily from the definitions.

To see (2), let $0 < p \in \mathbb{Z}$ such that

$$\mathcal{J}_{(D-S)|_S,\frac{1}{p}\cdot|pG|_S} = \mathcal{J}_{(D-S)|_S,||G||_S}$$

and consider $f : Y \to X$ a log resolution of $|G| + D$ and of $|pG| + D$. Let $T = (f^{-1})_*S$. We let $F_i = \mathrm{Fix}(f^*|iG|)$. Then by definition of F_1, we have

$$(\pi \circ f)_*\mathcal{O}_Y(f^*G - F_1) = \pi_*\mathcal{O}_X(G) = (\pi \circ f)_*\mathcal{O}_Y(E_Y(X, D) + f^*G).$$

We also have inequalities

$$f^*G - F_1 \leq f^*G - \llcorner \frac{1}{p} F_p \lrcorner \leq E_Y + f^*G - \llcorner \frac{1}{p} F_p \lrcorner \leq E_Y + f^*G.$$

Pushing forward, one sees that

$$\pi_* \mathcal{O}_X(G) = (\pi \circ f)_* \mathcal{O}_Y(E_Y + f^*G - \llcorner \frac{1}{p} F_p \lrcorner)$$

and so the image of $\pi_* \mathcal{O}_X(G)$ is contained in

$$(\pi \circ f)_* \mathcal{O}_T(E_Y + f^*G - \llcorner \frac{1}{p} F_p \lrcorner) = \pi_* \mathcal{J}_{(D-S)|_S, ||G||_S}(G).$$

□

Lemma 3.52. *Let $\pi : X \to Z$ be a projective morphism to a normal affine variety and $G \in \mathrm{Div}_{\mathbb{Q}}(X)$. If (X, D) is a log smooth pair, D is reduced, S is a component of D and $\mathbf{B}_+(G)$*[1] *contains no centers of* $\mathrm{NKLT}(X, D)$*, then for any $p \in \mathbb{N}$ sufficiently divisible and $B \in |pG|$ general, we have*

$$R^i \pi_* \mathcal{J}_{D-S, \frac{1}{p}B+S}(K_X + D + G) = 0 \qquad \text{for } i > 0, \text{ and}$$

$$\pi_* \mathcal{J}_{(D-S)|_S, ||G||_S}(K_S + (D-S)|_S + G|_S) \subset$$
$$\mathrm{Im}\left(\pi_* \mathcal{O}_X(K_X + D + G) \to \pi_* \mathcal{O}_S(K_S + (D-S)|_S + G|_S)\right).$$

Proof. If $0 < p \in \mathbb{Z}$ is sufficiently divisible and $B \in |pG|$ is general, then

$$\mathcal{J}_{D, ||G||} = \mathcal{J}_{D, \frac{1}{p}|pG|} = \mathcal{J}_{D, \frac{1}{p}B} \qquad \text{and}$$

$$\mathcal{J}_{(D-S)|_S, ||G||_S} = \mathcal{J}_{(D-S)|_S, \frac{1}{p}|pG|_S} = \mathcal{J}_{(D-S)|_S, \frac{1}{p}B|_S}.$$

By (3.44), we have a short exact sequence

$$0 \to \mathcal{J}_{D-S, \frac{1}{p}B+S} \to \mathcal{J}_{D, \frac{1}{p}B} \to \mathcal{J}_{(D-S)|_S, \frac{1}{p}B|_S} \to 0.$$

Let $f : Y \to X$ be a log resolution of $(X, D + |pG|)$ which is an isomorphism at a general point of each center of $\mathrm{NKLT}(X, D)$, then (as in the proof of (3.44))

$$\mathcal{J}_{D-S, \frac{1}{p}B+S} = f_* \mathcal{O}_Y(E_Y(X, D) - T - \llcorner \frac{1}{p} f^*B \lrcorner).$$

Notice that if $M = f^*|pG| - \mathrm{Fix}(f^*|pG|)$, then $\{\frac{1}{p} f^*B\} \geq \frac{1}{p}M$ and $\frac{1}{p}M$ is nef and its restriction to any center in $\mathrm{NKLT}(Y, \Gamma_Y)$ is big (we may

[1]Recall that $\mathbf{B}_+(G) = \cap_{\epsilon > 0} \mathrm{SBs}(G - \epsilon A) = \mathrm{SBs}(G - \epsilon' A)$ for any $0 < \epsilon' \ll 1$.

in fact assume that $M \sim_{\mathbb{Q}} f^*A + E$ where $A \in \mathrm{Div}_{\mathbb{Q}}(X)$ is ample and $E \geq 0$ contains no centers of $(Y, \Gamma_Y(X,D))$). Since

$$E_Y(X,D) - T - \llcorner \frac{1}{p} f^*B \lrcorner + f^*(K_X + D + G)$$

$$\sim_{\mathbb{Q}} K_Y + \Gamma_Y(X,D) - T + f^*(G - \frac{1}{p}B) + \{\frac{1}{p} f^*B\},$$

by (2.43), we have that for any $i > 0$

$$R^i \pi_* \mathcal{J}_{D-S, \frac{1}{p}B+S}(K_X + D + G) =$$

$$R^i (f \circ \pi)_* \mathcal{O}_Y(E_Y(X,D) - T - \llcorner \frac{1}{p} f^*B \lrcorner + f^*(K_X + D + G)) = 0.$$

□

4. Extension Theorems and applications

Theorem 4.1. *Let $\pi : X \to Z$ be a projective morphism to a normal affine variety. Let $(X, D = S + B)$ be a log smooth log canonical pair of dimension n, $0 < k \in \mathbb{Z}$ such that $k(K_X + D) \in \mathrm{Div}(X)$ and S an irreducible component of $\llcorner D \lrcorner$. If the stable base locus of $K_X + D$ contains no centers in $\mathrm{NKLT}(X, \ulcorner D \urcorner)$ and A is any sufficiently ample divisor on X then*

$$\sharp_m \qquad \mathcal{J}_{||mk(K_X+D)|_S||} \subset \mathcal{J}_{(\lceil D \rceil - S)|_S, ||mk(K_X+D)+A||_S}$$

holds for all $m \in \mathbb{N}$, and $\pi_* \mathcal{J}_{||mk(K_X+D)|_S||}(mk(K_X + D) + A)$ is contained in the image of the homomorphism

$$\pi_* \mathcal{O}_X(mk(K_X + D) + A) \to \pi_* \mathcal{O}_S(mk(K_X + D) + A).$$

Proof. We begin by proving the first statement by induction on $m \geq 0$. The case $m = 0$ is clear. We will show that $\sharp_m$ implies $\sharp_{m+1}$. Write $D = \sum d_i D_i$ and for $1 \leq s \leq k$ let

$$S \leq D^1 \leq D^2 \leq \ldots \leq D^k = \ulcorner D \urcorner$$

be the (uniquely defined) reduced divisors such that

$$kD = \sum_{s=1}^{k} D^s.$$

Let $N_s \in \mathrm{Div}(X)$ be the divisors defined by $N_0 = 0$ and

$$N_{s+1} = K_X + D^{s+1} + N_s \qquad \text{for } 0 \leq s \leq k-1.$$

In particular $N_k = k(K_X + D)$. We will show that there are inclusions

$$\star_s \qquad \mathcal{J}_{||mk(K_X+D)|_S||} \subset \mathcal{J}_{(D^{s+1}-S)|_S, ||mk(K_X+D)+N_s+A||_S} \qquad \text{for } 0 \leq s \leq k.$$

$\sharp_{m+1}$ then follows since by (3.32), we have

$$\mathcal{J}_{||(m+1)k(K_X+D)|_S||} \subset \mathcal{J}_{||mk(K_X+D)|_S||}.$$

$\star_0$ follows since by $\sharp_m$ and by (3.42), we have

$$\mathcal{J}_{||mk(K_X+D)|_S||} \subset \mathcal{J}_{(\ulcorner D \urcorner - S)|_S, ||mk(K_X+D)+A||_S} \subset \mathcal{J}_{(D^1-S)|_S, ||mk(K_X+D)+A||_S}.$$

Suppose now that $\star_{t-1}$ holds. We have

$$\begin{aligned}
&\pi_*\mathcal{J}_{||mk(K_X+D)|_S||}(mk(K_X+D)+N_t+A) \\
&\subset \pi_*\mathcal{J}_{(D^t-S)|_S, ||mk(K_X+D)+N_{t-1}+A||_S}(mk(K_X+D)+N_t+A) \\
&\subset \operatorname{Im}(\pi_*\mathcal{O}_X(mk(K_X+D)+N_t+A) \to \pi_*\mathcal{O}_S(mk(K_X+D)+N_t+A)) \\
&\subset \pi_*\mathcal{J}_{(D^{t+1}-S)|_S, ||mk(K_X+D)+N_t+A||_S}(mk(K_X+D)+N_t+A).
\end{aligned}$$

The first inclusion follows by $\star_{t-1}$, the second inclusion follows from (3.52) and the third one from (3.51). By (3.8), $\mathcal{J}_{||mk(K_X+D)|_S||}(mk(K_X+D)+N_t+A)$ is generated by global sections and so

$$\mathcal{J}_{||mk(K_X+D)|_S||} \subset \mathcal{J}_{(D^{t+1}-S)|_S, ||mk(K_X+D)+N_t+A||_S}.$$

This completes the proof. □

Theorem 4.2. *Let $\pi : X \to Z$ be a projective morphism to a normal affine variety. Let $(X, D = S + A + B)$ be a purely log terminal pair of dimension n where X and S are smooth, $D \in \operatorname{Div}_{\mathbb{Q}}(X)$, $\llcorner D \lrcorner = S$, A is a general ample $\mathbb{Q}$-divisor, $(S, \Omega = (D - S)|_S)$ is canonical and the stable base locus of $K_X + D$ does not contain S. For any sufficiently divisible $m > 0$, let*

$$F_m = \operatorname{Fix}(|m(K_X+D)|_S)/m$$

and $F = \lim F_{m!}$.

If $0 < \epsilon \in \mathbb{Q}$ is such that $\epsilon(K_X+D)+A$ is ample, $\Phi \in \operatorname{Div}_{\mathbb{Q}}(S)$ and $0 < k \in \mathbb{Z}$ such that

(1) *$kD \in \operatorname{Div}(X)$ and $k\Phi \in \operatorname{Div}(S)$, and*
(2) *$\Omega \wedge \lambda F \leq \Phi \leq \Omega$ where $\lambda = 1 - \epsilon/k$,*

then

$$|k(K_S + \Omega - \Phi)| + k\Phi \subset |k(K_X+D)|_S.$$

Proof. Pick a general ample divisor $C \sim_{\mathbb{Q}} A/k$ so that $(X, D+(k-1)C)$ is purely log terminal and $(S, \Omega + C|_S)$ is canonical. Pick $\epsilon/k < \eta \in \mathbb{Q}$ so that $\eta(K_X+D)+C$ is ample. If $0 < l \in \mathbb{Z}$ is sufficiently divisible so that $O = l(\eta(K_X+D)+C)$ is very ample, then

$$\begin{aligned}
\operatorname{Fix}(|l(K_X+D+C)|_S)/l &= \operatorname{Fix}(|l(1-\eta)(K_X+D)+O|_S)/l \\
&\leq \operatorname{Fix}(|l(1-\eta)(K_X+D)|_S)/l = (1-\eta)F_{(1-\eta)l}.
\end{aligned}$$

Therefore

$$\lim \mathrm{Fix}(|l!(K_X+D+C)|_S)/l! \le (1-\eta)F.$$

By (3.39), if P is a prime divisor on S that is not contained in the $\mathrm{Supp}(F)$, then for l sufficiently divisible, P is not contained in $\mathrm{Bs}(l(K_X+D+C))$. It follows that we may pick $0<l\in\mathbb{Z}$ sufficiently divisible such that

$$\mathrm{Fix}(|l(K_X+D+C)|_S)/l \le \lambda F.$$

Let $f: Y \to X$ be a log resolution of $(X, |l(K_X+D+C)|+\mathrm{Supp}(D+C))$ and write

$$K_Y+\Gamma = f^*(K_X+D+C)+E$$

where $\Gamma=\Gamma_Y(X,D+C)$ and $E=E_Y(X,D+C)$. We have that

$$\mathrm{Fix}(l(K_Y+\Gamma))/l = \mathrm{Fix}(lf^*(K_X+D+C))/l+E.$$

If $\Xi=\Gamma-\Gamma\wedge\mathrm{Fix}(l(K_Y+\Gamma))/l$, then

$$l(K_Y+\Xi)\in\mathrm{Div}(Y) \qquad \text{and} \qquad \mathrm{Fix}(l(K_Y+\Xi))\wedge\Xi=0.$$

Since $\mathrm{Mov}((K_Y+\Xi))$ is free and $\mathrm{Fix}(l(K_Y+\Xi))+\Xi$ has simple normal crossings support, it follows that $\mathrm{SBs}(K_Y+\Xi)$ contains no centers of $\mathrm{NKLT}(Y,\ulcorner\Xi\urcorner)$. Let $H\in\mathrm{Div}(Y)$ be an ample divisor and pick $0<m,l,q\in\mathbb{Z}$ such that l divides m and $Q=qH$ is sufficiently ample. We let $T=(f^{-1})_*S$, $\Gamma_T=(\Gamma-T)|_T$ and $\Xi_T=(\Xi-T)|_T$. For any section

$$\tau\in H^0(\mathcal{O}_T(m(K_T+\Xi_T)))=H^0(\mathcal{J}_{||m(K_T+\Xi_T)||}(m(K_T+\Xi_T))),$$

and any section $\sigma\in H^0(\mathcal{O}_T(Q))$, we have that

$$\sigma\cdot\tau\in H^0(\mathcal{J}_{||m(K_T+\Xi_T)||}(m(K_T+\Xi_T)+Q)).$$

By (4.1), $\sigma\cdot\tau$ is in the image of

$$H^0(\mathcal{O}_Y(m(K_Y+\Xi)+Q))\to H^0(\mathcal{O}_T(m(K_T+\Xi_T)+Q)).$$

Therefore, we have that

$$|m(K_T+\Xi_T)|+m(\Gamma_T-\Xi_T)+|Q|_T|\subset|m(K_Y+\Gamma)+Q|_T.$$

Notice that if $g=f|_T$, then $g_*\Gamma_T=\Omega+C|_S$. Since $g_*\Xi_T\le\Omega$, we have that $(S,g_*\Xi_T)$ is canonical and it follows that $|m(K_S+g_*\Xi_T)|=g_*|m(K_T+\Xi_T)|$. Pushing forward the above inclusion via f, one sees that

$$|m(K_S+g_*\Xi_T)|+m(\Omega+C|_S-g_*\Xi_T)+P|_S\subset|m(K_X+D+C)+P|_S$$

where $P=f_*Q$. For any prime divisor R on S we have

$$\mathrm{mult}_R\,\mathrm{Fix}(|l(K_X+D+C)|_S)=\mathrm{mult}_{R'}\,\mathrm{Fix}(|l(K_Y+\Gamma)|_T)$$

where $R' = (g^{-1})_* R$. Therefore

$$g_*\Xi_T - C|_S = \Omega - \Omega\wedge \mathrm{Fix}(|l(K_X+D+C)|_S)/l \geq \Omega - \Omega\wedge\lambda F \geq \Omega - \Phi,$$

and so

$$|m(K_S+\Omega-\Phi)| + m\Phi + (mC+P)|_S \subset |m(K_X+D)+mC+P|_S.$$

The result now follows from (3.46). □

4.1. **Deformation invariance of plurigenera.** Let $f : X \to Z$ be a smooth projective morphism from a smooth variety to an affine smmoth curve. Y.-T. Siu has shown that the plurigenera

$$P_m(X_z) = h^0(X_z, \mathcal{O}_{X_z}(mK_{X_z}))$$

are deformation invariant (do not depend on $z \in Z$). We will now give a proof of this beautiful result for the case of fibers of general type.

Theorem 4.3. *Let A be a sufficiently ample divisor on X. If $\kappa(K_{X_z}) \geq 0$ for general $z \in Z$, then $h^0(X_z, \mathcal{O}_{X_z}(mK_{X_z} + A|_{X_z}))$ does not depend on $z \in Z$.*

Proof. The function $h^0(X_z, \mathcal{O}_{X_z}(mK_{X_z} + A|_{X_z}))$ is upper semicontinuous (cf. [5, III.12.8]). Fix $z_0 \in Z$, we must show that $h^0(X_z, \mathcal{O}_{X_z}(mK_{X_z} + A|_{X_z})) = h^0(X_{z_0}, \mathcal{O}_{X_z}(mK_{X_z} + A|_{X_{z_0}}))$ this is equivalent to proving that $f_*\mathcal{O}_X(mK_X + A)$ is locally free (on a neighborhood of $z_0 \in Z$) or equivalently that

$$f_*\mathcal{O}_X(mK_X+A) \to H^0(X_z, \mathcal{O}_{X_z}(mK_{X_z}+A|_{X_z})) = f_*\mathcal{O}_{X_z}(mK_{X_z}+A|_{X_z})$$

is surjective (cf. [5, III.12.9]). Since Z is affine, this is equivalent to showing that $H^0(X, \mathcal{O}_X(mK_X + A) \to H^0(X_z, \mathcal{O}_{X_z}(mK_{X_z} + A|_{X_z}))$ is surjective.

We now apply (4.1) with $S = X_{z_0}$, $B = 0$, $k = 1$. We must check that the stable base locus of $K_X + S$ does not contain S. Note that $S \sim 0$ and hence it suffices to show that $H^0(X, (lK_X)) \neq 0$ for some $l > 0$. Since Z is affine, it is easy to see that this is equivalent to showing that $f_*(lK_X) \neq 0$ i.e. that $h^0(X_z, \mathcal{O}_{X_z}(lK_{X_z})) \neq 0$ for general $z \in Z$. But this is clear from the assumption that $\kappa(K_{X_z}) \geq 0$. □

Theorem 4.4. *If X_z is of general type for general $z \in Z$, then $h^0(X_z, \mathcal{O}_{X_z}(mK_{X_z}))$ does not depend on $z \in Z$ for any $m \geq 0$.*

Proof. The proof follows from (4.2), however we will give an elementary proof (using the techniques introduced above). We may assume that $m \geq 2$ (the case $m = 0$ is trivial and $m = 1$ is well known and follows from Hodge Theory: $h^0(X_z, \mathbb{C}_{X_z})$ is constant and given by

$\sum_{i=0}^{n} h^i(X_z, \Omega^{n-i}_{X_z})$ where $n = \dim X_z$. Since each $h^i(X_z, \Omega^{n-i}_{X_z})$ is upper semicontinuous, it must in fact be constant.)

Fix $\sigma \in H^0(X_z, \mathcal{O}_{X_z}(mK_{X_z}))$ with zero divisor Σ. We must show that σ extends to X.

By (4.3), there is an ample line bundle A such that $H^0(X, \mathcal{O}_X(lK_X + A) \to H^0(X_z, \mathcal{O}_{X_z}(lK_{X_z} + A|_{X_z}))$ is surjective for all $l > 0$. For any $l = km$, pick $D_l \in |lK_X + A|$ so that $D_l|_{X_z} = k\Sigma + A|_{X_z}$. Since X_z is of general type and Z is affine, it follows that K_X is of general type and hence we may write $K_X \sim_{\mathbb{Q}} E + \epsilon A$ where $\epsilon > 0$, and $E \geq 0$. Define $\Theta = \frac{m-1-\delta}{km} D_{km} + \delta\epsilon E$ for $0 < \delta \ll 1$, then

$$(m-1)K_X - (X_z + \Theta) \sim_{\mathbb{Q}} (\delta\epsilon - \frac{m-1-\delta}{km})A$$

is ample (for $k \gg 0$) and so by (3.43), $H^0(X_z, \mathcal{O}_{X_z}(mK_{X_z}) \otimes \mathcal{J}(X_z, \Theta|_{X_z}))$ is contained in the image of the restriction map

$$H^0(X, \mathcal{O}_X(mK_X)) \to H^0(X_z, \mathcal{O}_{X_z}(mK_{X_z})).$$

Thus it suffices to check that $\sigma \in H^0(X_z, \mathcal{O}_{X_z}(mK_{X_z}) \otimes \mathcal{J}(X_z, \Theta|_{X_z}))$ i.e. that σ vanishes along (the scheme defined by) $\mathcal{J}(X_z, \Theta|_{X_z}))$. Since

$$\Theta|_{X_z} - \Sigma \leq \frac{m-1-\delta}{km} A|_{X_z} + \delta\epsilon E|_{X_z}$$

and $(X_z, \frac{m-1-\delta}{km} A|_{X_z} + \delta\epsilon E|_{X_z})$ is klt, the claim follows from (3) of (3.42). □

Exercise 4.5. *Show that if X_z is of general type and Z is affine, then X is of general type.*

5. Pl-flips

In this section we will prove the existence of pl-flips.

5.1. pl-flips and the restricted algebra.

Definition 5.1. *Let (X, D) be a purely log terminal pair and $f : X \to Z$ be a projective morphism of normal varieties, then f is a* **pl-flipping contraction** *if*

(1) *X is $\mathbb{Q}$-factorial,*
(2) *$D \in \mathrm{Div}_{\mathbb{Q}}(X)$,*
(3) *f is small (i.e. $\dim \mathrm{Ex}(f) \leq \dim X - 2$) and $\rho(X/Z) = 1$,*
(4) *$-(K_X + D)$ is f-ample, and*
(5) *$S = \llcorner D \lrcorner$ is irreducible and $-S$ is f-ample.*

The **flip** *of a pl-flipping contraction if it exists is defined by*

$$f^+ : X^+ = \mathrm{Proj}_Z \mathfrak{R} \to Z \qquad \text{where} \qquad \mathfrak{R} = \bigoplus_{m\in\mathbb{N}} f_*\mathcal{O}_X(m(K_X+D)).$$

Remark 5.2. *Note the following:*

(1) *The flip exists if and only if it exists locally over Z. We may therefore assume that $Z = \mathrm{Spec}A$.*

(2) *Assuming that $Z = \mathrm{Spec}A$ then the flip exists if and only if*

$$R(K_X+D) = \bigoplus_{m\in\mathbb{N}} H^0(\mathcal{O}_X(m(K_X+D)))$$

is a finitely generated A-algebra.

(3) *It immediately follows that $f^+ : X^+ \to Z$ is also a small birational morphism with $\rho(X^+/Z) = 1$, X^+ is $\mathbb{Q}$-factorial and $K_{X^+} + D^+$ is f^+-ample where $D^+ = \phi_* D$ and $\phi = (f^+)^{-1} \circ f : X \dashrightarrow X^+$. ϕ restricts to an isomorphism over $Z - f(\mathrm{Ex}(f))$.*

(4) *It is easy to see that if in the above definition $D \in \mathrm{Div}_{\mathbb{R}}(X)$ instead of $D \in \mathrm{Div}_{\mathbb{Q}}(X)$, then one can choose $D' \in \mathrm{Div}_{\mathbb{Q}}(X)$ sufficiently close to D such that $f : X \to Z$ is a pl-flipping contraction with respect to (X, D). Similarly if $\llcorner D \lrcorner = S_1 + \ldots + S_r$ with $r > 1$, then there exists $S = S_i$ such that $-S$ is f-ample. Replacing D by $D - \epsilon(\llcorner D \lrcorner - S)$ we may assume that $\llcorner D \lrcorner$ is irreducible.*

Shokurov noticed that in order to prove the existence of flips, it suffices to prove the existence of pl-flips.

Definition 5.3. *If $f : X \to Z$ is a pl-flipping contraction and Z is affine, then we define the* **restricted algebra**

$$R_S(K_X+D) = \mathrm{Im}(R(X, K_X+D) \to R(S, K_S+\Omega))$$

where $\Omega \in \mathrm{Div}_{\mathbb{Q}}(S)$ is defined by $(K_X+D)|_S = K_S + \Omega$. Its m-th graded piece corresponds to the image of the homomorphism

$$H^0(\mathcal{O}_X(m(K_X+D))) \to H^0(\mathcal{O}_S(m(K_S+\Omega))).$$

In order to prove that $R(X, K_X+D)$ is finitely generated, Shokurov observed that it suffices to show that the restricted algebra $R_S(K_X+D)$ is finitely generated. We start by recalling the following well known result.

Lemma 5.4. *Let R be a graded algebra which is an integral domain and let $0 < d \in \mathbb{Z}$. Then R is a finitely generated algebra if and only if the algebra*

$$R_{(d)} = \bigoplus_{m\in\mathbb{N}} R_{md}$$

is a finitely generated algebra.

Proof. If R is finitely generated, then finite generation of $R_{(d)}$ follows since $R_{(d)}$ is the ring of invariants of R with respect to the obvious $\mathbb{Z}_d$ action on R and since by a theorem of E. Noether, the ring of invariants of a finitely generated ring under the action of a finite group is finitely generated.

Assume now that $R_{(d)}$ is finitely generated. Notice that if $f \in R_i$, then f is a root of the monic polynomial $x^d - f^d \in R_{(d)}[x]$ and hence R is integral over $R_{(d)}$. Finite generation of R now follows by E. Noether's theorem on the finiteness of integral closures. $\square$

Proposition 5.5. *If S is a normal prime divisor and $B \in \mathrm{WDiv}(X)$ is integral Weil and $\mathbb{Q}$-Cartier and its support does not contain S, then*

(1) *If $R(X, B)$ is finitely generated, then so is*

$$R_S(X, B) := \mathrm{Im}(\phi : R(X, B) \to R(S, B|_S)).$$

(2) *If $S \sim B$ and $R_S(X, B)$ is finitely generated then so is $R(X, B)$.*

Proof. (1) is clear. Assume now that $R_S(X, B)$ is finitely generated and $S \sim B$ so that $S - B = (g_1)$ for some rational function g_1 on X. We may identify $R(X, B)_m$ with the set of rational functions g on X such that $(g) + mB \geq 0$. Now if $g \in \ker(\phi)$, then $(g) + mB = S + S'$ where $S' \geq 0$. Then

$$(g/g_1) + (m - 1)B = S'$$

so that $g/g_1 \in R(X, B)_{m-1}$. In other words the kernel of ϕ is generated by g_1 and the result follows. $\square$

Theorem 5.6. *Let $f : X \to Z$ be a pl-flipping contraction with respect to (X, D) and $0 < k \in \mathbb{Z}$ such that $k(K_X + D) \in \mathrm{Div}(X)$. If $Z = \mathrm{Spec}(A)$, then the flip $f^+ : X^+ \to Z$ exists if and only if the restricted algebra $R_S(K_X + D)$ is finitely generated.*

Proof. By (5.2), the flip $f^+ : X^+ \to Z$ exists if and only if $R(K_X + D)$ is finitely generated. Since there are positive integers a and b such that $a(K_X + D) \sim bS$ (cf. (6.11)), by (5.4), $R(X, K_X + D)$ is finitely generated if and only if $R(X, S)$ is finitely generated. Let $S' \sim S$ be a divisor in $\mathrm{WDiv}(X)$ whose support does not contain S (this exists as $X \to Z$ is small and Z is affine). By (5.5) $R(X, S')$ is finitely generated if and only if $R_S(X, S')$ is finitely generated. By (5.4), $R_S(X, S')$ is finitely generated if and only if $R_S(k(K_X+D))$ is finitely generated. $\square$

5.2. **Zariski decomposition and pl-flips.** In order to prove the existence of pl-flips in dimension n, we will need to assume that we have constructed log terminal models in dimension $n-1$. Recall the following.

Definition 5.7. *Let (X, D) be a log canonical pair, Z be an affine variety and $f : X \dashrightarrow Y$ be a birational map over Z that extracts no divisor, then $f : X \dashrightarrow Y$ is a* **log terminal model of** (X, D) *if*

1. *$f : X \dashrightarrow Y$ extracts no divisor (i.e. f^{-1} contracts no divisors),*
2. *Y is $\mathbb{Q}$-factorial,*
3. *(Y, f_*D) is divisorially log terminal,*
4. *for any prime divisor E on X contracted by f, we have $a_E(X, D) < a_E(Y, f_*D)$, and*
5. *$K_Y + f_*D$ is nef over Z.*

Remark 5.8. *If f consists of a sequence of flips and divisorial contractions, then (1) and (4) follow. If moreover, X is $\mathbb{Q}$-factorial and (X, D) is divisorially log terminal, then (2) and (3) also follow.*

Theorem 5.9. *Let $\pi\colon X \longrightarrow Z$ be a projective morphism to a normal affine variety, $(X, D = A + B)$ be a kawamata log terminal pair of dimension n, where $A \geq 0$ is an ample $\mathbb{Q}$-divisor and $B \geq 0$. Then*

1. *The pair (X, D) has a log terminal model $\mu\colon X \dashrightarrow Y$. In particular if $K_X + D$ is $\mathbb{Q}$-Cartier then the log canonical ring*
$$R(X, K_X + D) = \bigoplus_{m \in \mathbb{N}} H^0(X, \mathcal{O}_X(\llcorner m(K_X + D) \lrcorner)),$$
is finitely generated.
2. *Let $V \subset \mathrm{Div}_{\mathbb{R}}(X)$ be the vector space spanned by the components of D. Then there is a constant $\delta > 0$ such that if G is a prime divisor contained in the stable base locus of $K_X + D$ and $\Xi \in V$ such that $\|\Xi - D\| < \delta$, then G is contained in the stable base locus of $K_X + \Xi$.*
3. *Let $W \subset V$ be the smallest rational affine space containing D. Then there is a constant $\eta > 0$ and a positive integer $r > 0$ such that if $\Xi \in W$ is any divisor and k is any positive integer such that $\|\Xi - D\| < \eta$ and $k(K_X + \Xi)/r$ is Cartier, then every component of $\mathrm{Fix}(k(K_X + \Xi))$ is a component of the diminished stable base locus of $K_X + D$.*

In this section we will prove

Theorem 5.10. $(5.9)_{n-1}$ *implies that pl-flips exist in dimension n.*

We begin by proving the following:

Theorem 5.11. *Assume (5.9) in dimension $n-1$. Let $\pi : X \to Z$ be a projective morphism to a normal affine variety. Let $(X, D = S + A + B)$ be a purely log terminal pair of dimension n where X and S are smooth, $D \in \mathrm{Div}_{\mathbb{Q}}(X)$, $\llcorner D \lrcorner = S$, A is a general ample divisor, $(S, \Omega = (D-S)|_S)$ is canonical and the stable base locus of $K_X + D$ does not contain S. For any sufficiently divisible $0 < m \in \mathbb{Z}$, let*

$$F_m = \mathrm{Fix}(|m(K_X + D)|_S)/m$$

and $F = \lim F_{m!}$.

Then $\Theta = \Omega - \Omega \wedge F$ is rational. In particular if $kD \in \mathrm{Div}(X)$ and $k\Theta \in \mathrm{Div}(S)$, then

$$|k(K_S + \Theta)| + k(\Omega - \Theta) = |k(K_X + D)|_S,$$

and

$$R_S(X, k(K_X + D)) \cong R(S, k(K_S + \Theta)).$$

Proof. Suppose that $\Theta \notin \mathrm{Div}_{\mathbb{Q}}(S)$. Let $V \subset \mathrm{Div}_{\mathbb{R}}(S)$ be the vector space spanned by the components of Θ. There is a constant $\delta > 0$ such that if $\Phi \in V$ and $||\Phi - \Theta|| < \delta$, then

(1) $\Phi \geq 0$,
(2) $\mathrm{Supp}(\Phi) = \mathrm{Supp}(\Theta)$ and
(3) any prime divisor contained in the stable base locus of $K_S + \Theta$ is also contained in the stable base locus of $K_S + \Phi$.

Notice that if $l(K_X + \Delta)$ is Cartier and $\Theta_l = \Omega - \Omega \wedge F_l$, then

$$|l(K_X + \Delta)|_S \subset |l(K_S + \Theta_l)| + l(\Omega \wedge F_l).$$

It follows that $\mathrm{Fix}(l(K_S + \Theta_l))$ does not contain any component of Θ_l. Therefore

$$\mathrm{SBs}(K_S + \Theta_l) \wedge \mathrm{Supp}(\Theta_l) = 0.$$

But for any $\delta > 0$ we may choose $l > 0$ sufficiently divisible so that $\Theta_l \in V$ and $||\Theta_l - \Theta|| < \delta$. Therefore

$$\mathrm{SBs}(K_S + \Theta) \wedge \mathrm{Supp}(\Theta) = 0.$$

We now consider $W \subset V$ the smallest rational affine vector space containing Θ. By assumption $\dim W > 0$. By (3) of (5.9), there are a positive integer $r > 0$ and a constant $0 < \eta \in \mathbb{R}$ such that for any $\Phi \in W$ with $k\Phi/r \in \mathrm{Div}(S)$ and $||\Phi - \Theta|| < \eta$ then

$$\mathrm{Fix}(k(K_S + \Phi)) \subset \mathrm{SBs}(K_S + \Theta).$$

We now pick $0 < \epsilon \in \mathbb{Q}$ such that $\epsilon(K_X + D) + A$ is ample. By Diophantine approximation, we may find a positive integer k, $\Phi \in \mathrm{Div}_{\mathbb{Q}}(S)$ and a component G of $\mathrm{Supp}(\Theta)$ (whose coefficient in Θ is irrational) such that

(1) $0 \leq \Phi \in W$,
(2) $k\Phi/r \in \mathrm{Div}(S)$ and $kD/r \in \mathrm{Div}(X)$,
(3) $||\Phi - \Theta|| < \min\{\delta, \eta, f\epsilon/k\}$ where f is the smallest non-zero coefficient of F, and
(4) $\mathrm{mult}_G \Phi > \mathrm{mult}_G \Theta$.

One sees that since $||\Phi - \Theta|| < f\epsilon/k$, then we have

$$\Omega \wedge \lambda F \leq \Omega - \Phi \leq \Omega$$

where $\lambda = 1 - \epsilon/k$. By (2) and (4.2) we have that

$$|k(K_S + \Phi)| + k(\Omega - \Phi) \subset |k(K_X + \Delta)|_S.$$

But by (4) G is a component of $\mathrm{Fix}(k(K_S+\Phi))$. Since $||\Phi-\Theta|| < \eta$, (2) implies that G is a component of $\mathrm{SBs}(K_S+\Theta)$. This is a contradiction. It follows that Θ is rational. The remaining assertions follow from (4.2). $\square$

We are now ready to prove the main theorem of this section which easily implies (5.10).

Theorem 5.12. *Assume that* (5.9) *holds in dimension* $n-1$.

Let $f\colon X \longrightarrow Z$ *be a projective morphism to a normal affine variety* Z*. Suppose that* $(X, D = S + A + B)$ *is a purely log terminal pair of dimension* n*,* $S = \llcorner D \lrcorner$ *is irreducible and not contained in the stable base locus of* $K_X + D$*,* $A \geq 0$ *is a general ample* $\mathbb{Q}$*-divisor and* $B \geq 0$ *is a* $\mathbb{Q}$*-divisor.*

Then there is a birational morphism $g\colon T \longrightarrow S$*, a positive integer* l *and a kawamata log terminal pair* (T, Θ) *such that*

$$R_S(X, l(K_X + D)) \cong R(T, l(K_T + \Theta)).$$

Proof. Let $\mu\colon Y \longrightarrow X$ be a log resolution of (X, D) then we may write

$$K_Y + \Gamma_Y(X, D) = \mu^*(K_X + D) + E_Y(X, D).$$

If T is the strict transform of S then we may choose μ so that $(T, \Psi = (\Gamma_Y(X, D) - T)|_T)$ is terminal. Note that T is not contained in the stable base locus of $K_Y + \Gamma_Y(X, D)$ as S is not contained in the stable base locus of $K_X + D$.

Pick a $\mathbb{Q}$-divisor F such that $\mu^*A - F$ is ample and $(Y, \Gamma_Y(X, D) + F)$ is purely log terminal. Pick $m > 1$ so that $m(\mu^*A - F)$ is very ample and pick $mC \in |m(\mu^*A - F)|$ very general. Then

$$(Y, \Gamma' = \Gamma_Y(X, D) - \mu^*A + F + C \sim_{\mathbb{Q}} \Gamma_Y(X, D)),$$

is purely log terminal and $(T, \Psi = (\Gamma' - T)|_T)$ is terminal.

On the other hand

$$R(X, k(K_X + D)) \cong R(Y, k(K_Y + \Gamma_Y(X, D))) \qquad \text{and}$$
$$R_S(X, k(K_X + D)) \cong R_T(Y, k(K_Y + \Gamma')),$$

for any k sufficiently divisible. Now apply (4.2) to (Y, Γ'). □

Proof of (5.10). We may assume that Z is affine and by (5.6), it suffices to prove that the restricted algebra is finitely generated. As Z is affine, S is mobile and as f is birational, the divisor $D - S$ is big. But then

$$D - S \sim_{\mathbb{Q}} A + B,$$

where A is a general ample $\mathbb{Q}$-divisor and $B \geq 0$. As S is mobile, we may assume that the support of B does not contain S. Now

$$K_X + D' = K_X + S + (1-\epsilon)(D - S) + \epsilon A + \epsilon B \sim_{\mathbb{Q}} K_X + D,$$

is purely log terminal, where ϵ is any sufficiently small positive rational number. By (5.4), we may replace D by D'. We may therefore assume that $D = S + A + B$, where A is a general ample $\mathbb{Q}$-divisor and $B \geq 0$. Since we are assuming (5.9) in dimension $n-1$, (5.12) implies that the restricted algebra is finitely generated. □

6. The cone theorem

The following results (toghether with the existance and termination of flips) constitute the heart of the minimal model program. These results are due to the contribuitions of many mathematicians in the 80's. In particular to Kawamata, Reid and Shokurov.

Theorem 6.1 (Non-vanishing Theorem)**.** *Let X be a projective variety, D a nef Cartier divisor and Δ a $\mathbb{Q}$-divisor such that (X, Δ) is sub kawamata log terminal (i.e. Δ is possibly not effective). Suppose that $aD - K_X - \Delta$ is $\mathbb{Q}$-Cartier, nef and big for some $a > 0$.*

Then, for all $m \gg 0$, we have $H^0(X, \mathcal{O}_X(mD - \llcorner \Delta \lrcorner)) \neq 0$.

Theorem 6.2 (Basepoint-free Theorem)**.** *Let (X, Δ) be a projective kawamata log terminal pair and D be a nef Cartier divisor such that $aD - K_X - \Delta$ is nef and big for some $a > 0$. Then $|bD|$ is basepoint-free for all $b \gg 0$.*

Theorem 6.3 (Rationality Theorem)**.** *Let (X, Δ) be a projective kawamata log terminal pair such that $K_X + \Delta$ is not nef. Let $a = a(X, \Delta) > 0$ be an integer such that $a(K_X + \Delta)$ is Cartier. Let H be a nef and big Cartier divisor, and define*

$$r = r(H) = \max\{t \in \mathbb{R} : H + t(K_X + \Delta) \text{ is nef}\}.$$

Then r is a rational number and it may be written as $r = u/v$ where u, v are integers and

$$0 < v < a(\dim X + 1).$$

Theorem 6.4 (Cone Theorem). *Let (X, Δ) be a projective kawamata log terminal pair. Then*

(1) *There are countably many rational curves $C_j \subset X$ such that $0 < -(K_X + \Delta) \cdot C_j \leq 2 \dim X$, and*

$$\overline{NE}(X) = \overline{NE}(X)_{(K_X+\Delta)\geq 0} + \sum \mathbb{R}_{\geq 0}[C_j].$$

(2) *For any $\epsilon > 0$, there are only finitely many rays*

$$[C_j] \in \overline{NE}(X)_{(K_X+\Delta+\epsilon H)<0}.$$

(3) *If $F \subset \overline{NE}(X)$ is a $(K_X+\Delta)$ negative extremal face, then there is a unique morphism $\operatorname{cont}_F : X \to Z$ such that $(\operatorname{cont}_F)_*\mathcal{O}_X = \mathcal{O}_Z$ (in particular Z is normal, cont_F is surjective with connected fibers) and an irreducible curve $C \subset X$ is contracted to a point if and only if $[C] \in F$.*

(4) *let L be a line bundle on X such that $L \cdot C = 0$ for all curves with $[C] \in F$. Then there is a line bundle L_Z on Z such that $(\operatorname{cont}_F)^* L_Z = L$.*

Proof. (We closely follow the proof in [10, §3.3].) If $K_X+\Delta$ is nef, (1) is clear and there is nothing to prove. If not, we must begin by choosing the countable collection of rays R_i (eventually we would like to show that $R_i = \mathbb{R}_{\geq 0}[C_i]$ for some rational curve C_i). We consider nef divisor classes L which are not ample so that $F_L = L^\perp \cap \overline{NE} \neq \{0\}$ (there are countably many of these). We claim that if $F_L \not\subset \overline{NE}_{K_X+\Delta\geq 0}$ then there is a nef divisor L' such that

$$F_L \supset F_{L'}, \qquad \dim F_{L'} = 1, \text{ and } F_{L'} \subset \overline{NE}_{K_X+\Delta<0}.$$

To see this, pick H an ample line bundle, $a = a(K_X + \Delta)$ such that $a(K_X + \Delta)$ is Cartier, $\alpha = (a(d + 1))!$ and let

$$r_L(n, H) := \max\{t \in \mathbb{R} : nL + H + t(K_X + \Delta) \text{ is nef}\}.$$

By (6.3) we have $\alpha r_L(n, H) \in \mathbb{N}$ and $r_L(n, H)$ is a non-decreasing function of n as L is nef. If $\xi \in F_L \setminus \overline{NE}_{(K_X+\Delta)_{\geq 0}}$, then

$$r_L(n, H) \leq \frac{H \cdot \xi}{-(K_X + \Delta) \cdot \xi}.$$

Thus $r_L(n, H)$ is bounded and hence $r_L(n, H) = r_L(H)$ for $n \gg 0$. So

$$D = D(nL, H) := \alpha(nL + H + r_L(H)(K_X + \Delta))$$

is a nef non-ample divisor (for $n \gg 0$) and

$$0 \neq F_D \subset F_L, \text{ and } \qquad F_D \subset \overline{NE}_{K_X+\Delta<0} \cup \{0\}.$$

To see the first inclusion, let $\xi \in F_D$, then $D \cdot \xi = 0$ and $(D-L)\cdot\xi \geq 0$ (as $D-L$ is nef for $n \gg 0$). But then $-L\cdot\xi = (D-L)\cdot\xi - D\cdot\xi \geq 0$ so that $L \cdot \xi = 0$ as L is nef. The second inclusion follows since by the first inclusion then we have $C \cdot D = C \cdot (H + \alpha r_L(H)(K_X + \Delta)) = 0$.

We must now show that by varying H we may assume that if $\dim F_L > 1$, then $\dim F_D < \dim F_L$. Let $V \subset N_1(X)$ be the subspace spanned by F_L, then we must show that

$$(nL + H + \alpha r_L(H)(K_X+\Delta))|_V = (H + \alpha r_L(H)(K_X+\Delta))|_V \neq 0.$$

If this were not the case, then the image of H via the projection $N^1(X) \to V$ is always contained in the linear subspace spanned by the image of $K_X + \Delta$. But since $N^1(X)$ is generated by classes of ample divisors, it follows that $\dim F_L = 1$ as required. Repeating the above argument we end up with an L' such that $\dim L' = 1$.

We must now show that

$$\overline{NE} = \overline{NE_{(K_X+\Delta)_{\geq 0}} + \sum_{\dim F_L = 1} F_L}.$$

The inclusion $\supset$ is clear. Suppose that the reverse inclusion does not hold, then there is a divisor M intersecting the interior of the left hand side and such that the right hand side is contained in $M_{<0}$. Let H be an ample divisor and

$$t = \max\{s > 0 | H + sM\}$$

is ample, then $t \in \mathbb{Q}$ (cf. (6.3)) and there is an element $Z \in \overline{NE} \setminus \{0\}$ such that $(H+tM)\cdot Z = 0$ and $Z \cdot (K_X+\Delta) < 0$. By what we have seen above, we can find a nef but not ample divisor L with $F_L \subset F_{H+tM}$ and $\dim F_L = 1$. By the definition of $H + tM$ it is clear that F_L is contained in $M_{>0}$ and this is the required contradiction.

We now check that the one dimensional rays F_L only accumulate in a neighborhood of $(K_X + \Delta)^\perp$, we proceed as follows. Let $K_X + \Delta, H_1, \ldots, H_d$ give a basis of $N^1(X)$ defined over $\mathbb{Z}$, where the H_i are ample divisors. Let $U \subset \mathbb{P}^d = \mathbb{P}(N_1(X))$ be the halfspace defined by $(K_X + \Delta)_{<0}$ and consider the coordinate system defined by

$$\xi \in U \to \phi(\xi) = (\frac{\xi \cdot H_1}{\xi \cdot (K_X+\Delta)}, \ldots, \frac{\xi \cdot H_d}{\xi \cdot (K_X+\Delta)}).$$

Let $N_{\mathbb{Z}} \subset N_1(X)$ be the set of integral classes, then $\Lambda = \phi(N_{\mathbb{Z}} \cap (K_X + \Delta)_{<0}) \subset \mathbb{A}^d$ is contained in a lattice. Thus if ξ is an integral class generating F_L, then $\phi(\xi) \in \Lambda$ and such classes do not accumulate in

U; i.e. the only accumulation points of F_L are near the hyperplane at infinity $(K_X+\Delta)^{\perp}$. But it is then clear that there are only finitely many F_L which are $K_X+\Delta+\epsilon H$ negative.

We next check that if F is a $K_X+\Delta$ negative extremal face, then $F=F_D=D^{\perp}\cap\overline{NE}\neq\{0\}$ for some Cartier divisor D. We let $<F>\subset N_1(X)$ be the linear subspace spanned by F and $V=<F>^{\perp}\subset N^1(X)$. F is spanned by the extremal rays that it contains and each extremal ray is defined over $\mathbb{Q}$ so that V is defined over $\mathbb{Q}$. We may pick $\epsilon>0$ such that $F\subset(K_X+\Delta+\epsilon H)_{<0}$. Note that as F is extremal, we have $<F>\cap\overline{NE}(X)=F$ and so

$$W_F:=\overline{NE}(X)_{(K_X+\Delta+\epsilon H)_{\geq 0}}+\sum_{\dim F_L=1,\ F_L\not\subset F}F_L$$

is a closed cone with

$$\overline{NE}(X)=W_F+F,\ \text{and}\qquad W_F\cap<F>=\{0\}.$$

Thus there is a hyperplane H containing F and not intersecting $W_F\setminus\{0\}$. Equivalently $H\in V$ and $H_{>0}\supset W_F\setminus\{0\}$. We may then find a rational hyperplane $H'\in V$ such that $H'_{>0}\supset W_F\setminus\{0\}$ and thus H' is a $\mathbb{Q}$-Cartier divisor with $F_{H'}=F$.

To see (3) and (4), let $F\subset\overline{NE}(X)$ be a $K_X+\Delta$ negative extremal face $F=F_D$ for some $\mathbb{Q}$-Cartier divisor D. For any $m\gg 0$, the]$\mathbb{Q}$-Cartier divisor $mD-(K_X+\Delta)$ is strictly positive on $\overline{NE}(X)-\{0\}$. Thus $mD-(K_X+\Delta)$ is ample and mD is nef and by (6.2) mD is base point free (for any $m\gg 0$). Let $g_F=g_{F,m}:X\to Z=Z_m$ be the Stein factorization of $X\to|mD|$ so that Z_m is normal and ${g_F}_*\mathcal{O}_X=\mathcal{O}_Z$. Let $M_{Z,m}$ be the pull back of the hyperplane bundle to Z_m so that $mD=g^*_{F,m}M_{Z,m}$. It is easy to see that a curve C is contracted by g_F if and only if $C\cdot D=0$, thus $g_{F,m}=g_F:X\to Z$ is independent of $m\gg 0$ (as it is determined by the curves it contracts). Hence $D=(m+1)D-mD\sim g^*_FM_{Z,m+1}-g^*_FM_{Z,m}$. Similarly if $L\cdot C=0$ for all $[C]\in F$, then $L+mD$ also supports F for $m\gg 0$ and so it defines $g_F:X\to Z$. By what we have seen above $L+mD\sim g^*_FN_Z$ for some Cartier divisor N_Z on Z, so that $L=g^*_F(N_Z-M_{Z,m})$.

□

Corollary 6.5. *If F is a negative extremal ray, inducing a morphism $g_F:X\to Z$, then there is a short exact sequence*

$$0\to\operatorname{Pic}(Z)\to\operatorname{Pic}(Z)\to\mathbb{Z}$$

where the first map is defined by $L\to g^*_FL$ and the second one by $M\to M\cdot C$ where C is any (fixed) contracted curve. In particular $\rho(X)=\rho(Z)+1$.

Proof. Exercise (cf. [10, §3]). □

Corollary 6.6. *If X is $\mathbb{Q}$-factorial, F is a negative extremal ray, and $g_F : X \to Z$ is of divisorial or Fano type, then Z is $\mathbb{Q}$-factorial.*

Proof. Suppose that g_F is divisorial (i.e. that the codimension of the exceptional locus is 1), and let E be an exceptional divisor such that $E \cdot C \neq 0$ for some exceptional curve C.Note that $F = \mathbb{R}[C]$ For any divisor B on Z, pick $s = -(g_F^{-1})_* B \cdot C / E \cdot C$ so that

$$((g_F^{-1})_* B + sE) \cdot C = 0.$$

Pick $m \in \mathbb{N}$ so that $m(g_F^{-1})_* B + sE)$ is Cartier. By (6.4), we have $m(g_F^{-1})_* B + sE) \sim g_F^* M_Z$ for some Cartier divisor M_Z on Z. Since

$$mB = g_{F*}(m(g_F^{-1})_* B + sE)) \sim M_Z$$

it follows that B is $\mathbb{Q}$-Cartier.

Suppose that g_F is Fano (i.e. that $\dim X > \dim Z$. Let B be a divisor on Z and let G be the closure of the pull back of B^0 the restriction of B to the smooth locus of Z (note that B^0 is a Cartier divisor on the smooth locus of Z and hence it's pull-back makes sense). Let $C \subset X_z$ be a curve on a general fiber X_z so that $F = \mathbb{R}[C]$. Clearly $G \cdot C = 0$. Pick $m \in \mathbb{N}$ so that mG is Cartier. By (6.4), we have $mG \sim g_F^* M_Z$ for some Cartier divisor M_Z on Z. Since $mB = g_{F*} mG \sim M_Z$, it follows that B is $\mathbb{Q}$-Cartier. □

Proof of the Non-vanishing Theorem. This proof is based on an argument of Shokurov cf. [10, §3.5]

Step 0. *We may assume that X is smooth, projective, (X, Δ) is sub kawamata log terminal and $aD - K_X - \Delta$ is ample for some $a > 0$.*

To see this, consider $f : X' \to X$ a birational map from a smooth projective variety. We write

$$K_{X'} + \Delta' = f^*(K_X + \Delta)$$

so that (X', Δ') is sub kawamata log terminal (Δ' is possibly not effective), and $af^*D - K_{X'} - \Delta' = f^*(aD - K_X - \Delta)$ is nef and big. So we may choose an effective divisor $F \in \mathrm{Div}_{\mathbb{Q}}(X')$ and an ample divisor $A \in \mathrm{Div}_{\mathbb{Q}}(X')$ such that $f^*(aD - K_X - \Delta) \sim_{\mathbb{Q}} A + F$. It follows that for any rational number $0 < \epsilon \ll 1$, we have that

$$f^*(aD - K_X - \Delta) - \epsilon F \sim_{\mathbb{Q}} (1 - \epsilon) f^*(aD - K_X - \Delta) + \epsilon A$$

is ample. We then have that $af^*D - K_{X'} - \Delta' - \epsilon F$ is ample and $(X', \Delta' + \epsilon F)$ is sub-kawamata log terminal. Let $\Delta'' := \Delta' + \epsilon F$, then $f_* \Delta'' \geq \Delta$ and one sees that

$$h^0(X', \mathcal{O}_X(mf^*D + \ulcorner -\Delta'' \urcorner)) \leq h^0(X, \mathcal{O}_X(mD + \ulcorner -\Delta \urcorner)).$$

Step 1. *We may assume that D is not numerically equivalent to* 0. If in fact $D \equiv 0$ (i.e. D is numerically trivial), for any $a, t \in \mathbb{Z}$ we have that

$$kD + \ulcorner -\Delta \urcorner \equiv K_X + \{\Delta\} + tD - (K_X + \Delta).$$

Since for $t \geq a$, $tD - (K_X + \Delta)$ is ample, by Kawamata Viehweg vanishing we compute

$$h^0(X, \mathcal{O}_X(mD + \ulcorner -\Delta \urcorner)) = \chi(X, \mathcal{O}_X(mD + \ulcorner -\Delta \urcorner)) =$$

$$\chi(X, \mathcal{O}_X(\ulcorner -\Delta \urcorner)) = h^0(X, \mathcal{O}_X(\ulcorner -\Delta \urcorner)) \neq 0.$$

Step 2. *For any point $x \in (X/\operatorname{Supp}(\Delta))$ there exists an integer q_0 such that for all integers $q \geq q_0$, there is a $\mathbb{Q}$-divisor*

$$M(q) \equiv (qD - K_X - \Delta)$$

with $\operatorname{mult}_x M(q) > 2 \dim X$.

To see this, let $d = \dim X$. D is nef so that $D^e \cdot A^{d-e} \geq 0$ for any ample divisor A, and so

$$(qD - K_X - \Delta)^d = ((q-a)D + aD - K_X - \Delta)^d \geq d(q-a)D \cdot (aD - K_X - \Delta)^{d-1}.$$

Since $D \not\equiv 0$, there is a curve $C \subset X$ such that $D \cdot C > 0$ and since D is big, there is an integer $p \gg 0$ such that $(p(aD - K_X - \Delta))^{d-1}$ may be represented by an effective cycle containing C. Therefore, $D \cdot (aD + G - K_X)^{d-1} > 0$. Therefore, the right hand side in the above equation goes to ∞ as q goes to ∞. So, by Serre-Vanishing and Riemann-Roch, we have

$$h^0(\mathcal{O}_X(e(qD - K_X - \Delta))) \geq \frac{e^d}{d!}(2d)^d + O(e^{d-1}).$$

Vanishing along x with multiplicity $> 2de$ imposes at most

$$\frac{(2de)^d}{d!} + O(e^{d-1})$$

conditions. So we can find a divisor

$$M(q, e) \in |e(qD - K_X - \Delta)| \qquad \text{with} \qquad \operatorname{mult}_x M(q, e) > 2de.$$

We set $M(q) = M(q, e)/e$.

Step 3. We pick a log resolution $f : Y \to X$ which dominates $\operatorname{Bl}_x(X)$. We set

(1) $K_Y \equiv f^*(K_X + \Delta) + \sum b_j F_j$, where $b_j > -1$,
(2) $(1/2)f^*(aD - K_X - \Delta) - \sum p_j F_j$ is ample for some $0 < p_j \ll 1$,
(3) $f^*M(q) = \sum r_j F_j$ with F_0 corresponding to the strict transform of the exceptional divisor of $\operatorname{Bl}_x(X) \to X$.

Step 4. We define

$$N(b,c) := bf^*D + \sum(-cr_j + b_j - p_j)F_j - K_Y.$$

We would like to arrange that $N(b,c)$ is ample. To this end we write

$$N(b,c) = bf^*D + \sum(-cr_j + b_j - p_j)F_j - K_Y$$

$$\equiv bf^*D - cf^*(qD - K_X - \Delta) - \sum p_j F_j - f^*(K_X + \Delta)$$

$$= (b - a - c(q-a))f^*D + (1-c)f^*(aD - K_X - \Delta) - \sum p_j F_j$$

$$= (b-a-c(q-a))f^*D + (\frac{1}{2}-c)f^*(aD-K_X-\Delta) + [\frac{1}{2}f^*(aD-K_X-\Delta) - \sum p_j F_j].$$

The first term is nef if $b - a - c(q-a) \geq 0$, the second term is nef if $c \leq 1/2$ and the remaining terms give an ample divisor. Thus, if these conditions are satisfied by b and c, then $N(b,c)$ is ample.

Step 5. We set

$$c = \min\{(1 + b_j - p_j)/r_j | r_j > 0\}.$$

Then $c > 0$ and we may assume that the p_j have been chosen so that the above minimum is achieved for exactly one value j' of j. We let $F = F_{j'}$. Now $x \notin \operatorname{Supp}\Delta$ and so $b_0 = d - 1$ and $r_0 > 2d$ so that $c < (1 + (d-1) - p_0)/2d < 1/2$. Therefore, $c < 1/2$ and so $N(b,c)$ is ample for any $b \geq a + c(q-a)$.

Step 6. We write

$$N(b,c) = bf^*D + A - F - K_Y$$

and $f^*\Delta = -\sum g_j F_j$. For any non-exceptional component F_j we have $b_j = g_j$. The coefficient of F_j in A is $(-cr_j + b_j - p_j) < b_j$ and so

$$\ulcorner A \urcorner \leq f^* \ulcorner -\Delta \urcorner + E$$

where E is f-exceptional and so

$$H^0(Y, \mathcal{O}_Y(bf^*D + \ulcorner A \urcorner)) \subset H^0(Y, \mathcal{O}_Y(bf^*D + f^*\ulcorner -\Delta \urcorner)) = H^0(X, \mathcal{O}_X(bD + \ulcorner -\Delta \urcorner)).$$

Now $N(b,c)$ is ample so that

$$H^1(Y, \mathcal{O}_Y(bf^*D + \ulcorner A \urcorner - F)) = H^1(Y, \mathcal{O}_Y(bf^*D + \ulcorner A - F \urcorner)) = 0.$$

Therefore since $H^0(F, \mathcal{O}_F((bf^*D + \ulcorner A \urcorner)|_F)) \neq 0$ (by induction on the dimension!), then $H^0(X, \mathcal{O}_X(bD + \ulcorner -\Delta \urcorner)) \neq 0$ as required

□

Proof of the Base Point Free Theorem. See [10, §3.2]. □

To prove the Rationality Theorem, we will need the following.

Lemma 6.7. *Let $0 \neq P(x,y) \in \mathbb{Z}[x,y]$ with $\deg P(x,y) \leq n$. Assume that there is a real number $r \in \mathbb{R}$, an integer a and a real number $\epsilon > 0$ such that $P(x,y) = 0$ for all sufficiently large integers x, y with $0 < ay - rx < \epsilon$.*

Then $r \in \mathbb{Q}$ and if $r = p/q$ with $(p,q) = 1$, then $q \leq a(n+1)/\epsilon$.

Proof. (See [10, 3.19]) Suppose $r \notin \mathbb{Q}$, then there is a pair of sufficiently big integers $(\bar{x}, \bar{y})$ with $0 < a\bar{y} - r\bar{x} < \epsilon/(n+2)$. Therefore

$$(\bar{x}, \bar{y}), (2\bar{x}, 2\bar{y}), \ldots, ((n+1)\bar{x}, (n+1)\bar{y})$$

are solutions of $P(x,y)$. It follows that $\bar{y}x - \bar{x}y$ divides $P(x,y)$. We may repeat this argument, choosing smaller ϵ so that we get a new pair of sufficiently big integers $(\bar{x}, \bar{y})$. It follows that $P(x,y)$ is divisible by infinitely many linear polynomials. This is the required contradiction.

Therefore $r = p/q \in \mathbb{Q}$ and we may assume $(p,q) = 1$. For any $j > 0$, there are integers (x_j, y_j) such that $ay_j - rx_j = aj/q$. We have that $a(y_j + kp) - r(x_j + akp) = aj/p$ for any $k \in \mathbb{Z}$. Therefore if $aj/p < \epsilon$, one sees (as above) that $(ay - rx) - aj/p$ divides $P(x,y)$. Since there are at most n such values, we have that $a(n+1)/p \geq \epsilon$. □

Lemma 6.8. *Let X be a smooth projective variety, $D_i \in \mathrm{Div}(X)$, $A \in \mathrm{Div}_{\mathbb{Q}}(X)$ such that $\mathrm{Supp}(A)$ has simple normal crossings and $\ulcorner A \urcorner \geq 0$. If $\sum u_i D_i$ is nef and $\sum u_i D_i + A - K_X$ is ample for some $u_i \in \mathbb{Z}$, then*

$$P(x_1, \ldots, x_k) = \chi(\sum x_i D_i + \ulcorner A \urcorner) \neq 0.$$

Proof. For any integer $m \gg 0$, $\sum m u_i D_i + A - K_Y$ is ample so that

$$\chi(\sum m u_i D_i + \ulcorner A \urcorner) = H^0(\mathcal{O}_X(\sum m u_i D_i + \ulcorner A \urcorner)).$$

The Lemma now follows from (6.1). □

Proof of the Rationality Theorem. We follow [10, §3.4].

Step 1. We may assume that H is base point free.

Assume that $a(K_X + \Delta)$ is Cartier, then by (6.2),

$$H' := m(cH + da(K_X + \Delta))$$

is base point free for any $m \gg c \gg d > 0$.

Note that

$$H + \frac{t' + mda}{mc}(K_X + \Delta) = \frac{1}{mc}(H' + t'(K_X + \Delta)).$$

Therefore $r(H) \in \mathbb{Q}$ if and only if $r(H') \in \mathbb{Q}$. Note that if $r(H')$ divides q, then $r(H)$ has denominator dividing mcv, but as $m \gg c \gg 0$, $r(H)$ has denominator dividing v.

Step 2. Fix $\epsilon > 0$. For any sufficiently large integers (p, q) with $0 < aq - rp < \epsilon$ let

$$L(p, q) = \mathrm{Bs}|pH + qa(K_X + \Delta)|.$$

(Here, $L(p, q) = X$ if $|pH + qa(K_X + \Delta)| = \emptyset$.) Then $L(p, q)$ is independent of (p, q) and non-empty. We let L_0 denote this set.

To see this, note that if $p', q' \gg p, q$, then we may write $(p', q') = (kp + kq) + (p'' + q'')$ where $p''H + q''a(K_X + \Delta)$ is ample. Therefore $L(p', q') \subset L(p, q)$ and by Noetherian induction, the sets $L(p, q)$ are independent of (p, q). Since $pH + qa(K_X + \Delta)$ is not nef, it is not base point free and so $L_0 \neq \emptyset$.

Step 3. Pick a log resolution $f : Y \to X$, let $D_1 = f^*H$, $D_2 = f^*(a(K_X + \Delta))$ and write

$$K_Y = f^*(K_X + \Delta) + A.$$

We have that $\ulcorner A \urcorner \geq 0$ is exceptional. Therefore

$$H^0(Y, \mathcal{O}_Y(pD_1 + qD_2 + \ulcorner A \urcorner)) = H^0(X, \mathcal{O}_X(pH + qa(K_X + \Delta))).$$

Define

$$P(x, y) = \chi(xD_1 + yD_2 + \ulcorner A \urcorner).$$

Since D_1 is nef and big, $P(x, y) \neq 0$.

Step 4. If $r \notin \mathbb{Q}$, then $L_0 \neq X$.

For any $0 < ay - rx < 1$, we have that

$$xD_1 + yD_2 + A - K_Y \equiv f^*(xH + (ay - 1)(K_X + \Delta))$$

is nef and big. Therefore, by (2.38)

$$H^i(Y, \mathcal{O}_Y(xD_1 + yD_2 + \ulcorner A \urcorner)) = 0 \qquad \forall\ i > 0.$$

By (6.7) there are sufficiently large integers p, q such that $0 < aq - rp < 1$ and

$$P(p, q) = h^0(Y, \mathcal{O}_Y(pD_1 + qD_2 + \ulcorner A \urcorner)) > 0.$$

Therefore $|pH + qa(K_X + \Delta)| \neq \emptyset$.

Step 5. Let $I \subset \mathbb{Z} \times \mathbb{Z}$ such that $0 < ay - rx < 1$ and $\mathrm{Bs}(xH + ya(K_X + \Delta)) = L_0$. For any $(p, q) \in I$ let $f : Y \to X$ be a log resolution and write:

(1) $K_Y = f^*(K_X + \Delta) + \sum a_j F_j$ where $a_j > 1$ as (X, Δ) is kawamata log terminal,
(2) $f^*(pH + (qa - 1)(K_X + \Delta)) - \sum p_j F_j$ is ample for some $0 < p_j \ll 1$ (this may be achieved as $pH + (qa - 1)(K_X + \Delta)$ is nef and big), and
(3) $f^*|pH + qa(K_X + \Delta)| = |L| + \sum r_j F_j$ where $r_j \geq 0$, $\sum r_j F_j = \mathrm{Fix}\, f^*|pH + qa(K_X + \Delta)|$ and $|L|$ is base point free.

We may choose $c > 0$ and $0 < p_j \ll 1$ so that

$$\sum(-cr_j + a_j - p_j)F_j = A' - F$$

where $\ulcorner A' \urcorner \geq 0$ and $A' \wedge F = 0$. Notice that $f(F) \subset \mathrm{Bs}(pH + qa(K_X + \Delta))$.

Step 6. If $p', q' \gg 0$ and $0 < aq' - rp' < aq - rp$, then the divisor

$$N(p', q') = f^*(p'H + q'a(K_X + \Delta)) + A' - F - K_Y$$

is ample.

We have

$$N(p', q') \equiv f^*(p' - (1+c)p)H + q' - (1+c)q)a(K_X + \Delta)) +$$

$$f^*((1+c)pH + (1+c)qa(K_X + \Delta)) + \sum(-cr_j + a_j - p_j)F_j - K_Y$$

$$\equiv cL + f^*(p' - (1+c)p)H + q' - (1+c)q)a(K_X + \Delta))$$

$$+ f^*(pH + (qa - 1)(K_X + \Delta)) - \sum p_j F_j.$$

But L is base point free and hence nef, $(p' - (1+c)p)H + (q' - (1+c)q)a(K_X + \Delta)$ is nef (as $(q' - (1+c)q)a < r(p' - (1+c)p)$) and $f^*(pH + (qa-1)(K_X + \Delta)) - \sum p_j F_j$ is ample.

Step 7. F is not a component of

$$\mathrm{Bs}|f^*(p'H + q'a(K_X + \Delta)) + \ulcorner A' \urcorner|.$$

By Step 6, the homomorphism

$$H^0(Y, \mathcal{O}_Y(f^*(p'H + q'a(K_X + \Delta)) + \ulcorner A' \urcorner)) \to$$

$$H^0(F, \mathcal{O}_F(f^*(p'H + q'a(K_X + \Delta)) + \ulcorner A' \urcorner))$$

is surjective. By (6.8), the polynomial

$$\chi(F, \mathcal{O}_F(f^*(p'H + q'a(K_X + \Delta)) + \ulcorner A' \urcorner))$$

is not identically zero and for $0 < aq' - rp' < aq - rp$, we have that

$$(f^*(p'H + q'a(K_X + \Delta)) + \ulcorner A' \urcorner)|_F - K_F = N(q', p')|_F$$

is ample so that

$$\chi(F, \mathcal{O}_F(f^*(p'H + q'a(K_X + \Delta)) + \ulcorner A' \urcorner)) = h^0(F, \mathcal{O}_F(f^*(p'H + q'a(K_X + \Delta)) + \ulcorner A' \urcorner)).$$

By (6.7) (with $\epsilon = aq - rp$), there are $p', q' \gg 0$ such that $0 < aq' - rp' < aq - rp$ and

$$h^0(F, \mathcal{O}_F(f^*(p'H + q'a(K_X + \Delta)) + \ulcorner A' \urcorner)) \neq 0.$$

The claim now follows.

It follows that $f(F)$ is not contained in L_0. This is a contradiction and so $r \in \mathbb{Q}$.

Step 8. We must now show that if $r = u/v$ where u, v are coprime integers, then $0 < v < a(\dim X + 1)$. See [10, §3.4].

$\square$

6.1. **Generalizations.** In what follows we will also need straightforward generalizations of the above results to the relative case with real divisors.

Theorem 6.9 (Basepoint-free Theorem). *Let (X, Δ) be a $\mathbb{Q}$-factorial kawamata log terminal pair a $\pi : X \to U$ a projective morphism (dominant with connected fibers) to a normal variety and D be a π-nef $\mathbb{R}$-Cartier divisor such that $aD - K_X - \Delta$ is π-nef and π-big for some $a > 0$.*

Then D is semiample over U (i.e. there is a morphism $f : X \to Z$ over U and a divisor $H \in \mathrm{Div}_{\mathbb{R}}(Z)$ ample over U such that $D = f^*H$.

Proof. As the property that D is semiample over U is local, we may assume that U is affine. This case is (7.1) of [4] (for example). $\square$

Corollary 6.10. *Let (X, Δ) be a $\mathbb{Q}$-factorial kawamata log terminal pair, where Δ is an $\mathbb{R}$-divisor. Let $f \colon X \longrightarrow U$ be a projective morphism of normal quasi-projective varieties such that $K_X + \Delta$ is nef over U and Δ is big over U.*

Then $K_X + \Delta$ is semiample over U.

Proof. We may assume that U is affine.

By (8.6) we may find $K_X + \Delta' = K_X + A + B \sim_{\mathbb{R}} K_X + \Delta$, where $A \geq 0$ is a general ample $\mathbb{Q}$-divisor and $B \geq 0$. As

$$(K_X + \Delta) - (K_X + B) \sim_{\mathbb{R},U} A,$$

is ample and $K_X + B$ is kawamata log terminal, (6.2) implies that $K_X + \Delta$ is semiample. $\square$

Exercise 6.11. *Use (6.2) to show that if $f \colon X \longrightarrow U$ is a projective morphism of normal quasi-projective varieties such that $K_X + \Delta$ is dlt and $-(K_X + \Delta)$ is ample over U, then if $B, C \in \mathrm{Div}_{\mathbb{Q}}(X)$ are numerically equivalent, they are $\mathbb{Q}$-linearly equivalent. (Hint. Reduce to the klt case and let $D = B - C$.)*

7. The minimal model program

7.1. **Types of models.**

Definition 7.1. *Let $\phi : X \dashrightarrow Y$ be a birational map that extracts no divisors (i.e. ϕ^{-1} contracts no divisors), $D \in \mathrm{Div}_{\mathbb{R}}(X)$ such that*

$D' = \phi_* D \in \mathrm{Div}_{\mathbb{R}}(Y)$. *Then* ϕ *is* D**-non-positive** *(resp.* D**-negative***) if for some common resolution* $p : W \to X$ *and* $q : W \to Y$*, we have*

$$p^* D = q^* D' + E$$

where $E \geq 0$ *is effective and* q*-exceptional (respectively* $E \geq 0$ *is* q*-exceptional and its support contains all* ϕ*-exceptional divisors).*

Lemma 7.2. *Let* $\phi : X \dashrightarrow Y$ *be a birational map that extracts no divisors (i.e.* ϕ^{-1} *contracts no divisors),* $D \in \mathrm{Div}_{\mathbb{R}}(X)$ *such that* $D' = \phi_* D \in \mathrm{Div}_{\mathbb{R}}(Y)$ *is nef.*

Then ϕ *is* D*-non-positive (resp.* D*-negative) if for some common resolution* $p : W \to X$ *and* $q : W \to Y$*, we have*

$$p^* D = q^* D' + E$$

where $p_* E \geq 0$ *(respectively* $p_* E \geq 0$ *and its support contains all* ϕ*-exceptional divisors).*

If $D = K_X + \Delta$ *and* $D' = K_Y + \phi_* \Delta$ *then this condition is equivalent to*

$$a_F(X, \Delta) \leq a_F(Y, \phi_* \Delta) \qquad (\textit{resp. } a_F(X, \Delta) < a_F(Y, \phi_* \Delta))$$

for all ϕ*-exceptional divisors* $F \subset X$.

Proof. By (2.7). ◻

Definition 7.3. *Let* $\pi : X \to U$ *be a projective morphism of normal quasi-projective varieties. If* $K_X + \Delta$ *is log canonical and* $\phi : X \dashrightarrow Y$ *is a birational map over* U *that extracts no divisors, then we say that*

1. Y *is a* **weak log canonical model** *for* $K_X + \Delta$ *over* U *(*$\mathrm{WLCM}(X, \Delta/U)$*) is* ϕ *if* $K_X + \Delta$*-non-positive and* $K_Y + \Phi_* \Delta$ *is nef over* U*.*
2. Y *is a* **log canonical model** *for* $K_X + \Delta$ *over* U *(*$\mathrm{LCM}(X, \Delta/U)$*) if* ϕ *is* $K_X + \Delta$*-non-positive and* $K_Y + \Phi_* \Delta$ *is ample over* U*.*
3. Y *is a* **log terminal model** *for* $K_X + \Delta$ *over* U *(*$\mathrm{LTM}(X, \Delta/U)$*) is* ϕ *if* $K_X + \Delta$*-negative and* $K_Y + \Phi_* \Delta$ *is divisorially log terminal nef over* U *and* Y *is* $\mathbb{Q}$*-factorial.*

If $\psi : X \dashrightarrow Z$ *is a rational map over* U*, then* Z *is an* **ample model** *for* $K_X + \Delta$ *over* U *if there is a log terminal model* $\phi : X \dashrightarrow Y$ *for* $K_X + \Delta$ *over* U*, a morphism* $f : Y \to Z$ *over* U *and a ample divisor* $H \in \mathrm{Div}_{\mathbb{R}}(Z)$ *such that* $K_Y + \Phi_* \Delta = f^* H$*.*

Lemma 7.4. *Let* $\pi : X \to U$ *be a projective morphism of normal quasi-projective varieties. If* $K_X + \Delta \equiv_U K_X + \Delta'$ *are log canonical (resp. divisorially log terminal) and* $\phi : X \dashrightarrow Y$ *is a birational map over* U *that extracts no divisors,* Y *is normal and* $\mathbb{Q}$*-factorial, then* Y *is a weak log canonical model (resp. a log terminal model) for* $K_X + \Delta$ *over*

U if and only if it is a weak log canonical model (resp. a log terminal model) for $K_X + \Delta'$ *over* U.

Proof. Let $p : W \to X$ and $q : W \to Y$ be a common resolution and write

$$p^*(K_X+\Delta) = q^*(K_Y+\phi_*\Delta)+E \qquad \text{and} \qquad p^*(K_X+\Delta') = q^*(K_Y+\phi_*\Delta')+E'.$$

Since $E - E' \equiv_Y 0$ is q-exceptional, by (2.7), we have $E = E'$ and the lemma follows. □

7.2. **The minimal model program (traditional).** Recall the following.

Definition 7.5. *Let* $\pi : X \to U$ *be a projective morphism of normal quasi-projective varieties,* (X, Δ) *a log canonical pair and* $f : X \to Z$ *be a morphism of normal varieties (surjective with connected fibers) over* U*. Then* f *is a* **flipping contraction** *over* U *if* f *is a small (i.e.* $\dim \operatorname{Ex}(f) < \dim X - 1$*) birational morphism of relative Picard number* $\rho(X/Z) = 1$*,* X *is* $\mathbb{Q}$*-factorial and* $-(K_X+\Delta)$ *is* f*-ample. The* **flip** $f^+ : X^+ \to Z$ *(if it exists) is given by* $X^+ = LCM(X, \Delta/Z)$*. In particular* f^+ *is a small birational morphism of relative Picard number* $\rho(X^+/Z) = 1$*,* X^+ *is* $\mathbb{Q}$*-factorial cf.* (7.7) *and* $(K_{X^+}+\Delta^+)$ *is* f^+*-ample where* $\Delta^+ = ((f^+)^{-1} \circ f)_*\Delta$*.*

Lemma 7.6. *Let* (X, Δ) *be a log canonical pair,* $f : X \to Z$ *be a flipping contraction and* $f^+ : X^+ \to Z$ *its flip. Then*

$$a_E(X, \Delta) \leq a_E(X^+, \Delta^+)$$

for any divisor E *over* X *and* $a_E(X, \Delta) < a_E(X^+, \Delta^+)$ *if the center of* E *is contained in the flipping or flipped locus. In particular* (X^+, Δ^+) *is log canonical.*

Proof. Let $p : W \to X$ and $q : W \to X^+$ be a common log resolution and write

$$p^*(K_X + \Delta) = q^*(K_{X^+} + \Delta^+) + F.$$

Since $K_X + \Delta$ is nef over Z (and hence is nef over X^+) and since F is q-exceptional, then by (2.7), $F \geq 0$. Suppose that E has center V contained in the flipping locus. If E is not contained in the support of F, then by (2.7), $W_v \cap F = \emptyset$ where $v \in V$ is a general point and W_v is the fiber over v. Let C be a curve in W_v such that either $p_*C \neq 0$ or $q_*C \neq 0$. Then as $C \cdot F = 0$, we have $p_*C \cdot (K_X+\Delta) = q_*C \cdot (K_{X^+}+\Delta^+)$ which is impossible. □

Lemma 7.7. *Let* (X, Δ) *be a dlt* $\mathbb{Q}$*-factorial pair,* $f : X \to Z$ *be a flipping contraction and* $f^+ : X^+ \to Z$ *its flip. Then* X^+ *is* $\mathbb{Q}$*-factorial and* $K_Z + f_*\Delta \notin \operatorname{Div}_{\mathbb{R}}(Z)$*.*

Proof. Replacing δ by $(1-\epsilon)\Delta$ for $0<\epsilon\ll 1$, we may assume that (X,Δ) is klt. Let $D^+\in \mathrm{WDiv}(X^+)$ and $D\in \mathrm{WDiv}(X)$ be its strict transform. As X is $\mathbb{Q}$-factorial, $D\in \mathrm{Div}_{\mathbb{Q}}(X)$. Pick a divisor $H = h(K_X+\Delta)\in \mathrm{Div}_{\mathbb{Q}}(X)$ such that $(D+H)\cdot C=0$ for any f-exceptional curve Then $D+H=f^*G$ for some $G\in \mathrm{Div}_{\mathbb{Q}}(Z)$ cf. (6.4). We have that $D^+ + \phi_*H = (f^+)^*G$ and $\phi_*H = h(K_{X^+}+\Delta^+)\in \mathrm{Div}_{\mathbb{Q}}(X^+)$.

Suppose now that $K_Z+f_*\Delta\in \mathrm{Div}_{\mathbb{R}}(Z)$, then $K_X+\Delta = f^*(K_Z+f_*\Delta)$ contradicting the fact that $-(K_X+\Delta)$ is f-ample. □

Definition 7.8. *Let $\pi : X\to U$ be a projective morphism of normal quasi-projective varieties, (X,Δ) a $\mathbb{Q}$-factorial log canonical pair and $f : X\to Z$ be a morphism of normal varieties (surjective with connected fibers) over U. Then f is a* **divisorial contraction** *over U if f is a birational morphism of relative Picard number $\rho(X/Z)=1$, $\dim \mathrm{Ex}(f)=\dim X-1$ and $-(K_X+\Delta)$ is f-ample.*

Lemma 7.9. *Let (X,Δ) be a dlt (resp. klt) $\mathbb{Q}$-factorial pair, $f : X\to Z$ be a divisorial contraction, then Z is $\mathbb{Q}$-factorial and $a_{\mathrm{Ex}(f)}(X,\Delta)< a_{\mathrm{Ex}(f)}(Z,f_*\Delta)$. In particular $(Z,f_*\Delta)$ is dlt (resp. klt). In particular $(Z,f_*\Delta)$ is log canonical.*

Proof. Let $E=\mathrm{Ex}(f)$, then $E\cdot C\neq 0$ were $R=\mathbb{R}^{\geq 0}[C]$ is the contracted negative extremal ray. Let $D\in \mathrm{WDiv}(Z)$ and $D'\in \mathrm{WDiv}(X)$ be its strict transform. As X is $\mathbb{Q}$-factorial, $D'\in \mathrm{Div}_{\mathbb{Q}}(X)$. Pick $h\neq 0$ such that $(D+hE)\cdot C=0$ for any f-exceptional curve. Then $D+hE=f^*G$ for some $G\in \mathrm{Div}_{\mathbb{Q}}(Z)$ cf. (6.4). But then $D=G$ and we are done.

We may now write $K_X+\Delta = f^*(K_Z+f_*\Delta)+aE$. By (2.7), $a>0$. □

Definition 7.10. *Let $\pi : X\to U$ be a projective morphism of normal quasi-projective varieties, (X,Δ) a log canonical pair and $f : X\to Z$ be a morphism of normal varieties (surjective with connected fibers) over U. Then f is a* **Mori fiber space** *if $\rho(X/Z)=1$ and $-(K_X+\Delta)$ is ample over Z.*

Let $\pi : X\to U$ be a projective morphism of normal quasi-projective varieties. Assume that (X,Δ) is a divisorially log terminal $\mathbb{Q}$-factorial pair. We would like to find a finite sequence of well understood geometric operations (flips and divisorial contractions) whose output is a log terminal model for $K_X+\Delta$ over U or a Mori fiber space.

Step 1. If $K_X+\Delta$ is nef over U stop (this is a minimal model over U). Otherwise, pick a $K_X+\Delta$ negative extremal ray R and consider the corresponding contraction morphism $f=\mathrm{cont}_R : X\to Z$ over U.

Note that $\rho(X/Z)=1$ and $-(K_X+\Delta)$ is ample over Z.

Step 2. If $\dim X>\dim Z$ stop (this is a Mori fiber space).

If $\dim X = \dim Z$ and $\dim \mathrm{Ex}(f) = \dim X - 1$, we say that f is a **divisorial contraction**. In this case Z is $\mathbb{Q}$-factorial, $(Z, f_*\Delta)$ is divisorially log terminal cf. (7.9). Replace (X, Δ) by $(Z, f_*\Delta)$ and go back to Step 1.

If $\dim X = \dim Z$ and $\dim \mathrm{Ex}(f) > \dim X - 1$, we have a **small contraction**. In this case Z is not $\mathbb{Q}$-factorial (cf. (**??**)) so we may not replace (X, Δ) by $(Z, f_*\Delta)$. Instead we replace (X, Δ) by the flip (X^+, Δ^+) of $f : X \to Z$ and go back to Step 1.

Step 3. For this procedure to be succesful, we must show that flips exist and that it terminates after finitely many steps. Notice that if $f : X \to Z$ is divisorial, then $\rho(Z) = \rho(X) - 1$ and if $X \dashrightarrow X^+$ is a flip, then $\rho(X) = \rho(X^+)$. Since $\rho(X)$ is a positive integer, there are finitely many divisorial contractions. We must therefore show that there are no infinite sequences of flips.

Conjecture 7.11. *Let (X, Δ) be a log canonical pair.*

(1) *If $f : X \to Z$ is a flipping contraction, then the flip of f exists.*
(2) *There are no infinite sequences of flips $\phi_i : X_i \dashrightarrow X_{i+1}$ for (X_i, Δ_i) where $\Delta_{i+1} = (\phi_i)_*\Delta_i$ and $(X, \Delta) = (X_0, \Delta_0)$ is log canonical.*

Remark 7.12. *We will show that kawamata log terminal flips exist in all dimensions. Termination of flips is known in dimension 3 and there are partial results in dimension 4. We will show that sequences of flips for the minimal model program with scaling terminate when (X, Δ) is kawamata log terminal and Δ is big.*

Remark 7.13. *If $\dim(X) = 2$, then there are no flips. Starting from a smooth surface X (and $\Delta = 0$), one proceeds by contracting the extremal rays corresponding to -1* **curves** *i.e. rational curves $C \cong \mathbb{P}^1_{\mathbb{C}}$ with $C^2 = K_{\mathbb{P}^1_{\mathbb{C}}} \cdot C = -1$. Each time, one obtains a morphism $X_i \to X_{i+1}$ were X_{i+1} is a smooth surface and $\rho(X_{i+1}) = \rho(X_i) - 1$ is a positive integer. After contracting finitely many -1 curve, we therefore obtain a* **minimal surface** *i.e. a surface X_{min} birational to X that contains no -1 curves. We then have that either $K_{X_{min}}$ is nef (this happens when $\kappa(X) \geq 0$) or that there is a negative extremal ray R. Contracting this ray we obtain a Mori fiber space $X_{min} \to Z$ and so X is covered by rational curves. If $\dim Z = 0$, then $X_{min} = \mathbb{P}^2_{\mathbb{C}}$, and if $\dim Z = 1$ then $X_{min} \to Z$ is a ruled surface.*

7.3. **The minimal model with scaling.** In this version of the minimal model program, we start with a $\mathbb{Q}$-factorial kawamata log terminal pairs (X, Δ) and $(X, \Delta + C)$ where $K_X + \Delta + C$ is nef and Δ is big.

We pick

$$\lambda = \sup\{t | K_X + \Delta + tC \text{ is nef}\}.$$

If $\lambda = 0$, then $K_X + \Delta$ is nef and we stop.

Assume $\lambda > 0$. Since Δ is big, there is a $K_X + \Delta$ negative extremal ray $R = \mathbb{R}^{\geq 0}[\Sigma]$ such that $(K_X + \Delta + \lambda C) \cdot \Sigma = 0$ cf. (6.4). Let $f : X \to Z$ be the corresponding contraction.

If $\dim Z < \dim X$ we have a Mori fiber space and we stop.

Otherwise, we replace (X, Δ) by the corresponding flip or divisorial contraction. Notice that f is $K_X+\Delta+\lambda C$-trivial so that $K_X+\Delta+\lambda C$ is nef even after peforming the flip or divisorial contraction. We may therefore repeat the above procedure.

In this way we obtain a sequence of weak log canonical models for $(X, \Delta + tC)$ for $t \in [0, 1]$.

If we can show that there are only finitely many such models, then we can show that the minimal model program with scaling terminates.

7.4. **Minimal models for varieties of general type.**

Theorem 7.14. *Let (X, Δ) be a projective $\mathbb{Q}$-factorial kawamata log terminal pair.*

If Δ is big then any minimal model program with scaling for $K_X + \Delta$ terminates. That is, if $K_X + \Delta$ is pseudo-effective then $K_X + \Delta$ has a log terminal model and if $K_X + \Delta$ is not pseudo-effective then $K_X + \Delta$ has a Mori fiber space.

We have the following immediate consequence.

Corollary 7.15. *Let (X, Δ) be a projective kawamata log terminal pair.*

If $K_X + \Delta$ is big, then $K_X + \Delta$ has a log terminal model and a log canonical model.

Proof. Since $K_X + \Delta$ is big, there is an effective divisor $D \in \mathrm{Div}_{\mathbb{R}}(X)$ such that $K_X + \Delta \sim_{\mathbb{R}} D \geq 0$. Let $0 < \epsilon \ll 1$, then $K_X + \Delta + D \sim_{\mathbb{R}} (1+\epsilon)(K_X + \Delta)$ is kawamata log terminal and $\Delta + D$ is big. It follows that $K_X + \Delta + D$ has a log terminal model which is also a log terminal model for $K_X + \Delta$. Since $\Delta + D$ is big, by (6.2), $K_X + \Delta + D$ is semiample so that there is a projective morphism $g : X \to Z$ such that $K_X + \Delta + D = g^*A$ and $A \in \mathrm{Div}_{\mathbb{R}}(Z)$ is ample. It follows that $g : X \to Z$ is the log canonical model of $K_X + \Delta$. □

Remark 7.16. *The above results also hold in the relative case.*

Corollary 7.17. *Let $f : X \to Z$ be a flipping contraction, then the flip of f exists.*

Corollary 7.18. *Let (X,Δ) be a klt pair and $\mathcal{E}$ be a set of exceptional divisors over X such that if $E\in\mathcal{E}$, then $a_E(X,\Delta)\leq 0$. Then there exists a birational morphism $\nu: X'\to X$ such that X is $\mathbb{Q}$-factorial and $\operatorname{Ex}(\nu)=\mathcal{E}$. If $\mathcal{E}=\emptyset$, then ν is small and we say that $X'\to X$ is a $\mathbb{Q}$-factorialization of (X,Δ) and if $\mathcal{E}$ contains all exceptional divisors such that $a_E(X,\Delta)\leq 0$, then $K_{X'}+\Delta'=\nu^*(K_X+\Delta)$ is terminal and we say that $X'\to X$ is a terminalization of (X,Δ).*

Proof. Let $f: Y\to X$ be a log resolution of (X,Δ) and write $K_Y+\Gamma=f^*(K_X+\Delta)+E$ where $\Gamma=\Gamma(X,\Delta)$ and $E=E(X,\Delta)$. Let F be the reduced divisor consisting of all exceptional divisors not contained in $\mathcal{E}$ and $0<\epsilon\ll 1$. Then $(Y,\Gamma+\epsilon F)$ is klt and so there is $\phi: Y\dashrightarrow X'$ a log terminal model for $(Y,\Gamma+\epsilon F)$ over X. In particular X' is $\mathbb{Q}$-factorial. Then $K_{X'}+\phi_*(\Gamma+\epsilon F)$ is nef over X and hence so is $\phi_*(E+\epsilon F)$ (since $K_{X'}+\phi_*(\Gamma-E)\equiv_X 0$). By the negativity lemma $E+\epsilon F\leq 0$ and hence $E+\epsilon F=0$ so that the divisors in $\mathcal{E}$ are contracted by $Y\dashrightarrow X'$. It is also easy to see that if P is a prime divisor contracted by $Y\dashrightarrow X'$, then

$$a_P(Y,\Gamma-E)=a_P(X,\Delta)=a_P(X',\phi_*(\Gamma-E))=a_P(X',\phi_*(\Gamma+\epsilon F))>a_P(Y,\Gamma+\epsilon F)$$

and hence that P is contained in $\operatorname{Supp}(E+F)=\mathcal{E}$. □

7.5. **The main induction.** We begin with the following.

Definition 7.19. *Let $\pi\colon X\longrightarrow U$ be a projective morphism of normal quasi-projective varieties, and let V be a finite dimensional affine subspace of the real vector space of Weil divisors on X. Define*

$$\mathcal{L}=\{\,\Delta\in V\,|\,K_X+\Delta \text{ is log canonical}\},$$
$$\mathcal{N}_\pi=\{\,\Delta\in \mathcal{L}\,|\,K_X+\Delta \text{ is nef over } U\}.$$

Moreover, fixing an $\mathbb{R}$-divisor $A\geq 0$, define

$$V_A=\{\,\Delta\,|\,\Delta=A+B, B\in V\,\},$$
$$\mathcal{L}_A=\{\,\Delta=A+B\in V_A\,|\,K_X+\Delta \text{ is log canonical and } B\geq 0\,\},$$
$$\mathcal{E}_{A,\pi}=\{\,\Delta\in \mathcal{L}_A\,|\,K_X+\Delta \text{ is pseudo-effective over } U\},$$
$$\mathcal{N}_{A,\pi}=\{\,\Delta\in \mathcal{L}_A\,|\,K_X+\Delta \text{ is nef over } U\}.$$

Given a birational map $\phi\colon X\dashrightarrow Y$ over U, whose inverse does not contract any divisors, define

$$\mathcal{W}_{Y,\pi}=\{\,\Delta\in\mathcal{E}_{A,\pi}\,|\,(Y,\Gamma=\phi_*\Delta) \text{ is a weak log canonical model for } (X,\Delta) \text{ over } U\,\},$$

and given a rational map $\psi\colon X\dashrightarrow Z$ over U, define

$$\mathcal{A}_Z=\{\,\Delta\in\mathcal{E}_{A,\pi}\,|\,Z \text{ is an ample model for } (X,\Delta) \text{ over } U\,\},.$$

Almost invariably, the support of A will have no components in commnon with V. In this case the condition that $B \geq 0$ is vacuous. In nearly all applications, A will be an ample $\mathbb{Q}$-divisor over U. In this case, we often assume that A is *general* in the sense that we fix a positive integer such that kA is very ample, and we assume that $A = \frac{1}{k}A'$, where $A' \in |kA|$ is very general. With this choice of A, we have

$$\mathcal{N}_{A,\pi} \subset \mathcal{E}_{A,\pi} \subset \mathcal{L}_A = \mathcal{L} + A \subset V_A = V + A,$$

and the condition that the support of A has no common components with any element of V is then automatic.

The proof of (7.14) is by induction on the dimension. We brake it down in to several statements of independent interest.

Theorem 7.20 (Existence of pl-flips). *Let $f\colon X \longrightarrow Z$ be a pl-flipping contraction for an n-dimensional purely log terminal pair (X, Δ).*

Then the flip $f^+\colon X^+ \longrightarrow Z$ of f exists.

Theorem 7.21 (Existence of log terminal models). *Let $\pi\colon X \longrightarrow U$ be a projective morphism of normal quasi-projective varieties, where X has dimension n. Suppose that $K_X + \Delta$ is kawamata log terminal, where Δ is big over U.*

If there exists an $\mathbb{R}$-divisor D such that $K_X + \Delta \sim_{\mathbb{R},U} D \geq 0$, then $K_X + \Delta$ has a log terminal model over U.

Theorem 7.22 (Finiteness of models (big case)). *Let $\pi\colon X \longrightarrow U$ be a projective morphism of normal quasi-projective varieties, where X has dimension n. Fix A, a general ample $\mathbb{Q}$-divisor over U. Suppose that $K_X + \Delta_0$ is kawamata log terminal, for some Δ_0.*

Let $\mathcal{C} \subset \mathcal{L}_A$ be a rational polytope such that $K_X + \Delta$ is π-big, for every $\Delta \in \mathcal{C}$.

Then the set of isomorphism classes

$$\{\, Y \mid Y \text{ is a log terminal model over } U \text{ of a pair } (X, \Delta), \text{ where } \Delta \in \mathcal{C}\,\},$$

is finite.

Theorem 7.23 (Non-vanishing theorem). *Let $\pi\colon X \longrightarrow U$ be a projective morphism of normal quasi-projective varieties, where X has dimension n. Suppose that $K_X + \Delta$ is kawamata log terminal, where Δ is big over U.*

If $K_X + \Delta$ is π-pseudo-effective, then there exists an $\mathbb{R}$-divisor D such that $K_X + \Delta \sim_{\mathbb{R},U} D \geq 0$.

Theorem 7.24 (Finiteness of models). *Let $\pi\colon X \longrightarrow U$ be a projective morphism of normal quasi-projective varieties, where X has dimension* *n. Fix A, a general ample $\mathbb{Q}$-divisor over U. Suppose that $K_X + \Delta_0$ is kawamata log terminal, for some Δ_0.*

Then the set of isomorphism classes

$$\{\, Y \,|\, Y \text{ is the weak log canonical model over } U \text{ of a pair } (X, \Delta), \text{ where } \Delta \in \mathcal{L}_A \,\},$$

is finite.

Theorem 7.25 (Effective Zariski decomposition)**.** *Let $\pi\colon X \longrightarrow Z$ be a projective morphism to a normal affine variety. Let $(X, \Delta = A + B)$ be a kawamata log terminal pair of dimension n, where $A \geq 0$ is an ample $\mathbb{Q}$-divisor and $B \geq 0$. If $K_X + \Delta$ is pseudo-effective, then*

(1) *The pair (X, Δ) has a log terminal model $\mu\colon X \dashrightarrow Y$. In particular if $K_X + \Delta$ is $\mathbb{Q}$-Cartier then the log canonical ring*

$$R(X, K_X + \Delta) = \bigoplus_{m \in \mathbb{N}} H^0(X, \mathcal{O}_X(\llcorner m(K_X + \Delta)\lrcorner)),$$

is finitely generated.

(2) *Let $V \subset \operatorname{Div}_{\mathbb{R}}(X)$ be the vector space spanned by the components of Δ. Then there is a constant $\delta > 0$ such that if G is a prime divisor contained in the stable base locus of $K_X + \Delta$ and $\Xi \in V$ such that $\|\Xi - \Delta\| < \delta$, then G is contained in the stable base locus of $K_X + \Xi$.*

(3) *Let $W \subset V$ be the smallest rational affine space containing Δ. Then there is a constant $\eta > 0$ and a positive integer $r > 0$ such that if $\Xi \in W$ is any divisor and k is any positive integer such that $\|\Xi - \Delta\| < \eta$ and $k(K_X + \Xi)/r$ is Cartier, then every component of $\operatorname{Fix}(k(K_X + \Xi))$ is a component of the stable base locus of $K_X + \Delta$.*

The proof of Theorem 7.20, Theorem 7.21, Theorem 7.22, Theorem 7.23, Theorem 7.24 and Theorem 5.9 proceeds by induction:

- Theorem 5.9_{n-1} implies Theorem 7.20_n, see (5.12).
- Theorem 7.21_{n-1}, Theorem 7.24_{n-1} and Theorem 7.20_n imply Theorem 7.21_n, cf. (9.4).
- Theorem 7.21_n implies Theorem 7.22_n, cf. (10.4).
- Theorem 7.23_{n-1}, Theorem 7.24_{n-1}, Theorem 7.21_n and Theorem 7.22_n imply Theorem 7.23_n, cf. (11.4).
- Theorem 7.21_n and Theorem 7.23_n imply Theorem 7.24_n, cf. (10.4).
- Theorem 7.21_n, Theorem 7.23_n and Theorem 7.24_n imply Theorem 5.9_n, cf. (**??**).

8. Special termination with scaling

In this section we show that we have special termination of the MMP with scaling.

Lemma 8.1. *Assume Theorem 5.12_n and Theorem 7.24_n.*

Let $\pi_i \colon X_i \longrightarrow U$ be a sequence of projective morphisms of normal quasi-projective varieties, where X_i and X_j are isomorphic in codimension one. Suppose that there exist $\mathbb{R}$-divisors Δ_i such that $K_{X_i} + \Delta_i$ is $\mathbb{R}$-Cartier and nef over U. Suppose that there are fixed $\mathbb{R}$-divisors $A_1 \geq 0$ and $B_1 \geq 0$ on $X = X_1$, with transforms A_i and B_i on X_i, such that

$$A_i \leq \Delta_i \leq B_i,$$

where A_1 is big over U and $K_{X_1} + B_1$ is kawamata log terminal. Let n be the dimension of X.

Then the set of isomorphism classes of birational maps

$$\{\, X \dashrightarrow X_i \,|\, i \in \mathbb{N} \,\},$$

is finite.

Here, two birational maps $\phi \colon X \dashrightarrow X_i$ and $\psi \colon X \dashrightarrow X_j$ are isomorphic if there exists an isomorphism $\eta \colon X_i \longrightarrow X_j$ such that $\psi = \eta \circ \phi$.

Proof. As we are assuming Theorem 5.12_n, replacing X by a log terminal model we may assume that X is $\mathbb{Q}$-factorial (cf. (7.18).

Let Δ'_i be the strict transform of Δ_i on X. By (8.6) we may assume that A_1 is an ample $\mathbb{Q}$-divisor over U and B_1 is an effective $\mathbb{Q}$-divisor. As (X_i, Δ_i) is a weak log canonical model of (X, Δ'_i), the result follows as we are assuming Theorem 7.24_n. □

Lemma 8.2. *Assume Theorem 5.12_{n-1} and Theorem 7.24_{n-1}.*

Let $\pi \colon X \longrightarrow U$ be a projective morphism of normal quasi-projective varieties, where X is a $\mathbb{Q}$-factorial variety of dimension n. Suppose that

$$K_X + \Delta + C = K_X + S + A + B + C,$$

is divisorially log terminal and nef over U, where $S = \llcorner \Delta \lrcorner$, $\mathbf{B}_+(A/U)$ does not contain any log canonical centres of $(X, \Delta + C)$ and B, $C \geq 0$.

Then every $(K_X + \Delta)$-MMP over U with scaling of C is eventually disjoint from S.

Proof. Suppose not. Let $X_i \dashrightarrow X_{i+1}$ be an infinite sequence of flips and divisorial contractions over U, starting with $X_1 := X$, for the $(K_X + \Delta)$-MMP with scaling of C, which meets S infinitely often. We may write $A \sim_{\mathbb{R},U} A' + B'$ where A' is ample over U and the support of B' contains no log canonical centre of $(X, \Delta + C)$. Replacing A by $\epsilon A'$

and B by $B+(1-\epsilon)A+\epsilon B'$ where $0<\epsilon \ll 1$, we may therefore assume that A is ample over U. Let T be a component of S that intersects the flipping locus infinitely many times. Pick a rational number $\epsilon > 0$ such that $A' = A + \epsilon(S-T)$ is ample over U. Replacing A' by an $\mathbb{R}$-linearly equivalent over U divisor, we may assume that

$$K_X + S - \epsilon(S-T) + A' + B \sim_{\mathbb{R},U} K_X + S + A + B,$$

is purely log terminal. Note that every step of the $(K_X+S+A+B)$-MMP over U is a step of the $(K_X+S-\epsilon(S-T)+A'+B)$-MMP over U, and $\llcorner S-\epsilon(S-T)+A'+B \lrcorner = T$. Thus, replacing $K_X+\Delta$ by $K_X+S-\epsilon(S-T)+A'+B$ we may assume that S is irreducible and $K_X+\Delta$ is purely log terminal. Let S_i be the strict transform of S in X_i and let $S_i \dashrightarrow S_{i+1}$ be the induced birational map.

Since there are at most finitely many divisorial contractions, we may assume that there is an integer $k>0$ such that for any $i \geq k$ the rational map $X_i \dashrightarrow X_{i+1}$ is a flip.

By (4.2.14) of [3], we may also assume that for any $i \geq k$, the rational map $S_i \dashrightarrow S_{i+1}$ does not extract any divisors. By (1.6) of [1], we may therefore assume that for any $i \geq k$, the rational map $S_i \dashrightarrow S_{i+1}$ is an isomorphism at the generic point of every divisor on S_i and S_{i+1}. In particular we may assume that $S_i \dashrightarrow S_{i+1}$ is an isomorphism in codimension one.

Let Δ_i and C_i be the strict transforms of Δ and C on X_i. Since S_i is the unique log canonical centre of (X_i,Δ_i), S_i is normal. By adjunction we may write

$$(K_{X_i}+\Delta_i)|_{S_i} = K_{S_i}+\Theta_i.$$

Then, for any $i \geq k$, Θ_i is the strict transform of Θ_k. By definition of the MMP with scaling there is a $t_i \geq 0$ such that $K_{X_i}+\Delta_i+t_iC_i$ is nef and log canonical. It follows that $K_{S_i}+\Theta_i+t_iC_i|_{S_i}$ is also nef and log canonical. Let $t_\infty = \lim t_i$, then $X_i \dashrightarrow X_{i+1}$ is $K_{X_k}+\Delta_k+t_\infty C_k$ non positive for all $i \geq 0$. Proceeding as above, one sees that

$$K_X+\Delta+t_\infty C \sim_{\mathbb{R},U} K_X+S+A'+B'$$

where $K_X+S+A'+B'$ is purely log terminal and A' is ample over U. By (8.8), we have that

$$K_{X_k}+\Delta_k+t_\infty C_k \sim_{\mathbb{R},U} K_{X_k}+S_k+A''+B'',$$

where $K_{X_k}+S_k+A''+B''$ is purely log terminal and A'' is ample over U. In particular

$$K_{S_k}+\Theta_k+t_\infty C_k|_{S_k} \sim_{\mathbb{R},U} (K_{X_k}+S_k+A''+B'')|_{S_k} = K_{S_k}+A''|_{S_k}+\Xi_k,$$

where $K_{S_k}+A''|_{S_k}+\Xi_k$ is kawamata log terminal. It then follows that $K_{S_k}+A''|_{S_k}+\Xi_k+(t_i-t_\infty)C_k$ is kawamata log terminal for any $i \gg 0$.

Thus the hypotheses of (8.1) are satisfied, and the set of isomorphism classes

$$\{ S \dashrightarrow S_i \mid i \in \mathbb{N} \},$$

is finite, so that the set of pairs

$$\{ (S_i, \Theta_i) \mid i \in \mathbb{N} \},$$

is also finite.

On the other hand, let $X_i \longrightarrow Z_i$ be the flipping contraction and let T_i be the normalisation of the image of S_i in Z_i, so that there are birational morphisms $p_i \colon S_i \longrightarrow T_i$ and $q_i \colon S_{i+1} \longrightarrow T_i$. Note that $-(K_{S_i} + \Theta_i)$ is p_i-ample whilst $(K_{S_{i+1}} + \Theta_{i+1})$ is q_i-ample. By assumption infinitely often the flipping locus intersects S_i. If p_i is an isomorphism and the flipping locus intersects S_i, then $S_i \cdot \Sigma_i > 0$, where Σ_i is a flipping curve. But then S_{i+1} must intersect any flipped curve negatively, so that all flipped curves lie in S_{i+1} and q_i is not an isomorphism. In particular infinitely often one of the birational morphisms p_i or q_i is not an isomorphism.

Thus we may assume that p_k or q_k is not an isomorphism, where the isomorphism class of S_k is repeated infinitely often. Pick any valuation ν whose centre is contained in the locus where $S_k \dashrightarrow S_{k+1}$ is not an isomorphism. By (2.28) of [9]

$$a(\nu, S_k, \Theta_k) < a(\nu, S_{k+1}, \Theta_{k+1}) \qquad \text{and} \qquad a(\nu, S_i, \Theta_i) \leq a(\nu, S_{i+1}, \Theta_{i+1}),$$

for all $i \geq k+1$, a contradiction. □

We use (8.2) to run a special MMP:

Lemma 8.3. *Assume Theorem 5.12$_{n-1}$, Theorem 7.24$_{n-1}$ and Theorem 7.20$_n$.*

Let $\pi \colon X \longrightarrow U$ be a projective morphism of normal quasi-projective varieties, where X is $\mathbb{Q}$-factorial of dimension n. Suppose that $(X, \Delta + C = S + A + B + C)$ is a divisorially log terminal pair, such that $\llcorner \Delta + C \lrcorner = S$, $\mathbf{B}_+(A/U)$ does not contain any log canonical centres of $(X, \Delta + C)$, and $B \geq 0$, $C \geq 0$. Suppose that there is an $\mathbb{R}$-divisor $D \geq 0$ whose support is contained in S and a real number $\alpha \geq 0$, such that

$$(*) \qquad K_X + \Delta \sim_{\mathbb{R},U} D + \alpha C.$$

If $K_X + \Delta + C$ is nef over U then there is a log terminal model $\phi \colon X \dashrightarrow Y$ for $K_X + \Delta$ over U, where $\mathbf{B}_+(\phi_* A/U)$ does not contain any log canonical centres of $(Y, \Gamma = \phi_* \Delta)$.

Proof. By (8.7) and (8.9) we may run the $(K_X+\Delta)$-MMP with scaling of C over U, and this will preserve the condition that $\mathbf{B}_+(A/U)$ does not contain any log canonical centre of (X,Δ). Pick $t\in[0,1]$ minimal such that $K_X+\Delta+tC$ is nef over U. If $t=0$ we are done. Otherwise we may find a $(K_X+\Delta)$-negative extremal ray R over U, such that $(K_X+\Delta+tC)\cdot R=0$. Let $f\colon X\longrightarrow Z$ be the associated contraction over U. As $t>0$, $C\cdot R>0$ and so $D\cdot R<0$. In particular f is always birational.

If f is divisorial, then we can replace X, S, A, B, C and D by their images in Z. Note that $(*)$ continues to hold.

Otherwise f is small. As $D\cdot R<0$, R is spanned by a curve Σ which is contained in a component T of S, where $T\cdot\Sigma<0$. Note that $K_X+S+A+B-\epsilon(S-T)$ is purely log terminal for any positive $\epsilon\ll 1$, and so f is a pl-flip.

As we are assuming Theorem 7.20_n, the flip $f'\colon X'\longrightarrow Z$ of $f\colon X\longrightarrow Z$ exists. Again, if we replace X, S, A, B, C and D by their images in X', then $(*)$ continues to hold. On the other hand this flip is certainly not an isomorphism in a neighbourhood of S and so the MMP terminates by (8.2). □

8.1. **lemmas.**

Lemma 8.4. *Let $\pi\colon X\longrightarrow U$ be a projective morphism of normal quasi-projective varieties. Let V be a finite dimensional affine linear subspace of the space of Weil divisors on X and let A be a big $\mathbb{Q}$-divisor over U. Let $\mathcal{C}\subset\mathcal{L}_A(V)$ be a polytope.*

If $\mathbf{B}_+(A/U)$ does not contain any log canonical centres of (X,Δ), for every $\Delta\in\mathcal{C}$, then we may find a general ample $\mathbb{Q}$-divisor A' over U and a translation

$$L\colon A^1_{\mathbb{R}}(X)\longrightarrow A^1_{\mathbb{R}}(X),$$

by a divisor $\mathbb{Q}$-linearly equivalent to zero over U such that $L(\mathcal{C})\subset\mathcal{L}_{A'}(V')$, where $V'=L(V)$.

Proof. Let $\Delta_1,\Delta_2,\dots,\Delta_l$ be the vertices of the polytope $\mathcal{C}$, and let Z the union of the non kawamata log terminal locus of each (X,Δ_i). Then Z contains the non kawamata log terminal locus of (X,Δ), for any $\Delta\in\mathcal{C}$.

By assumption, we may write $A\sim_{\mathbb{R},U}C+D$, where C is an ample $\mathbb{R}$-divisor over U and $D\geq 0$ does not contain any component of Z. Then we may find rational functions $f_1,f_2,\dots,f_k$, real numbers $r_1,r_2,\dots,r_k$ and an $\mathbb{R}$-Cartier divisor E on U such that

$$A=C+D+\sum r_i(f_i)+\pi^*E.$$

Replacing C by $C+\sum r_i(f_i)+\pi^*E$ we may assume that $A \sim_{\mathbb{Q},U} C+D$. Replacing C by $C' \leq C$ sufficiently close to C and D by $D+(C-C')$ we may assume that C is $\mathbb{Q}$-Cartier. Replacing C by a $\mathbb{Q}$-linearly equivalent divisor over U, we may assume that C is a general ample $\mathbb{Q}$-divisor over U. Given any rational number $\delta > 0$, let

$$L\colon A^1_{\mathbb{R}}(X) \longrightarrow A^1_{\mathbb{R}}(X) \qquad \text{given by} \qquad L(\Delta) = \Delta + \delta(C+D-A),$$

be translation by the divisor $\delta(C+D-A) \sim_{\mathbb{Q},U} 0$. As $C+D$ does not contain Z, if δ is sufficiently small then

$$K_X + L(\Delta_i) = K_X + \Delta_i + \delta(C+D-A) = K_X + \delta C + (\Delta_i - \delta A + \delta D),$$

is log canonical for every $1 \leq i \leq l$. But then $L(\mathcal{C}) \subset \mathcal{L}_{A'}(V')$, where $A' = \delta C$ and $V' = L(V)$. □

Lemma 8.5. *Let $\pi\colon X \longrightarrow U$ be a projective morphism of normal quasi-projective varieties. Let V be a finite dimensional rational affine subspace of the space of Weil divisors on X and let A be a general ample $\mathbb{Q}$-divisor over U. Let $G \geq 0$ be a $\mathbb{Q}$-Cartier divisor whose support does not contain any log canonical centre of (X,Δ), for any $\Delta \in \mathcal{L}_A(V)$.*

If there is a kawamata log terminal pair (X,Δ_0) then we may find a general ample $\mathbb{Q}$-divisor A' over U, a rational affine space V' and an injective rational affine linear map

$$L\colon V_A \longrightarrow V'_{A'},$$

which preserves $\mathbb{Q}$-linear equivalence over U such that $L(\mathcal{L}_A(V))$ is contained in the interior of $\mathcal{L}_{A'}(V')$ and there is a divisor $\Delta'_0 \in \mathcal{L}_{A'}(V')$, whose support contains the support of G such that $K_X+\Delta'_0$ is kawamata log terminal.

Proof. Let W be the vector space spanned by the components of Δ_0. Since $\Delta_0 \in \mathcal{L}(W)$ is a non-empty rational polytope, possibly replacing Δ_0, we may assume that $K_X + \Delta_0$ is $\mathbb{Q}$-Cartier.

As $\mathcal{L}_A(V)$ is a rational polytope, possibly replacing V by the span of $\mathcal{L}_A(V)$ we may assume that $K_X + \Delta$ is $\mathbb{R}$-Cartier for ever $\Delta \in \mathcal{L}_A(V)$. By compactness, we may pick $\mathbb{Q}$-divisors $\Delta_1, \Delta_2, \dots, \Delta_l$ such that $\mathcal{L}_A(V)$ is contained in the simplex spanned by $\Delta_1, \Delta_2, \dots, \Delta_l$ (we do not assume that $\Delta_i \geq 0$). As

$$\Delta_i - \Delta_0 = (K_X + \Delta) - (K_X + \Delta_0),$$

is $\mathbb{Q}$-Cartier, we may pick a positive rational number $\epsilon > 0$ such that

$$\epsilon(\Delta_i - \Delta_0) + (1-\epsilon)A,$$

is an ample $\mathbb{Q}$-divisor over U, for $1 \leq i \leq l$. Pick

$$A'_i \sim_{\mathbb{Q},U} \epsilon(\Delta_i - \Delta_0) + (1-\epsilon)A,$$

general ample $\mathbb{Q}$-divisors over U. Let V' be the space spanned by V, A and A', the $\mathbb{Q}$-divisors $A'_1, A'_2, \dots, A'_l$ and the components of Δ_0 and G. If we define $L\colon V_A \longrightarrow V'_{A'}$ by

$$L(\Delta_i) = (1-\epsilon)\Delta_i + A'_i + \epsilon\Delta_0 + A' - A,$$

and extend to the whole of V_A by linearity then L is an injective rational linear map, which preserves $\mathbb{Q}$-linear equivalence over U. If we set

$$\Delta'_0 = \Delta_0 + A',$$

then $K_X + \Delta'_0$ is kawamata log terminal and $\Delta'_0 \in \mathcal{L}_{A'}(V')$. Note that L is the composition of

$$L_1\colon V_A \longrightarrow V'_{A'} \qquad \text{and} \qquad L_2\colon V'_{A'} \longrightarrow V'_{A'},$$

given by

$$L_1(\Delta_i) = \Delta_i + A'_i/(1-\epsilon) + A' - A \quad \text{and} \quad L_2(\Delta) = (1-\epsilon)\Delta + \epsilon\Delta'_0.$$

As $L_1(\mathcal{L}_A(V)) \subset \mathcal{L}_{A'}(V')$, it follows that if $\Delta \in \mathcal{L}_A(V)$ then $K_X + L(\Delta)$ is kawamata log terminal. Pick a $\mathbb{Q}$-Cartier divisor $H \geq 0$ which contains every component of every element of V'. Let V'' be the span of V and the components of H. Let $\delta > 0$ by any rational number and let

$$T\colon V'_A \longrightarrow V''_A,$$

be tranlsation by δH. If $\delta > 0$ is sufficiently small then $T(L(\mathcal{L}_A(V)))$ is contained in the interior of $\mathcal{L}_{A'}(V')$ and $T(\Delta'_0)$ contains the support of G. Replacing L by $T \circ L$ and V' by V'' we are done. □

Lemma 8.6. *Let $\pi\colon X \longrightarrow U$ be a projective morphism of normal quasi-projective varieties. Let $(X, \Delta = A + B)$ is a log canonical pair, where $A \geq 0$ and $B \geq 0$.*

If $\mathbf{B}_+(A/U)$ does not contain any log canonical centres of (X, Δ) and there is a kawamata log terminal pair (X, Δ_0) then we may find a kawamata log terminal pair $(X, \Delta' = A' + B')$, where $A' \geq 0$ is an ample $\mathbb{Q}$-divisor, general over U, $B' \geq 0$ and $K_X + \Delta \sim_{\mathbb{Q},U} K_X + \Delta'$.

Proof. By (8.4) we may assume that A is a general ample $\mathbb{Q}$-divisor over U. Let V be the vector space spanned by the components of Δ. As $\Delta \in \mathcal{L}_A(V)$ the result follows by (8.5). □

Lemma 8.7. *Let $\pi\colon X \longrightarrow U$ be a projective morphism of normal quasi-projective varieties. Suppose the pair $(X, \Delta = A + B)$ has kawamata log terminal singularities, where A is big over U, $B \geq 0$, and C is an $\mathbb{R}$-Cartier divisor such that $K_X + \Delta$ is not nef over U, but $K_X + \Delta + C$ is nef over U.*

Then there is a $(K_X + \Delta)$-negative extremal ray R and a real number $0 < \lambda \leq 1$ such that $K_X + \Delta + \lambda C$ is nef over U but trivial on R.

Lemma 8.8. *Let $\pi : X \to U$ be a projective morphism of normal varieties and $\phi : X \dashrightarrow Y$ be a rational map over U of $\mathbb{Q}$-factorial varieties such that ϕ^{-1} contracts no divisors. Assume that $(X, \Delta = T + A + B)$ is a purely log terminal pair such that $T = \llcorner \Delta \lrcorner$, $A \geq 0$ is ample over U and $B \geq 0$. Assume moreover that $T' = \phi_* T \neq 0$ and ϕ is $K_X + \Delta$ non-positive.*

Then, $(Y, \phi_*\Delta)$ is purely log terminal and we have that $\phi_*(\Delta) \sim_{\mathbb{R},U} T' + A' + B'$ where $(Y, T' + A' + B')$ is purely log terminal, $A' \geq 0$ is a general ample $\mathbb{Q}$-divisor over U and $B' \geq 0$.

Proof. As (X, Δ) is purely log terminal, ϕ is $K_X + \Delta$-non-positive and $\llcorner \phi_*\Delta \lrcorner = \phi_* T = T'$, it follows immediately that $(Y, \phi_*\Delta)$ is purely log terminal.

Let $A' \geq 0$ be a general ample $\mathbb{Q}$-divisor over U such that $A - \phi_*^{-1} A'$ is ample over U. Pick a $A_1 \sim_{\mathbb{Q},U} A - \phi_*^{-1} A'$ a general ample $\mathbb{Q}$-divisor over U. As the support of $\phi_*^{-1} A'$ does not contain T, we have that

$$K_X + T + (1 - \epsilon)A + \epsilon(\phi_*^{-1} A' + A_1) + B$$

is purely log terminal for any $0 < \epsilon \ll 1$. By what we have allready proved, it follows that $K_Y + T' + \epsilon A' + (1-\epsilon)\phi_* A + \phi_*(\epsilon A_1 + B)$ is also purely log terminal. Replacing $\epsilon A'$ by A' and $(1-\epsilon)\phi_* A + \phi_*(\epsilon A_1 + B)$ by B' the lemma follows. □

Lemma 8.9. *Let $\pi \colon X \longrightarrow U$ be a projective morphism of normal quasi-projective varieties. Suppose that $K_X + \Delta$ is divisorially log terminal, X is $\mathbb{Q}$-factorial and $\phi \colon X \dashrightarrow Y$ is a $(K_X + \Delta)$ flip or divisorial contraction.*

If $\Gamma = \phi_*\Delta$ then

1. *ϕ is an isomorphism at the generic point of every log canonical centre of $K_Y + \Gamma$. In particular (Y, Γ) is divisorially log terminal.*
2. *If $\Delta = S + A + B$, where $S = \llcorner \Delta \lrcorner$, A is big over U and $\mathbf{B}_+(A/U)$ does not contain any log canonical centres of $K_X + \Delta$, then $\phi_* S = \llcorner \Gamma \lrcorner$, $\phi_* A$ is big over U and $\mathbf{B}_+(\phi_* A/U)$ does not contain any log canonical centres of $K_Y + \Gamma$.*

   In particular $\Gamma \sim_{\mathbb{R},U} \Gamma'$ where (Y, Γ') is kawamata log terminal and Γ' is big.

9. Log terminal models

Definition 9.1. *Let $\pi \colon X \longrightarrow U$ be a projective morphism of normal varieties. Let $(X, \Delta = A + B)$ be a $\mathbb{Q}$-factorial divisorially log terminal pair and let D be an $\mathbb{R}$-divisor, where $A \geq 0$, $B \geq 0$ and $D \geq 0$. A* ***nice model*** *over U for (X, Δ), with respect to A and D, is any birational map $f \colon X \dashrightarrow Y$ over U, such that*

- *f^{-1} does not contract any divisors,*
- *the only divisors contracted by f are components of D,*
- *Y is $\mathbb{Q}$-factorial,*
- *$K_Y + f_*\Delta$ is divisorially log terminal and nef over U, and*
- *$\mathbf{B}_+(f_*A/U)$ does not contain any log canonical centres of $(Y, \Gamma = f_*\Delta)$.*

Lemma 9.2. *Assume Theorem 5.12$_{n-1}$, Theorem 7.24$_{n-1}$ and Theorem 7.20$_n$.*

Let $\pi\colon X \longrightarrow U$ be a morphism of normal projective varieties, where X has dimension n. Let $(X, \Delta = A + B)$ be a divisorially log terminal log pair and let D be an $\mathbb{R}$-divisor, where $A \geq 0$, $B \geq 0$ and $D \geq 0$.

If

(i) $K_X + \Delta \sim_{\mathbb{R},U} D \geq 0$,
(ii) *(X, G) is log smooth, where G is the support of $\Delta + D$, and*
(iii) *$\mathbf{B}_+(A/U)$ does not contain any log canonical centres of (X, G)*

then (X, Δ) has a nice model over U, with respect to A and D.

Proof. We may write $K_X + \Delta \sim_{\mathbb{R},U} D_1 + D_2$, where every component of D_1 is a component of $\llcorner\Delta\lrcorner$ and no component of D_2 is a component of $\llcorner\Delta\lrcorner$. We proceed by induction on the number of components of D_2.

If $D_2 = 0$ then pick any divisor H such that $K_X + \Delta + H$ is divisorially log terminal and ample over U (for example take for H any sufficiently ample, general ample $\mathbb{Q}$-divisor over U). As the support of D is contained in $\llcorner\Delta\lrcorner$, (8.3) implies that (X, Δ) has a nice model $f\colon X \dashrightarrow Y$ over U, with respect to A and D.

Now suppose that $D_2 \neq 0$. Let

$$\lambda = \sup\{\, t \in [0,1] \,|\, (X, \Delta + tD_2) \text{ is log canonical}\,\},$$

be the log canonical threshold of D_2. Then $\lambda > 0$ and $(X, \Theta = \Delta + \lambda D_2)$ is divisorially log terminal and log smooth, $K_X + \Theta \sim_{\mathbb{R},U} D + \lambda D_2$ and the number of components of $D + \lambda D_2$ that are not components of $\llcorner\Theta\lrcorner$ is smaller than the number of components of $D_1 + D_2$ that are not components of $\llcorner\Delta\lrcorner$. By induction there is a nice model $f\colon X \dashrightarrow Y$ over U for (X, Θ), with respect to A and D. Since $\mathbf{B}_+(f_*A/U)$ does not contain any log canonical centres of $(Y, f_*\Theta)$,

$$\begin{aligned} K_Y + f_*\Delta &\sim_{\mathbb{R},U} f_*D_1 + f_*D_2, \\ K_Y + f_*\Theta &= K_Y + f_*\Delta + \lambda f_*D_2, \end{aligned}$$

where $K_Y + f_*\Theta$ is divisorially log terminal and nef over U, and the support of f_*D_1 is contained in $\llcorner f_*\Delta\lrcorner$, (8.3) implies that $(Y, f_*\Delta)$ has a nice model $g\colon Y \dashrightarrow Z$ over U, with respect to f_*A and f_*D. The composition $g \circ f\colon X \dashrightarrow Z$ is then a nice model over U for (X, Δ), with respect to A and D. $\square$

Lemma 9.3. *Let $\pi\colon X \longrightarrow U$ be a morphism of normal projective varieties. Let $(X, \Delta = A + B)$ be a divisorially log terminal log pair and let D be an $\mathbb{R}$-divisor, where $A \geq 0$, $B \geq 0$ and $D \geq 0$.*

If every component of D is either semiample or a component of $\mathbf{B}((K_X + \Delta)/U)$ and $f\colon X \dashrightarrow Y$ is a nice model over U for (X, Δ), with respect to A and D, then f is a log terminal model for (X, Δ) over U.

Proof. By hypothesis the only divisors contracted by f are components of $\mathbf{B}((K_X + \Delta)/U)$. Since the question is local over U, we may assume that U is affine. Since $\mathbf{B}_+(f_*A/U)$ does not contain any log canonical centres of $(Y, \Gamma = f_*\Delta)$, we may find $K_Y + \Gamma' \sim_{\mathbb{R},U} K_Y + \Gamma$ where $K_Y + \Gamma'$ is kawamata log terminal and Γ' is big over U. (6.10) implies that $K_Y + \Gamma$ is semiample.

If $p\colon W \longrightarrow X$ and $q\colon W \longrightarrow Y$ resolve the indeterminacy of f then we may write

$$p^*(K_X + \Delta) + E = q^*(K_Y + \Gamma) + F,$$

where $E \geq 0$ and $F \geq 0$ have no common components, and both E and F are exceptional for q.

As $K_Y+\Gamma$ is semiample, $\mathbf{B}((q^*(K_Y+\Gamma)+F)/U)$ and F have the same support. On the other hand, every component of E is a component of $\mathbf{B}((p^*(K_X+\Delta)+E)/U)$. Thus $E = 0$ and any divisor contracted by f is contained in the support of F. Therefore f is a log terminal model of (X, Δ). $\square$

Lemma 9.4. *Theorem 5.12_{n-1}, Theorem 7.24_{n-1} and Theorem 7.20_n imply Theorem 5.12_n.*

Proof. Pick any ample $\mathbb{R}$-Cartier divisors H_i on U such that

$$K_X + \Delta + \pi^*H_1 \sim_{\mathbb{R}} D + \pi^*H_2 \geq 0.$$

Replacing Δ by $\Delta + \pi^*H_1$ and D by $D + \pi^*H_2$, we may assume that $K_X + \Delta \sim_{\mathbb{R}} D \geq 0$. By (8.6) we may assume that $\Delta = A + B$, where A is a general ample $\mathbb{Q}$-divisor over U and $B \geq 0$. By (9.6) we may assume that $D = M + F$, where every component of F is a component of the stable fixed divisor and if L is a component of M then mL is mobile. Pick a log resolution $f\colon Y \longrightarrow X$ of the support of D and Δ, which resolves the base locus of each linear system $|mL|$, for every component L of M. Let Γ be the divisor defined in (9.10). By definition of Γ, every component of the exceptional locus belongs to $\mathbf{B}((K_Y + \Gamma)/U)$. Replacing Γ by an $\mathbb{R}$-linearly equivalent divisor, we may assume that Γ contains an ample divisor over U. In particular, replacing $m\pi^*L$ by a general element of the linear system $|m\pi^*L|$, we may assume that $K_Y + \Gamma \sim_{\mathbb{R},U} N + G$, where every component of N is semiample, every component of G is a component of $\mathbf{B}((K_Y + \Gamma)/U)$, and $(Y, \Gamma + N + G)$ is log smooth. By (9.10), we may replace X by Y and the result follows by (9.2) and (9.3). □

9.1. **lemmas.**

Lemma 9.5. *Let D be an integral Weil divisor on a normal variety X.*

Then the stable base locus coincides with the usual definition of the stable base locus.

Proof. Let

$$|D|_{\mathbb{Q}} = \{\, C \geq 0 \,|\, C \sim_{\mathbb{Q}} D \,\}.$$

Let R be the intersection of the elements of $|D|_{\mathbb{R}}$ and let Q be the intersection of the elements of $|D|_{\mathbb{Q}}$. It suffices to prove that $Q = R$. As $|D|_{\mathbb{Q}} \subset |D|_{\mathbb{R}}$, it is clear that $R \subset Q$.

Suppose that $x \notin R$. We want to show that $x \notin Q$. Replacing X by the blow up of X at x, we may assume that there is a divisor C which is not a component of R and it suffices to prove that C is not a component of Q. We may find $D' \in |D|_{\mathbb{R}}$ such that C is not a component of D'. But then

$$D = D' + \sum r_i(f_i),$$

where f_i are rational functions on X and r_i are real numbers. Let V be the real subspace of the vector space of all Weil divisors on X spanned by the components of D, D' and (f_i) and let W be the span of the (f_i). Then $W \subset V$ are defined over the rationals. Set

$$\mathcal{P} = \{\, D'' \in V \,|\, D'' \geq 0,\ \operatorname{mult}_C(D'') = 0,\ D'' - D \in W \,\} \subset |D|_{\mathbb{R}}.$$

Then $\mathcal{P}$ is a rational polyhedron. As $D' \in \mathcal{P}$, $\mathcal{P}$ is non-empty, and so it must contain a rational point D''. We may write

$$D'' = D + \sum s_i(f_i),$$

where s_i are real numbers. Since D'' and D have rational coefficients, it follows that we may find s_i which are rational. But then $D'' \in |D|_{\mathbb{Q}}$, and so C is not a component of D'', nor therefore of Q. □

Proposition 9.6. *Let $\pi\colon X \longrightarrow U$ be projective morphism of normal varieties and let $D \geq 0$ be an $\mathbb{R}$-divisor. Then we may find $\mathbb{R}$-divisors M and F such that*

(1) *$M \geq 0$ and $F \geq 0$,*
(2) *$D \sim_{\mathbb{R},U} M + F$,*
(3) *every component of F is a component of $\mathbf{B}(D/U)$, and*
(4) *if B is a component of M then some multiple of B is mobile.*

We need two basic results:

Lemma 9.7. *Let X be a normal variety and let D and D' be two $\mathbb{R}$-divisors such that $D \sim_{\mathbb{R}} D'$.*

Then we may rational functions $f_1, f_2, \dots, f_k$ and real numbers $r_1, r_2, \dots, r_k$ which are independent over the rationals such that

$$D = D' + \sum_i r_i(f_i).$$

In particular every component of (f_i) is either a component of D or of D'.

Proof. By assumption we may find $f_1, f_2, \dots, f_k$ and real numbers $r_1, r_2, \dots, r_k$ such that

$$D = D' + \sum_{i=1}^{k} r_i(f_i).$$

Pick k minimal with this property. Suppose that the real numbers r_i are not independent over $\mathbb{Q}$. Then we can find rational numbers d_i, not all zero, such that

$$\sum_i d_i r_i = 0.$$

Possibly re-ordering we may assume that $d_k \neq 0$. Multiplying through by an integer we may assume that $d_i \in \mathbb{Z}$. Possibly replacing f_i by f_i^{-1}, we may assume that $d_i \geq 0$. Let d be the least common multiple of the non-zero d_i. If $d_i \neq 0$, we replace f_i by f_i^{d/d_i} (and hence r_i by $d_i r_i/d$) so that we may assume that either $d_i = 0$ or 1. For $1 \leq i < k$, set

$$g_i = \begin{cases} f_i/f_k & \text{if } d_i = 1 \\ f_i & \text{if } d_i = 0. \end{cases}$$

Then

$$D = D' + \sum_{i=1}^{k-1} r_i(g_i),$$

which contradicts our choice of k.

Now suppose that B is a component of (f_i). Then

$$\operatorname{mult}_B(D) = \operatorname{mult}_B(D') + \sum r_j n_j,$$

where $n_j = \operatorname{mult}_B(f_j)$ is an integer and $n_i \neq 0$. But then one of $\operatorname{mult}_B(D) - \operatorname{mult}_B(D') \neq 0$, so that one of $\operatorname{mult}_B(D)$ and $\operatorname{mult}_B(D') \neq 0$ must be non zero. $\square$

Lemma 9.8. *Let $\pi\colon X \longrightarrow U$ be a projective morphism of normal varieties and let $0 \leq D' \sim_{\mathbb{R},U} D \geq 0$ be two $\mathbb{R}$-divisors on a normal variety X with no common components.*

Then we may find $D'' \in |D/U|_{\mathbb{R}}$ such that a multiple of every component of D'' is mobile.

Proof. Pick ample $\mathbb{R}$-divisors on U, H and H' such that $D + \pi^* H \sim_{\mathbb{R}} D' + \pi^* H'$ and $D + \pi^* H$ and $D' + \pi^* H'$ have no common components. Replacing D by $D + \pi^* H$ and D' by $D' + \pi^* H'$, we may assume that $D' \sim_{\mathbb{R}} D$.

We may write

$$D' = D + \sum r_i(f_i) = D + R,$$

where $r_i \in \mathbb{R}$ and f_i are rational functions on X. By (9.7) we may assume that every component of R is a component of $D + D'$.

We proceed by induction on the number of components of $D + D'$. If $q_1, q_2, \dots, q_k$ are any positive rational numbers then we may always write

$$C' = C + Q = C + \sum q_i(f_i),$$

where $C \geq 0$ and $C' \geq 0$ have no common components. But now if we suppose that q_i is sufficiently close to r_i then C is supported on D and C' is supported on D'. We have that $mC \sim mC'$ for some integer $m > 0$. By Bertini we may find $C'' \sim_{\mathbb{Q}} C$ such that every component of C'' has a multiple which is mobile. Pick $\lambda > 0$ maximal such that $D_1 = D - \lambda C \geq 0$ and $D'_1 = D' - \lambda C' \geq 0$. Note that

$$0 \leq D_1 \sim_{\mathbb{R}} D'_1 \geq 0$$

are two $\mathbb{R}$-divisors on X with no common components, and that $D_1 + D'_1$ has fewer components than $D + D'$. By induction we may then find

$$D''_1 \in |D_1|_{\mathbb{R}},$$

such that a multiple of every component of D''_1 is mobile. But then

$$D'' = \lambda C'' + D''_1 \in |D|_{\mathbb{R}},$$

and every component of D'' has a multiple which is mobile. $\square$

Proof of (9.6). We may write $D = M + F$, where every component of F is contained in $\mathbf{B}(D/U)$ and no component of M is contained in $\mathbf{B}(D/U)$. We proceed by induction on the number of components B of M such that no multiple of B is mobile.

Fix one such component B. It suffices to find $D' \in |D/U|_{\mathbb{R}}$ such that B is not a component of D' and with the property that if B' is a component of D' such that no multiple of B' is mobile, then B' is also a component of D. Now we may find $D_1 \in |D/U|_{\mathbb{R}}$ such that B is not a component of D_1. Cancelling common components of D_1 and D, by (9.8), one sees that $D \sim_{\mathbb{R},U} D' = D'' + E$ where every component of D'' has a multiple which is mobile and $\operatorname{Supp} E \subset \operatorname{Supp} D \cap \operatorname{Supp} D_1$. □

Lemma 9.9. *Let $\pi \colon X \longrightarrow U$ be a proper morphism of normal quasi-projective varieties. Let D be a $\mathbb{R}$-Cartier divisor on X and let D' be its restriction to the generic fibre of π.*

If $D' \sim_{\mathbb{R}} B' \geq 0$ for some $\mathbb{R}$-divisor B' on the generic fibre of π, then there is a divisor B on X such that $D \sim_{\mathbb{R},U} B \geq 0$.

Proof. Taking the closure of the generic points of B', we may assume that there is an $\mathbb{R}$-divisor $B_1 \geq 0$ on X such that the restriction of B_1 to the generic fibre is B'. As

$$D' - B' \sim_{\mathbb{R}} 0,$$

it follows that there is an open subset U_1 of U, such that

$$(D - B_1)|_{V_1} \sim_{\mathbb{R}} 0,$$

where V_1 is the inverse image of U_1. But then there is a divisor G on X such that

$$D - B_1 \sim_{\mathbb{R}} G,$$

where $Z = \pi(\operatorname{Supp} G)$ is a proper closed subset. As U is quasi-projective, there is an ample divisor H on U which contains Z. Possibly rescaling, we may assume that $F = \pi^* H \geq -G$. But then

$$D \sim_{\mathbb{R}} (B_1 + F + G) - F,$$

so that

$$D \sim_{\mathbb{R},U} (B_1 + F + G) \geq 0.$$

□

Lemma 9.10. *Let $\pi \colon X \longrightarrow U$ be a proper morphism of normal quasi-projective varieties. Let (X, Δ) be a kawamata log terminal pair. Let $f \colon Z \longrightarrow X$ be any log resolution of (X, Δ) and suppose that we write*

$$K_Z + \Phi_0 = f^*(K_X + \Delta) + E,$$

where $\Phi_0 \geq 0$ and $E \geq 0$ have no common components, $f_*\Phi_0 = \Delta$ and E is exceptional. Let $F \geq 0$ be any divisor whose support is equal to the exceptional locus of f.

Then we may find $\eta > 0$ such that if $\Phi = \Phi_0 + \eta F$ then

- *$f_*\Phi = \Delta$,*
- *$K_Z + \Phi$ is kawamata log terminal,*

- *if Δ is big over U then so is Φ, and*
- *the log terminal models (respectively weak log canonical models) over U of $K_X + \Delta$ and the log terminal models (respectively weak log canonical models) over U of $K_Z + \Phi$ are the same.*

Proof. Everything is clear, apart from the fact that if $\phi\colon Z \dashrightarrow W$ is a log terminal model (respectively weak log canonical model) over U of $K_Z + \Phi$ then it is a log terminal model (respectively weak log canonical model) over U of $K_X + \Delta$.

Let $\psi\colon X \dashrightarrow W$ be the induced birational map and set $\Psi = \phi_* \Phi$. By what we have already observed, possibly blowing up more, we may assume that ϕ is a morphism. By assumption if we write

$$K_Z + \Phi = \phi^*(K_W + \Psi) + G,$$

then $G > 0$ and the support of G is the full ϕ-exceptional locus (respectively G is exceptional). Thus

$$f^*(K_X + \Delta) + E + \eta F = \phi^*(K_W + \Psi) + G.$$

By negativity of contraction (cf. (2.7)) $G - E - \eta F \geq 0$, so that in particular ϕ must contract every f-exceptional divisor and ψ^{-1} does not contract any divisors. But then, ψ is a log terminal model (respectively weak log canonical model) over U by (**??**). □

10. Finiteness of models, the big case

Lemma 10.1. *Assume Theorem 5.12$_n$.*

Let $\pi\colon X \longrightarrow U$ be a projective morphism of normal quasi-projective varieties, where X has dimension n. Suppose that there is a kawamata log terminal pair (X, Δ'_0). Let V be a finite dimensional rational affine subspace of the space of Weil divisors on X. Fix a general ample $\mathbb{Q}$-divisor A over U. Let $\mathcal{C} \subset \mathcal{L}_A(V)$ be a rational polytope. Suppose that either

1. *if $\Delta \in \mathcal{C}$ then $K_X + \Delta$ is π-big, or*
2. *Theorem 7.23$_n$ holds.*

Then there are finitely many rational maps $\phi_i\colon X \dashrightarrow Y_i$ over U, $1 \leq i \leq k$, with the property that if $\Delta \in \mathcal{C} \cap \mathcal{E}_{A,\pi}(V)$ then there is an index $1 \leq i \leq k$ such that ϕ_i is a log terminal model of $K_X + \Delta$ over U.

Proof. As $\mathcal{L}_A(V)$ is a rational polytope, we may assume that $\mathcal{C}$ spans V_A. We proceed by induction on the dimension of V.

First suppose that $\dim V > 0$ and that there is a divisor $\Delta_0 \in \mathcal{C}$ such that $K_X + \Delta_0 \sim_{\mathbb{R},U} 0$. Pick $\Theta \in \mathcal{C}$, $\Theta \neq \Delta_0$. Then there is a divisor Δ

on the boundary of $\mathcal{C}$ such that

$$\Theta - \Delta_0 = \lambda(\Delta - \Delta_0),$$

for some $0 < \lambda \leq 1$. Now

$$\begin{aligned} K_X + \Theta =& \lambda(K_X + \Delta) + (1 - \lambda)(K_X + \Delta_0) \\ \sim_{\mathbb{R},U}& \lambda(K_X + \Delta). \end{aligned}$$

In particular $\Delta \in \mathcal{E}_{A,\pi}(V)$ if and only if $\Theta \in \mathcal{E}_{A,\pi}(V)$ and (11.7) implies that $K_X + \Delta$ and $K_X + \Theta$ have the same log terminal models. On the other hand the boundary of $\mathcal{C}$ is contained in finitely many affine hyperplanes defined over the rationals, and we are done by induction on the dimension of V in this case.

We now prove the general case. By (8.5), we may assume that $K_X+\Delta$ is kawamata log terminal for all $\Delta \in \mathcal{C}$ and that if $\dim V > 0$ then $\mathcal{C}$ is contained in the interior of $\mathcal{L}_A(V)$. Since $\mathcal{L}_A(V)$ is compact and $\mathcal{C} \cap \mathcal{E}_{A,\pi}(V)$ is closed, it suffices to prove this result locally about any divisor $\Delta_0 \in \mathcal{C} \cap \mathcal{E}_{A,\pi}(V)$. By assumption there is a divisor $D_0 \geq 0$ such that $K_X + \Delta \sim_{\mathbb{R},U} D_0$ and so, as we are assuming Theorem 5.12_n, there is a log terminal model $\phi\colon X \dashrightarrow Y$ over U for $K_X + \Delta_0$. In particular we may assume that $\dim V > 0$.

As ϕ is $(K_X + \Delta_0)$-negative there is a neighbourhood $\mathcal{C}_0$ of Δ_0 in $\mathcal{L}_A(V)$, which we may assume to a be rational polytope, such that for any $\Delta \in \mathcal{C}_0$ and any ϕ-exceptional divisor E contained in X, we have $a(E, X, \Delta) < a(E, Y, \phi_*\Delta)$. If $K_X+\Delta_0$ is π-big then, possibly shrinking $\mathcal{C}_0$, we may assume that $K_X+\Delta$ is π-big for all $\Delta \in \mathcal{C}_0$. Since $K_Y+\phi_*\Delta_0$ is kawamata log terminal and Y is $\mathbb{Q}$-factorial, possibly shrinking $\mathcal{C}_0$, we may assume that $K_Y+\phi_*\Delta$ is kawamata log terminal for all $\Delta \in \mathcal{C}_0$. In particular, replacing $\mathcal{C}$ by $\mathcal{C}_0$, we may assume that the rational polytope $\mathcal{C}' = \phi_*(\mathcal{C})$ is contained in $\mathcal{L}_{\phi_*A}(W)$, where $W = \phi_*V$. By (8.4), there is a rational affine linear map $L\colon W \longrightarrow V'$ and a general ample $\mathbb{Q}$-divisor A' over U such that $L(\mathcal{C}') \subset \mathcal{L}_{A'}(V')$. Replacing $V'_{A'}$ by the span of $L(\mathcal{C}')$, we may assume that $\dim V' \leq \dim V$. By (**??**) and (10.7), any log terminal model of $(Y, \phi_*\Delta)$ over U is a log terminal model of (X, Δ) over U, for any $\Delta \in \mathcal{C}$. Replacing X by Y we may therefore assume that $K_X + \Delta_0$ is π-nef.

By (6.10) $K_X + \Delta_0$ has an ample model $\psi\colon X \longrightarrow Z$ over U. In particular $K_X + \Delta_0 \sim_{\mathbb{R},Z} 0$. By what we have already proved there are finitely many birational maps $\phi_i\colon X \dashrightarrow Y_i$ over Z, $1 \leq i \leq k$, such that for any $\Delta \in \mathcal{C} \cap \mathcal{E}_{A,\psi}$, there is an index i such that ϕ_i is a log terminal model of $K_X + \Delta$ over Z. But then, since there are only finitely many models Y_i, possibly shrinking $\mathcal{C}$, (10.8) implies that if

$\Delta \in \mathcal{C}$ then there is an index $1 \leq i \leq k$ such that ϕ_i is a log terminal model for $K_X + \Delta$ over U. □

Lemma 10.2. *Assume Theorem 5.12$_n$.*

Let $\pi\colon X \longrightarrow U$ be a projective morphism of normal quasi-projective varieties, where X has dimension n. Suppose that there is a kawamata log terminal pair (X, Δ_0). Let V be a finite dimensional rational affine subspace of the space of Weil divisors on X. Fix a general ample $\mathbb{Q}$-divisor A over U. Let $\mathcal{C} \subset \mathcal{L}_A(V)$ be a rational polytope. Suppose that either

(1) *if $\Delta \in \mathcal{C}$ then $K_X + \Delta$ is π-big, or*
(2) *Theorem 7.23$_n$ holds.*

Then we can find finitely many rational maps $\phi_i\colon X \dashrightarrow Y_i$ over U, $1 \leq i \leq k$, such that if $\phi\colon X \dashrightarrow Y$ is a log terminal model of $K_X + \Delta$, where $\Delta \in \mathcal{C}$, then there is an index $1 \leq i \leq k$ and an isomorphism $\eta\colon Y_i \longrightarrow Y$ such that $\phi = \eta \circ \phi_i$.

Proof. Pick $\mathbb{Q}$-Cartier divisors $B_1, B_2, \dots, B_l \geq 0$ which generate the group of Weil divisors modulo relative numerical equivalence. By (8.5) we may assume that there is a rational number $\epsilon > 0$ such that $\epsilon C \leq \Delta$ and $K_X + \Delta + \epsilon C$ is kawamata log terminal, where $C = \sum B_j$. Further, in case (1), possibly replacing ϵ by a smaller number, we may assume that if $\Delta \in \mathcal{C}$ then $K_X + \Delta - \epsilon C$ is π-big. Let W be the rational affine space spanned by V and the divisors $B_1, B_2, \dots, B_l$. If we set

$$\mathcal{C}' = \{\, \Delta + \sum b_i B_i \,|\, \Delta \in \mathcal{C} \text{ and } |b_i| \leq \epsilon \,\},$$

then $\mathcal{C}' \subset \mathcal{L}_A(W)$ is a rational polytope and if $\Delta' \in \mathcal{C}'$ then $K_X + \Delta'$ is kawamata log terminal and π-big in case (1).

By (10.1) there are rational maps $\phi_i\colon X \dashrightarrow Y_i$ over U, $1 \leq i \leq k$, such that given any $\Theta \in \mathcal{C}' \cap \mathcal{E}_{A,\pi}(W)$, we may find an index $1 \leq i \leq k$ such that ϕ_i is a log terminal model of $K_X + \Theta$ over U. Pick $\Delta \in \mathcal{C}$ and let $\phi\colon X \dashrightarrow Y$ be a log terminal model of $K_X + \Delta$ over U. Let $G_i = \phi_* B_i$ and $\Gamma = \phi_* \Delta$. Then the divisors $G_1, G_2, \dots, G_l$ span the group of Weil divisors of Y modulo relative numerical equivalence. Since Y is $\mathbb{Q}$-factorial, we may find rational numbers $b_1, b_2, \dots, b_l$ such that $G = \sum b_i G_i$ is ample over U. If we set $B = \sum b_i B_i$ then, possibly replacing G by a small multiple, we may assume that $|b_i| \leq \epsilon$ and ϕ is $(K_X + \Delta + B)$-negative. But then $\Delta + B \in \mathcal{C}' \cap \mathcal{E}_{A,\pi}(W)$ and ϕ is a log terminal model for $K_X + \Delta + B$ over U. As $K_Y + \Gamma + G$ is ample over U, (10.9) implies that there is an index $1 \leq i \leq k$ and an isomorphism $\eta\colon Y_i \longrightarrow Y$ such that $\phi = \eta \circ \phi_i$. □

Lemma 10.3. *Assume Theorem 5.12$_n$.*

Let $\pi\colon X \longrightarrow U$ be a projective morphism of normal quasi-projective varieties, where X has dimension n. Suppose that there is a kawamata log terminal pair (X, Δ_0). Let V be a finite dimensional rational affine subspace of the space of Weil divisors on X. Fix a general ample $\mathbb{Q}$-divisor A over U. Let $\mathcal{C} \subset \mathcal{L}_A(V)$ be a rational polytope. Suppose that either

1. *if $\Delta \in \mathcal{C}$ then $K_X + \Delta$ is π-big, or*
2. *Theorem 7.23_n holds.*

Then we can find finitely many rational maps $\phi_i\colon X \dashrightarrow Y_i$ over U, $1 \leq i \leq k$, such that if $\phi\colon X \dashrightarrow Y$ is a $\mathbb{Q}$-factorial weak log canonical model of $K_X + \Delta$, where $\Delta \in \mathcal{C}$, then there is an index $1 \leq i \leq k$ and an isomorphism $\eta\colon Y_i \longrightarrow Y$ such that $\phi = \eta \circ \phi_i$.

Proof. By (8.5) we may assume that if $\Delta \in \mathcal{C}$ then $K_X+\Delta$ is kawamata log terminal. Possibly enlarging V, we may assume that V is the vector space spanned by a finite set of prime divisors I. Let J be the set of prime divisors contracted by any log terminal model of $K_X + \Delta$ over U, for any divisor $\Delta \in \mathcal{L}_A(V)$. By (10.2) we may find finitely many birational maps $\phi_i\colon X \dashrightarrow Y_i$, $1 \leq i \leq k$, over U such that if $\phi\colon X \dashrightarrow Y$ is a log terminal model of $K_X + \Delta$ over U, for some $\Delta \in \mathcal{L}_A(V)$, then there is an index $1 \leq i \leq k$ and an isomorphism $\eta\colon Y_i \longrightarrow Y$ such that $\phi = \eta \circ \phi_i$. In particular the set J is finite. Let W be the vector space spanned by $I \cup J$. Note that the set of prime divisors contracted by any log terminal model of $K_X + \Delta$ over U, for any $\Delta \in \mathcal{L}_A(W)$, is also equal to J. Thus replacing V by W we may assume that $J \subset I$.

Pick $\Delta \in \mathcal{L}_A(V)$ and let $\phi\colon X \dashrightarrow Y$ be any $\mathbb{Q}$-factorial weak log canonical model of $K_X + \Delta$ over U. Pick $F \geq 0$, with support equal to the exceptional locus of ϕ such that $K_X + \Delta + F$ is kawamata log terminal. In particular $\Delta + F \in \mathcal{L}_A(V)$. On the other hand ϕ is negative with respect to $K_X + \Delta + F$ and

$$K_Y + \Gamma = \phi_*(K_X + \Delta) = \phi_*(K_X + \Delta + F),$$

is nef over U. Thus ϕ is a log terminal model of $K_X + \Delta + F$ over U and so there is an index $1 \leq i \leq k$ and an isomorphism $\eta\colon Y_i \longrightarrow Y$ such that $\phi = \eta \circ \phi_i$. □

Lemma 10.4. *Theorem 5.12_n implies Theorem 7.22_n.*

Proof. This is (1) of (10.3). □

In terms of induction, we will need a version of Theorem 7.24_n locally around the locus where $K_X + \Delta$ is not kawamata log terminal. To this end we need a version of (8.2) for a convex set of divisors:

Proposition 10.5. *Assume Theorem 5.12_n and Theorem 7.22_n.*

Let $(X, \Delta_0 = S + A + B_0 \in \mathcal{E}_A)$ be a log smooth projective purely log terminal pair of dimension n, where $\llcorner \Delta_0 \lrcorner = S$, A is ample and $B \geq 0$. Let V_0 be the vector space of Weil divisors on X generated by the components of B_0. Fix a general ample divisor H such that $K_X + \Delta_0 + H$ is ample, and let V be the translate by $S + A$, of the vector space spanned by V_0 and H. Given any polytope F in V, the cone $\mathcal{C}(F)$ over F (with vertex Δ_0) is the polytope spanned by F and Δ_0.

Pick a constant $\alpha > 0$ such that

$$F = \{\, \Delta_0 + E + H \in V \mid \|E\| \leq \alpha, E \in V_0 \,\} \subseteq \mathcal{N}_A$$

and let $\mathcal{C}_0 = \mathcal{C}(F)$. If $K_X + \Delta_0$ does not have a log terminal model then there is a countable collection of polytopes $\mathcal{P}_i$ and birational maps $\phi_i \colon X \dashrightarrow Y_i$ such that

1. *$\mathcal{P}_i^\circ \cap \mathcal{P}_j^\circ = \varnothing$ for any $i \neq j$ such that $\mathcal{P}_i$ and $\mathcal{P}_j$ are of maximal dimension,*
2. *$\mathcal{P}_i \subset \mathcal{W}_{Y_i}$,*
3. *for any $\Delta = \Delta_0 + E + H \in F$, a $(K_X + \Delta_0)$-MMP with scaling of $E + H$ is given by*

   $$X \dashrightarrow Y_{i_1} \dashrightarrow Y_{i_2} \dashrightarrow Y_{i_3} \cdots,$$

   for appropriate indices i_j,
4. *for all $\epsilon > 0$, the set*

   $$\{\, i \in I \mid \exists \Delta = \Delta_0 + t(E + H) \in \mathcal{P}_i, t > \epsilon \,\},$$

   is finite, and
5. *if $\mathcal{C}_i$ denotes the cone over $\mathcal{P}_i$, then*

   $$\mathcal{C}_i - \{\Delta_0\} = \bigcup \{\, \mathcal{P}_j \mid \mathcal{P}_j \subset \mathcal{C}_i \,\}.$$

Proof. By assumption the set

$$\mathcal{P}_0 = \mathcal{C}_0 \cap \mathcal{N}_A,$$

does not contain Δ_0. By (**??**), $\mathcal{P}_0$ is a polytope. Let $\phi_0 \colon X \dashrightarrow Y_0 = X$ be the identity map. Note that $\mathcal{C}_0$ is indeed the cone over $\mathcal{P}_0$.

Suppose that we have defined $\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_k$, satisfying (1), (2) and a modified version of (3), where we stop the MMP in (3) when Δ lies on the boundary of $\cup \mathcal{P}_i$. Let $F_{i,j}$ be the faces of $\mathcal{P}_i$, whose associated cones are of maximal dimension. For any such face, let $F = F_{i,j}$ and let $\mathcal{C}_{k+1}$ be the corresponding cone. If $\mathcal{P}_i$ is of maximal dimension, we discard F whenever $\mathcal{C}_{k+1} = \mathcal{C}_i$, for some $i < k$. Let R be a $(K_X + \Delta_0)$-extremal ray of Y_i, which cuts out F. Let $Y_i \dashrightarrow Y_{k+1}$ be the corresponding step of the $(K_X + \Delta_0)$-MMP.

Let $\mathcal{P}_{k+1}$ be the nef cone of Y_{k+1} intersected with $\mathcal{C}_{k+1}$. As before $\mathcal{P}_{k+1}$ is a polytope that, by assumption, does not contain Δ_0.

Thus, by induction, we may assume that we have constructed a countable set of polytopes $\mathcal{P}_i$ and rational maps $\phi_i \colon X \dashrightarrow Y_i$ satisfying (1), (2) and the modified version of (3) described above. Note that property (4) follows from Theorem 7.22$_n$, and (3) and (5) follow from (4), since each $\mathcal{C}_i$ contains infinitely many $\mathcal{P}_j$. □

Lemma 10.6. *Assume Theorem 7.24$_{n-1}$, Theorem 5.12$_n$ and Theorem 7.22$_n$. Let $(X, \Delta_0 = S + A + B_0)$ be a purely log terminal pair, where X is projective of dimension n, A is ample, $\llcorner \Delta_0 \lrcorner = S$ and $B_0 \geq 0$. Suppose that $K_X + \Delta_0$ is pseudo-effective and S is not a component of $N_\sigma(K_X + \Delta_0)$.*

Let V_0 be the span of the components of B_0. Then we may find an ample divisor H, a positive constant α, and a log pair (W, R) with the following properties. Let V be the translate by $S + A$ of the vector space of Weil divisors on X generated by H and the components of B_0.

Then for every $B \in V_0$ such that

$$\|B - B_0\| < \alpha t,$$

for any $t \in (0, 1]$, we may find a log terminal model $\phi \colon X \dashrightarrow Y$ of $(X, \Delta = S + A + B + tH)$ which does not contract S and such that the pairs (W, R) and $(Y, T = \phi_* S)$ have isomorphic neighbourhoods of R and T.

Proof. Suppose not. Passing to a log resolution, we may assume that (X, Δ_0) is log smooth cf. (9.10). Using the notation established in (10.5) and possibly relabelling, by assumption there is no $\mathcal{C}_k$ such that for any two elements Δ_1 and Δ_2 in $\mathcal{C}_k$, the corresponding models have isomorphic neighbourhoods of T. Hence we may find a sequence of polytopes $\mathcal{P}_i$, such that the corresponding cones are nested $\mathcal{C}_i \subset \mathcal{C}_{i-1}$, and moreover the corresponding Y_i are not eventually isomorphic in a neighbourhood of T. By compactness of F, we may find $\Delta \in F$ such that $(\Delta_0, \Delta] \cap \mathcal{C}_i \neq \varnothing$ for every i. By (3) of (10.5), the corresponding $(K_X + \Delta_0)$-MMP with scaling of $E + H = \Delta - \Delta_0$ is not eventually an isomorphism in a neighbourhood of S and this contradicts (8.2). □

10.1. **Lemmas.**

Lemma 10.7. *Let $\pi \colon X \longrightarrow U$ be a projective morphism of normal quasi-projective varieties. Suppose that $K_X + \Delta$ is divisorially log terminal and let $\phi \colon X \dashrightarrow Y$ be a birational map over U such that ϕ^{-1}*

does not contract any divisors, $K_Y + \phi_*\Delta$ is divisorially log terminal and $a(F, X, \Delta) < a(F, Y, \phi_*\Delta)$ for all ϕ-exceptional divisors $F \subset X$.

If $\varphi\colon Y \dashrightarrow Z$ is a log terminal model of $(Y, \phi_*\Delta)$ over U, then $\eta = \varphi \circ \phi\colon X \dashrightarrow Z$ is a log terminal model of (X, Δ) over U.

Proof. Clearly η^{-1} contracts no divisors and Z is $\mathbb{Q}$-factorial and $K_Z + \eta_*\Delta$ is nef. By (**??**), it suffices to show that $a(F, X, \Delta) < a(F, Z, \eta_*\Delta)$ for all η-exceptional divisors $F \subset X$.

Let $p\colon W \longrightarrow X$, $q\colon W \longrightarrow Y$ and $r\colon W \longrightarrow Z$ be a common resolution. As φ is a log terminal model of $(Y, \phi_*\Delta)$ we have that $q^*(K_Y + \phi_*\Delta) - r^*(K_Z + \eta_*\Delta) = E \geq 0$ and its support contains the exceptional divisors of φ. Since

$$p^*(K_X + \Delta) - r^*(K_Z + \eta_*\Delta) = p^*(K_X + \Delta) - q^*(K_Y + \phi_*\Delta) + E$$

the assertion follows easily. □

Corollary 10.8. *Let $\pi\colon X \longrightarrow U$ be a projective morphism of normal quasi-projective varieties. Let V be a finite dimensional affine subspace of the real vector space of Weil divisors on X, which is defined over the rationals. Fix a general ample $\mathbb{Q}$-divisor A over U. Let (X, Δ_0) be a kawamata log terminal pair, let $f\colon X \longrightarrow Z$ a morphism over U such that $\Delta_0 \in V_A$ and $K_X + \Delta_0 = f^*H$, for some ample divisor H. Let $\phi\colon X \dashrightarrow Y$ be a birational map over Z.*

Then there is a neighbourhood P_0 of Δ_0 in $\mathcal{L}_A(V)$ such that for all $\Delta \in P_0$

- *$\phi\colon X \dashrightarrow Y$ is a log terminal model of $K_X + \Delta$ over Z if and only if ϕ is a log terminal model of $K_X + \Delta$ over U.*

Lemma 10.9. *Let $\pi\colon X \longrightarrow U$ be a projective morphism of normal quasi-projective varieties.*

If $K_X + \Delta$ is kawamata log terminal then the ample model of $K_X + \Delta$ over U is unique, if it exists at all.

Proof. Let Z_1 and Z_2 be ample models for $K_X + \Delta$ over U. By definition we may find log terminal models $\phi_i\colon X \dashrightarrow Y_i$ for $K_X + \Delta$ over U and morphisms $f_i\colon Y_i \longrightarrow Z_i$ over U such that, $K_{Y_i} + \Gamma_i \sim_{\mathbb{R}} f_i^* H_i$, where H_i is an ample divisor over U and $\Gamma_i = \phi_{i*}\Delta$, for $i = 1$ and 2.

Suppose that $g\colon W \longrightarrow X$ resolves the indeterminacy of ϕ_i, for $i = 1$ and 2. Let $\Psi \geq 0$ be the divisor on W whose existence is guaranteed by (9.10). Then (Y_i, Γ_i) is also a log terminal model for $K_W + \Psi$, for $i = 1$ and 2.

Thus replacing (X, Δ) by (W, Ψ) we may assume that ϕ_i is a morphism over U. In particular, there are divisors $E_i \geq 0$, exceptional for

ϕ_i, such that

$$K_X + \Delta = \phi_i^*(K_{Y_i} + \Gamma_i) + E_i,$$

for $i = 1$ and 2. As $K_{Y_i} + \Gamma_i$ is nef, for $i = 1$ and 2, negativity of contraction implies that $E_1 = E_2$. In particular $g_1^* H_1 \sim_{\mathbb{R}} g_2^* H_2$, where $g_i = f_i \circ \phi_i$ and so $Z_1 \simeq Z_2$. □

11. Non-vanishing

We follow the general lines of the proof of the non-vanishing theorem, see for example Chapter 3, §5 of [10]. In particular there are two cases:

Lemma 11.1. *Assume Theorem 5.12$_n$. Let (X, Δ) be a projective, log smooth, kawamata log terminal pair of dimension n, such that $K_X + \Delta$ is pseudo-effective and $\Delta - A \geq 0$ for an ample $\mathbb{Q}$-divisor A. Suppose that for every positive integer k such that kA is integral,*

$$h^0(X, \mathcal{O}_X(\llcorner mk(K_X + \Delta) \lrcorner + kA)),$$

is a bounded function of m.

Then there is an $\mathbb{R}$-divisor D such that $K_X + \Delta \sim_{\mathbb{R}} D \geq 0$.

Proof. By (11.6) it follows that $K_X + \Delta$ is numerically equivalent to $N_\sigma(K_X + \Delta)$. Since $N_\sigma(K_X + \Delta) - (K_X + \Delta)$ is numerically trivial and ampleness is a numerical condition, it follows that

$$A'' = A + N_\sigma(K_X + \Delta) - (K_X + \Delta),$$

is ample and numerically equivalent to A. Thus if $A' \sim_{\mathbb{R}} A''$ is general

$$K_X + \Delta' = K_X + A' + (\Delta - A),$$

is kawamata log terminal and numerically equivalent to $K_X + \Delta$, and

$$K_X + \Delta' \sim_{\mathbb{R}} N_\sigma(K_X + \Delta) \geq 0.$$

Thus by Theorem 5.12$_n$, $K_X + \Delta'$ has a log terminal model $\phi\colon X \dashrightarrow Y$, which, by (11.7), is also a log terminal model for $K_X + \Delta$. Replacing (X, Δ) by (Y, Γ) we may therefore assume that $K_X + \Delta$ is nef and the result follows by the base point free theorem, cf. (**??**). □

Lemma 11.2. *Let (X, Δ) be a projective, log smooth, kawamata log terminal pair such that $\Delta = A + B$, where A is a general ample $\mathbb{Q}$-divisor and $B \geq 0$. Suppose that there is a positive integer k such that kA is integral and*

$$h^0(X, \mathcal{O}_X(\llcorner mk(K_X + \Delta) \lrcorner + kA)),$$

is an unbounded function of m.

Then we may find a projective, log smooth, purely log terminal pair (Y, Γ) and a general ample $\mathbb{Q}$-divisor C on Y, where

- *Y is birational to X,*
- *$\Gamma - C \geq 0$,*
- *$T = \llcorner \Gamma \lrcorner$ is an irreducible divisor, and*
- *Γ and $N_\sigma(K_Y + \Gamma)$ have no common components.*

Moreover the pair (Y, Γ) has the property that $K_X + \Delta \sim_{\mathbb{R}} D \geq 0$ for some $\mathbb{R}$-divisor D if and only if $K_Y + \Gamma \sim_{\mathbb{R}} G \geq 0$ for some $\mathbb{R}$-divisor G.

Proof. Pick m large enough so that

$$h^0(X, \mathcal{O}_X(\llcorner mk(K_X + \Delta)\lrcorner + kA)) > \binom{kn+n}{n}.$$

By standard arguments, given any point $x \in X$, we may find an effective divisor which is $\mathbb{R}$-linearly equivalent to

$$\llcorner mk(K_X + \Delta)\lrcorner + kA,$$

of multiplicity greater than kn at x. In particular, we may find an $\mathbb{R}$-divisor

$$0 \leq H \sim_{\mathbb{R}} m(K_X + \Delta) + A,$$

of multiplicity greater than n at x. Given $t \in [0, m]$, consider

$$\begin{aligned}(t+1)(K_X + \Delta) &= K_X + \frac{m-t}{m}A + B + t(K_X + \Delta + \frac{1}{m}A)\\ &\sim_{\mathbb{R}} K_X + \frac{m-t}{m}A + B + \frac{t}{m}H\\ &= K_X + \Delta_t.\end{aligned}$$

Fix $0 < \epsilon \ll 1$, let $A' = (\epsilon/m)A$ and $u = m - \epsilon$. We have:

(1) $K_X + \Delta_0$ is kawamata log terminal,
(2) $\Delta_t \geq A'$, for any $t \in [0, u]$ and
(3) the locus of log canonical singularities of (X, Δ_u) contains a very general point x of X.

Let $\pi\colon Y \longrightarrow X$ be a common log resolution of (X, Δ_t). We may write

$$K_Y + \Psi_t = \pi^*(K_X + \Delta_t) + E_t,$$

where $E_t \geq 0$ and $\Psi_t \geq 0$ have no common components, $\pi_*\Psi_t = \Delta_t$ and E_t is exceptional. Pick an exceptional divisor $F \geq 0$ and a positive integer l such that $l(\pi^*A' - F)$ is very ample and let lC be a very general element of the linear system $|l(\pi^*A' - F)|$. For any $t \in [0, u]$, let

$$\Phi_t = \Psi_t - \pi^*A' + C + F \sim_{\mathbb{R}} \Psi_t \qquad \text{and} \qquad \Gamma_t = \Phi_t - \Phi_t \wedge N_\sigma(K_Y + \Phi_t).$$

Then properties (1-3) above become

(1) $K_Y + \Gamma_0$ is kawamata log terminal,

(2) $\Gamma_t \geq C$, for any $t \in [0, u]$, and
(3) (Y, Γ_u) is not kawamata log terminal.

Moreover

(4) (Y, Γ_t) is log smooth, for any $t \in [0, u]$, and
(5) Γ_t and $N_\sigma(K_Y + \Gamma_t)$ do not have any common components.

Let

$$s = \sup\{\, t \in [0, u] \mid K_Y + \Gamma_t \text{ is log canonical}\,\}.$$

Note that

$$\begin{aligned} N_\sigma(K_Y + \Phi_t) &= N_\sigma(K_Y + \Psi_t) \\ &= N_\sigma(\pi^*(K_X + \Delta_t)) + E_t \\ &= N_\sigma((t+1)\pi^*(K_X + \Delta)) + E_t \\ &= (t+1)N_\sigma(\pi^*(K_X + \Delta)) + E_t. \end{aligned}$$

Thus $K_Y + \Gamma_t$ is a continuous, piecewise linear function of t. Setting $\Gamma = \Gamma_s$, we may write

$$\Gamma = T + C + B',$$

where $\llcorner \Gamma \lrcorner = T$, C is ample and $B' \geq 0$. Possibly perturbing Γ, we may assume that T is irreducible, so that $K_Y + \Gamma$ is purely log terminal. $\square$

We will need the following consequence of Kawamata-Viehweg vanishing:

Lemma 11.3. *Let $(X, \Delta = S + A + B)$ be a $\mathbb{Q}$-factorial projective purely log terminal pair and let $m > 1$ be an integer. Suppose that*

(1) $S = \llcorner \Delta \lrcorner$ *is irreducible,*
(2) $m(K_X + \Delta)$ *is integral,*
(3) $m(K_X + \Delta)$ *is Cartier in a neighbourhood of S,*
(4) $h^0(S, \mathcal{O}_S(m(K_X + \Delta))) > 0$,
(5) $K_X + G + B$ *is kawamata log terminal, where $G \geq 0$,*
(6) $A \sim_{\mathbb{Q}} (m-1)tH + G$ *for some t,*
(7) $K_X + \Delta + tH$ *is big and nef.*

Then $h^0(X, \mathcal{O}_X(m(K_X + \Delta))) > 0$.

Proof. Considering the long exact sequence associated to the restriction exact sequence,

$$0 \longrightarrow \mathcal{O}_X(m(K_X+\Delta)-S) \longrightarrow \mathcal{O}_X(m(K_X+\Delta)) \longrightarrow \mathcal{O}_S(m(K_X+\Delta)) \longrightarrow 0,$$

it suffices to observe that

$$H^1(X, \mathcal{O}_X(m(K_X + \Delta) - S)) = 0,$$

by Kawamata-Viehweg vanishing, since

$$\begin{aligned} m(K_X+\Delta)-S &= (m-1)(K_X+\Delta)+K_X+A+B \\ &\sim_{\mathbb{Q}} K_X+G+B+(m-1)(K_X+\Delta+tH) \end{aligned}$$

and $K_X+\Delta+tH$ is big and nef. □

Lemma 11.4. *Theorem 7.23$_{n-1}$, Theorem 7.24$_{n-1}$, Theorem 5.12$_n$ and Theorem 7.22$_n$, imply Theorem 7.23$_n$.*

Proof. By (9.9), it suffices to prove this result for the generic fibre of U. Thus we may assume that U is a point, so that X is a projective variety.

By (9.10) we may assume that (X,Δ) is log smooth. By (8.6) we may assume that $\Delta = A+B$, where $A \geq 0$ is a general ample $\mathbb{Q}$-divisor and $B \geq 0$. By (11.1) and (11.2), we may therefore assume that $\Delta = S+A+B$, where (X,Δ) is a log smooth purely log terminal pair, A is a general ample $\mathbb{Q}$-divisor, $B \geq 0$ and $\llcorner \Delta \lrcorner = S$ is irreducible and not a component of $N_\sigma(K_X+\Delta)$.

Let H be the ample divisor on X and $\alpha > 0$ be the constant whose existence is guaranteed by (10.6). Possibly replacing A by an $\mathbb{R}$-linearly equivalent divisor, we may assume that there is a positive constant ϵ such that $A-\epsilon H \geq 0$. Let V_0 be the vector space of Weil divisors spanned by the components of B and let V be the translate by $S+A$ of the span of V_0 and H.

Given $t>0$ and any $B' \in V_0$, $\|B'-B\| < \alpha t$, let

$$\Psi = S+A+B'+tH.$$

Let $\phi\colon X \dashrightarrow Y$ be the log terminal model of $K_X+\Psi$, whose existence is guaranteed by (10.6). Let T be the strict transform of S, let $\Gamma = \phi_*\Psi$ and define Θ by adjunction

$$(K_Y+\Gamma)|_T = K_T+\Theta.$$

By linearity we may formally extend the assignment $\Psi \longrightarrow \Theta$ to a rational affine linear map

$$L\colon V \longrightarrow W,$$

to the whole of V, where W is an appropriate finite dimensional rational affine space of Weil divisors on T. In particular, $L(\Delta)$ is big and by (10.6) it follows that $K_T+L(\Delta)$ is nef.

Now by taking $A' = \phi_*A|_T$, there is a rational polytope $\mathcal{C}_T \subset \mathcal{N}_{A'}$ containing $L(\Delta)$ such that $K_T+\Theta$ is kawamata log terminal for any $\Theta \in \mathcal{C}_T$, by (8.5) and (**??**). Moreover, we can find a rational polytope $\mathcal{C} \subset \mathcal{L}_A$ containing Δ such that $L(\mathcal{C}) = \mathcal{C}_T$.

We may find a positive integer k such that if $r(K_Y+\Gamma)$ is integral then $rk(K_Y+\Gamma)$ is Cartier in a neighbourhood of T. By Kollár's effective base point free theorem, [8], we may find a positive integer M' such that if D is a nef Cartier divisor on T such that $D-(K_T+\Omega)$ is nef and big, where $K_T+\Omega$ is kawamata log terminal, then $M'D$ is base point free, where M' does not depend on either D or on Ω. Set $M=kM'$. Suppose that $\Theta \in \mathcal{C}_T$. Then $\Theta \geq A'$, so that we may write

$$K_T+\Theta \sim_{\mathbb{R}} K_T+G+\Theta',$$

where G is ample and $K_T+G+\Theta'$ is kawamata log terminal. Now if $rk(K_Y+\Gamma)|_T = rk(K_T+\Theta)$, then

$$rk(K_T+\Theta)-(K_T+\Theta') \sim_{\mathbb{R}} (rk-1)(K_T+\Theta)+G.$$

Thus, if $r(K_Y+\Gamma)$ is integral, then $Mr(K_T+\Theta)$ is base point free.

By (**??**), there are real numbers $r_i>0$ with $\sum r_i = 1$, positive integers $p_i>0$ and $\mathbb{Q}$-divisors $\Delta_i \in \mathcal{C}$ such that

$$p_i(K_X+\Delta_i),$$

is integral,

$$K_X+\Delta = \sum r_i(K_X+\Delta_i),$$

and

$$\|\Delta-\Delta_i\| \leq \frac{\alpha\epsilon}{m\,i},$$

where $m_i = Mp_i$. Let $\Theta_i = L(\Delta_i)$.

By our choice of k, $p_ik(K_T+\Theta_i)$ is Cartier. So, $m_i(K_T+\Theta_i)$ is base point free and so

$$h^0(T,\mathcal{O}_T(m_i(K_T+\Theta_i)))>0.$$

(11.3) implies that $h^0(Y,\mathcal{O}_Y(m_i(K_Y+\Gamma_i)))>0$, where $\Gamma_i=\phi_*\Delta_i$. Notice that the pair $(Y,\Gamma_i = T+\phi_*A+\phi_*B)$ clearly satisfies conditions (1-4) of (11.3). We then let $t=\epsilon/m_i$ so that $\phi_*A \geq (m_i-1)t\phi_*H$ and condition (6) of (11.3) holds. Conditions (5) and (7) of (11.3) are now easy to check.

As ϕ is $(K_X+\Delta_i+tH)$-negative, it is certainly $(K_X+\Delta_i)$-negative. But then

$$h^0(X,\mathcal{O}_X(m_i(K_X+\Delta_i))) = h^0(Y,\mathcal{O}_Y(m_i(K_Y+\Gamma_i)))>0.$$

In particular there is an $\mathbb{R}$-divisor D such that

$$K_X+\Delta = \sum r_i(K_X+\Delta_i) \sim_{\mathbb{R}} D \geq 0. \qquad \square$$

11.1. **Lemmas.** We will need some definitions and results from [13].

Definition-Lemma 11.5. *Let X be a smooth projective variety, B be a big $\mathbb{R}$-divisor and let C be a prime divisor. Let*

$$\sigma_C(B) = \inf\{\, \operatorname{mult}_C(B') \,|\, B' \sim_{\mathbb{Q}} B, B' \geq 0 \,\}.$$

Then σ_C is a continuous function on the cone of big divisors.

Now let D be any pseudo-effective $\mathbb{R}$-divisor and let A be any ample $\mathbb{Q}$-divisor. Let

$$\sigma_C(D) = \lim_{\epsilon \to 0} \sigma_C(D + \epsilon A).$$

Then $\sigma_C(D)$ exists and is independent of the choice of A.

There are only finitely many prime divisors C such that $\sigma_C(D) > 0$ and the $\mathbb{R}$-divisor $N_\sigma(D) = \sum_C \sigma_C(D) C$ is determined by the numerical equivalence class of D.

Proof. See §III.1 of [13]. □

Proposition 11.6. *Let X be a smooth projective variety and let D be a pseudo-effective $\mathbb{R}$-divisor. Let B be any big $\mathbb{R}$-divisor.*

If D is not numerically equivalent to $N_\sigma(D)$, then there is a positive integer k and a positive rational number β such that

$$h^0(X, \mathcal{O}_X(\llcorner mD \lrcorner + \llcorner kB \lrcorner)) > \beta m, \qquad \textit{for all} \qquad m \gg 0.$$

Proof. Let A be any integral divisor. Then we may find a positive integer k such that

$$h^0(X, \mathcal{O}_X(\llcorner kB \lrcorner - A)) \geq 0.$$

Thus it suffices to exhibit an ample divisor A and a positive rational number β such that

$$h^0(X, \mathcal{O}_X(\llcorner mD \lrcorner + A)) > \beta m \qquad \text{for all} \qquad m \gg 0.$$

Replacing D by $D - N_\sigma(D)$, we may assume that $N_\sigma(D) = 0$. Now apply (V.1.12) of [13]. □

Lemma 11.7. *Let $\pi \colon X \longrightarrow U$ be a projective morphism of normal quasi-projective varieties. Let $\phi \colon X \dashrightarrow Y$ be a birational map over U such that ϕ^{-1} does not contract any divisors. Suppose that $K_X + \Delta' \sim_{\mathbb{R},U} \mu(K_X + \Delta)$ for some $\mu > 0$, where both $K_X + \Delta$ and $K_X + \Delta'$ are log canonical (resp. Y is $\mathbb{Q}$-factorial, $K_X + \Delta \equiv_U K_X + \Delta'$ and both $K_X + \Delta$ and $K_X + \Delta'$ are divisorially log terminal). Set $\Gamma = \phi_* \Delta$ and $\Gamma' = \phi_* \Delta'$.*

Then Y is a weak log canonical model (respectively a log terminal model) for $K_X + \Delta$ over U if and only if it is a weak log canonical model (resp. a log terminal model) for $K_X + \Delta'$ over U.

Proof. Note first that either $K_Y+\Gamma' \sim_{\mathbb{R},U} \mu(K_Y+\Gamma)$ or Y is $\mathbb{Q}$-factorial. In particular $K_Y+\Gamma$ is $\mathbb{R}$-Cartier if and only if $K_Y+\Gamma'$ is $\mathbb{R}$-Cartier. If $p\colon W \longrightarrow X$ and $q\colon W \longrightarrow Y$ resolve the indeterminancy of $\phi = q\circ p^{-1}$, then we may write

$$p^*(K_X+\Delta) = q^*(K_Y+\Gamma)+E \quad \text{and} \quad p^*(K_X+\Delta') = q^*(K_Y+\Gamma')+E'.$$

Since $\mu E - E' \equiv_Y 0$ is q-exceptional, $\mu E = E'$ by (2.7). Therefore, ϕ is $(K_X+\Delta)$-non-positive (respectively $(K_X+\Delta)$-negative) if and only if ϕ is $(K_X+\Delta')$-non-positive (respectively $(K_X+\Delta')$-negative).

Finally, one sees that since $K_X+\Delta' \equiv_U \mu(K_X+\Delta)$ it follows that $K_Y+\Gamma' \equiv_U \mu(K_Y+\Gamma)$, so that $K_Y+\Gamma$ is nef over U if and only if $K_Y+\Gamma'$ is nef over U. □

12. Finiteness of models, the general case

Lemma 12.1. *Theorem 5.12$_n$ and Theorem 7.23$_n$ imply Theorem 7.24$_n$.*

Proof. By (10.3), we can find finitely many rational maps $\phi_i\colon X \dashrightarrow Y_i$ over U, $1 \leq i \leq k$, such that if $\phi\colon X \dashrightarrow Y$ is a $\mathbb{Q}$-factorial weak log canonical model of $K_X+\Delta$, where $\Delta \in \mathcal{C}$ then there is an index $1 \leq i \leq k$ and an isomorphism $\eta\colon Y_i \longrightarrow Y$ such that $\phi = \eta\circ\phi_i$. By (**??**) for each index $1 \leq i \leq k$ there are finitely many contraction morphisms $f_{ij}\colon Y_i \longrightarrow Z_{ij}$ over U such that if $\Delta \in \mathcal{W}_{A,\pi}(V)$ and there is a contraction morphism $f\colon Y \longrightarrow Z$ over U, with

$$K_Y+\Gamma = K_Y+\phi_*\Delta = f^*D,$$

for some divisor D on Z, then there is an isomorphism $\xi\colon Z_{ij} \longrightarrow Z$ such that $f = \xi\circ f_{ij}$.

Pick $\Delta \in \mathcal{L}_A(V)$ and let $\psi\colon X \dashrightarrow W$ be any weak log canonical model of $K_X+\Delta$ over U. By (8.6) we may find a kawamata log terminal pair (X,Δ') such that $K_X+\Delta' \sim_{\mathbb{R},U} K_X+\Delta$. But then

$$K_W+\Psi = K_W+\psi_*\Delta' \sim_{\mathbb{R},U} K_W+\psi_*\Delta,$$

and $K_W+\Psi$ is kawamata log terminal. As we are assuming Theorem 5.12$_n$ and Theorem 7.23$_n$, there is a log terminal model $g\colon W \dashrightarrow Y$ of $K_W+\Psi$. But then $K_Y+\Gamma = K_Y+g_*\Psi$ is divisorially log terminal terminal, Y is $\mathbb{Q}$-factorial and the inverse of f does not contract any divisors. But the inverse f of g is the structure map. Thus $f\colon Y \longrightarrow W$ is a small morphism and

$$K_Y+\Gamma = f^*(K_W+\Phi),$$

is nef over U. If $\phi\colon X \dashrightarrow Y$ is the induced rational map then ϕ is a $\mathbb{Q}$-factorial weak log canonical model of $K_X+\Delta'$ over U. But then (11.7) implies that ϕ is also a $\mathbb{Q}$-factorial weak log canonical model of

$K_X + \Delta$ over U. Thus there is an index $1 \leq i \leq k$ and an isomorphism $\eta\colon Y_i \longrightarrow Y$ such that $\phi = \eta \circ \phi_i$. Via the isomorphism η, and an isomorphism ξ, the contraction f corresponds to one of the finitely many contractions f_{ij}. □

13. The Sarkisov Program

13.1. **Introduction.** Recall the following.

Conjecture 13.1. *Let (X, B) be a kawamata log terminal pair.*

Then we may run a $K_X + B$ minimal model program, $p : X \dashrightarrow X'$ such that either

1. *$(X', B' = p_*B)$ is a minimal model (that is $K_{X'} + B'$ is nef), or*
2. *there is a Mori fiber space $\phi : X' \to S$ (that is $\rho(X'/S) = 1$ and $-(K_{X'} + B')$ is ϕ-ample).*

A $K_X + B$ minimal model program is a finite sequence of well understood birational maps $X_i \dashrightarrow X_{i+1}$ (known as $K_X + B$ flips and divisorial contractions) inducing a rational map $p : X = X_0 \dashrightarrow X' = X_N$. If $K_X + B$ is pseudo-effective (resp. not pseudo-effective), then (X', p_*B) is a minimal model (resp. there is a Mori fiber space $X' \to S$). By [**?**], the only case in which (13.1) is not known to hold is when $K_X + B$ is pseudo-effective and neither B nor $K_X + B$ are big. It is not the case that a given pair (X, B) has a unique minimal model (resp. a unique Mori fiber space), however the minimal model program predicts that any two minimal models (resp. Mori fiber spaces) should be related in a very precise manner (the terminalizations of two minimal models are related by a sequence of flops, resp. two Mori fiber spaces are related by a sequence of Sarkisov links cf. (13.11)).

Recently Kawamata [6] has proved:

Theorem 13.2. *Let (X, B) be a $\mathbb{Q}$-factorial terminal log-pair. Suppose that (X', B') and (X'', B'') are two minimal models of (X, B).*

Then the birational map $X' \dashrightarrow X''$ may be factored as a sequence of $K_X + B$-flops.

Note that if neither B nor $K_X + B$ are big, then the existence and finiteness of minimal models for (X, B) is still conjectural.

The Sarkisov program predicts that a result similar to (13.2) should also hold in the case when $K_X + B$ is not pseudo-effective. The purpose of this paper is to show that this is indeed the case.

Theorem 13.3. *Let (X, B) be a $\mathbb{Q}$-factorial kawamata log terminal pair such that $K_X + B$ is not pseudo-effective.*

Then any two corresponding Mori fiber spaces are related by a sequence of Sarkisov links.

We say that two pairs (X', B') and (X'', B'') are birational if there exists a birational map $g : X' \dashrightarrow X''$ such that $g_*B' = B''$ and $(g^{-1})_*B'' = B'$. Note that if (X', B') and (X'', B'') are birational $\mathbb{Q}$-factorial terminal pairs, then it is easy to see that (X', B') and (X'', B'') are the output of running a minimal model program for an appropriate log pair (X, B). Consider in fact X a resolution of the indeterminacies of ϕ so that X is smooth and we have morphisms $p : X \to X'$ and $q : X \to X''$ with $\phi = q \circ p^{-1}$. Then, we may write $K_X + (p^{-1})_*B' = p^*(K_{X'} + B') + E'$ where E' is effective and its support equals $\mathrm{Ex}(p)$. Since $X' \to X$ is birational, we may run the (X, B) minimal model program over X'. It is easy to see that the output of this minimal model program is X'. Since $(p^{-1})_*B' = (q^{-1})_*B''$, a similar statement holds for (X'', B''). We therefore have the following:

Corollary 13.4. *Let $\phi' : (X', B') \to S'$ and $\phi'' : (X'', B'') \to S''$ be two Mori fiber spaces of $\mathbb{Q}$-factorial terminal pairs.*

Then (X', B') and (X'', B'') are birational if and only if they are related by a finite sequence of Sarkisov links.

In fact, the most interesting case of (13.3) is when $B' = B'' = 0$ and X' and X'' have terminal singularities.

We now recall the definition of Sarkisov links:

Links of type (I)

$$\begin{array}{ccccc} & & Z & \dashrightarrow & X_1 \\ & \swarrow & & & \\ X & & & & \downarrow \\ \downarrow & & & & \\ S & & \leftarrow & & S_1 \end{array}$$

where $Z \dashrightarrow X_1$ is a sequence of flips, $Z \to X$ is a extremal divisorial contraction, $X_1 \to S_1$ is a Mori fiber space and $\rho(S_1/S) = 1$.

Links of type (II)

$$\begin{array}{ccccccc} & & Z & \dashrightarrow & Z' & & \\ & \swarrow & & & & \searrow & \\ X & & & & & & X_1 \\ \downarrow & & & & & & \downarrow \\ S & & & \overset{\sim}{\leftarrow} & & & S_1 \end{array}$$

where $Z \dashrightarrow Z'$ is a sequence of flips, $Z \to X$ and $Z' \to X_1$ are extremal divisorial contractions and $X_1 \to S_1$ is a Mori fiber space.

Links of type (III)

$$
\begin{array}{ccccc}
X & \dashrightarrow & Z' & & \\
 & & & \searrow & \\
\downarrow & & & & X_1 \\
 & & & & \downarrow \\
S & & \rightarrow & & S_1
\end{array}
$$

where $X \dashrightarrow Z'$ is a sequence of flips, $Z' \to X_1$ is an extremal divisorial contraction, $X_1 \to S_1$ is a Mori fiber space and $\rho(S/S_1) = 1$.

Links of type (IV)

$$
\begin{array}{ccccc}
X & & \dashrightarrow & & X_1 \\
\downarrow & & & & \downarrow \\
S & & & & S_1 \\
 & \searrow & & \swarrow & \\
 & & T & &
\end{array}
$$

where $X \dashrightarrow X'$ is a sequence of flips, $X_1 \to S_1$ is a Mori fiber space and $\rho(S/T) = \rho(S_1/T) = 1$.

In order to explain the origins of this program let us recall the well known case of rational surfaces. In this case the minimal surfaces are $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{F}_n$ for $n \geq 2$. A link of type I (resp. type III) relates the Mori fiber spaces $\mathbb{P}^2 \to \mathrm{Spec}\mathbb{C}$ and $\mathbb{F}_1 \to \mathbb{P}^1$ by blowing up a point on $\mathbb{P}^2$ (resp. blowing down the -1 curve on $\mathbb{F}_1$). A link of type II relates the Mori fiber spaces $\mathbb{F}_n \to \mathbb{P}^1$ and $\mathbb{F}_{n\pm1} \to \mathbb{P}^1$ by an elementary transformation i.e. by blowing up a point on a fiber and then blowing down the strict transform of this fiber. A link of type IV is the identity on $\mathbb{P}^1 \times \mathbb{P}^1$ and interchanges the two Mori fiber space structures. The content of (13.3) is not only that $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{F}_n$ are all related by links as above, but also that any birational map $p : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ may be obtained by a sequence of such links. This last statement is equivalent to the Nöether-Castelnuovo Theorem which says that the group of birational transformations of $\mathbb{P}^2$ is generated by isomorphisms of $\mathbb{P}^2$ and a single Cremona transformation.

The proof of (13.3) is based on the original ideas of the Sarkisov program (as explained by Corti and Reid [2]). The main twist is that we are unable to show termination of an arbitrary sequence of flips, nor the acc property for log canonical thresholds. Instead our argument is based of the principle of finiteness of minimal models for kawamata log terminal pairs (Y, Γ) such that Γ varies in a compact subset of the big cone (cf. (13.6)).

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13.2. **Notation and conventions.** We work over the field of complex numbers $\mathbb{C}$. An $\mathbb{R}$-Cartier divisor D on a normal variety X is *nef* if $D\cdot C \geq 0$ for any curve $C \subset X$. We say that two $\mathbb{R}$-divisors D_1, D_2 are $\mathbb{R}$-linearly equivalent ($D_1 \sim_{\mathbb{R}} D_2$) if $D_1 - D_2 = \sum r_i(f_i)$ where $r_i \in \mathbb{R}$ and f_i are rational functions on X. We say that a $\mathbb{R}$-Weil divisor D is *big* if we may find an ample $\mathbb{R}$-divisor A and an effective $\mathbb{R}$-divisor B, such that $D \sim_{\mathbb{R}} A + B$. A divisor D is *pseudo-effective*, if for any ample divisor A and any rational number $\epsilon > 0$, the divisor $D + \epsilon A$ is big. If A is a $\mathbb{Q}$-divisor, we say that A is a *general ample divisor* if A is ample and there is a sufficiently divisible integer $M > 0$ such that MA is very ample and $MA \in |MA|$ is very general. If A is a $\mathbb{R}$-divisor, we say that A is a *general ample* $\mathbb{R}$*-divisor* if $A = \sum r_i A_i$ where $r_i \in \mathbb{R}$ and A_i are general ample $\mathbb{Q}$-divisors.

A *log pair* (X, Δ) is a normal variety X and an effective $\mathbb{R}$-Weil divisor Δ such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. We say that a log pair (X, Δ) is *log smooth*, if X is smooth and the support of Δ is a divisor with global normal crossings. A projective morphism $g\colon Y \to X$ is a *log resolution* of the pair (X, Δ) if Y is smooth and $g^{-1}(\Delta) \cup \{$ exceptional set of $g\}$ is a divisor with normal crossings support. We write $g^*(K_X+\Delta) = K_Y+\Gamma$ and $\Gamma = \sum b_i \Gamma_i$ where Γ_i are distinct reduced irreducible divisors and $g_*\Gamma = \Delta$. The discrepancy of Γ_i is $a(\Gamma_i, X, \Delta) = -b_i$.

A pair (X, Δ) is *kawamata log terminal* (klt) if $b_i < 1$ for all i. We say that the pair (X, Δ) is *log canonical* if $b_i \leq 1$ for all i. We say that the pair (X, Δ) is *terminal* if the discrepancy of any exceptional divisor is greater than zero.

Let (X, B) and (X', B') be kawamata log terminal pairs, then (X', B') is a *minimal model* of (X, B) if there is a birational map $p : X \dashrightarrow X'$ that extracts no divisors such that $B' = p_*B$ and $K_{X'} + B'$ is nef.

If $p : X \dashrightarrow X'$ is a rational map of normal varieties over a normal variety S and H is a $\mathbb{R}$-Cartier divisor on X which is the pull back of a $\mathbb{R}$-Cartier divisor on S, then we say that p is an *H-trivial rational map*.

13.3. **Preliminaries.** We begin by recalling several results from [**?**] that we will need in what follows.

Theorem 13.5. *Let (X, B) be a klt pair such that either one of the following three conditions hold: i) B is big, or ii) $K_X + B$ is big, or iii) $K_X + B$ is not pseudo-effective.*

Then, any minimal model program with scaling for $K_X + B$ exists and terminates after finitely many steps.

Proof. See [**?**]. □

Given a klt pair (X, B), we will say that a projective morphism of normal varieties $f : Y \to Z$ is a *nef model* of (X, B) if $\phi : X \dashrightarrow Y$ is a minimal model of (X, B) and f is surjective with connected fibers and $K_Y + \phi_* B = f^* H$ for some nef $\mathbb{R}$-divisor H on Z.

Theorem 13.6. *Let X be a normal projective variety and V a finite dimensional subspace of $\mathrm{Div}_{\mathbb{R}}(X)$. Let B_0 be a big $\mathbb{Q}$-divisor on X and $\mathcal{B}$ be a compact subset of V such that for any $B \in \mathcal{B}$, one has that (X, B) is klt and $B \geq B_0$.*

Then the set

$$\{f : Y \to Z | f \text{ is a nef model of } (X, B),\ B \in \mathcal{B}\}$$

is finite.

Proof. See [**?**]. □

Corollary 13.7. *Let (X, B) be a klt pair and $\mathcal{E}$ be any set of exceptional divisors such that $\mathcal{E}$ contains only exceptional divisors E of discrepancy $a(E, X, B) \leq 0$.*

Then there exists a birational morphism $\mu : X' \to X$ and a $\mathbb{Q}$-divisor B' on X' such that:

1. *(X', B') is a $\mathbb{Q}$-factorial klt pair,*
2. *E is an exceptional divisor for μ if and only if $E \in \mathcal{E}$,*
3. *$B' = \sum_{E \subset X} -a(E; X, B)E$ so that $\mu_* B' = B$ and $K_{X'} + B' = \mu^*(K_X + B)$.*

Proof. See [**?**]. □

If $\mathcal{E} = \{E | a(E, X, B) \leq 0\}$, we say that X' is a *terminalization* of X. If $\mathcal{E}$ contains a unique divisor say E, then we say that $\mu : X' \to X$ is a *divisorial extraction* of E.

Lemma 13.8. *Let (W, B_W) be a terminal pair, (X, B_X) a log pair and $f : W \dashrightarrow X$ a birational map that extracts no divisors. If $K_X + B_X$ is nef and $a(E, X, B_X) \geq a(E, W, B_W)$ for all divisors $E \subset W$ then $a(E, X, B_X) \geq a(E, W, B_W)$ for all divisors E over W. In particular*

1. *(X, B_X) is kawamata log terminal,*
2. *if $X' \to X$ is a divisorial extraction of a divisor E', with $a(E', X, B_X) \leq 0$ then E' is a divisor contained in W, and*
3. *if $X' \to X$ is a terminalization of (X, B_X), then $f' : W \dashrightarrow X'$ extracts no divisors.*

Proof. Let Z be a common log resolution and $p : Z \to X$ and $q : Z \to W$ be the induced morphisms. Then we may write

$$q^*(K_W + B_W) = p^*(K_X + B_X) + F$$

where

$$q_*F = \sum_{E \subset W} (a(E, X, B_X) - a(E, W, B_W))E \geq 0.$$

By the Negativity Lemma, it follows that $F \geq 0$ and hence that $a(E, X, B_X) \geq a(E, W, B_W)$ for all divisors E over W. As $a(E, W, B_W) > -1$ for all divisors E over W, it follows immediately that (X, B_X) is kawamata log terminal. To see the second assertion, it suffices to notice that as (W, B_W) is terminal and $a(E', W, B_W) \leq a(E', X, B_X) \leq 0$, then E' is not exceptional over W. Similarly, a terminalization only extracts divisors E'_i of discrepancy $a(E'_i, X, B_X) \leq 0$. □

Definition 13.9. *Let (X, B_X) and (W, B_W) two $\mathbb{Q}$-factorial log pairs and $f : W \dashrightarrow X$ a birational map. We will write $(W, B_W) \geq (X, B_X)$ if:*

(1) *f extracts no divisors,*
(2) *$a(E, W, B_W) \leq a(E, X, B_X)$ for all divisors $E \subset W$.*

Definition 13.10. *Let (X, B_X) and $(X', B_{X'})$ be kawamata log terminal pairs. Then (X, B_X) and $(X', B_{X'})$ are Sarkisov related if there exists a kawamata log terminal pair (W, B_W) such that (X, B_X) and $(X', B_{X'})$ are the output of a (W, B_W)-MMP.*

Notice that if (X, B_X) and $(X', B_{X'})$ are Sarkisov related, then we may find a log smooth terminal pair $(W', B_{W'})$ and morphisms $p : W' \to X$ and $q : W' \to X'$ such that (X, B_X) and $(X', B_{X'})$ are the output of a $(W', B_{W'})$-MMP.

13.4. **The main result.**

Theorem 13.11. *Let*

$$\begin{array}{ccc} X & \overset{\Phi}{\dashrightarrow} & X' \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\phi'} \\ S & & S' \end{array}$$

be a birational map of Sarkisov related kawamata log terminal pairs (X, B_X) and $(X', B_{X'})$ where ϕ is a $K_X + B_X$-Mori fiber space and ϕ' is a $K_{X'} + B_{X'}$-Mori fiber space.

Then Φ is given by a finite sequence of links of type I, II, III or IV.

Proof. Let $H' \sim_{\mathbb{R}} -(K_{X'} + B_{X'}) + (\phi')^* A'$, where A' is an ample $\mathbb{R}$-divisor on S' which is very general in $\mathrm{NS}_{\mathbb{R}}(S')$ and H' is a general ample $\mathbb{R}$-divisor. Therefore, $K_{X'} + B_{X'} + H' \sim_{\mathbb{R}} (\phi')^* A'$ is nef and $(X', B_{X'} + H')$ is kawamata log terminal. Similarly, let C be a general ample $\mathbb{R}$-divisor on X such that $(X, B_X + C)$ is kawamata log terminal and $K_X + B_X + C \sim_{\mathbb{R}} \phi^* A$ where A is an ample $\mathbb{R}$-divisor on S which is very general in $\mathrm{NS}_{\mathbb{R}}(S)$. We may assume that (W, B_W) is log smooth, $p : W \to X$ and $q : W \to X'$ are morphisms and $B_W + p^*C + q^*H' + \mathrm{Ex}(p) + \mathrm{Ex}(q)$ has simple normal crossings. We let $C_W = (p^{-1})_* C = p^*C$ and $H_W = (q^{-1})_* H = q^* H'$, then we may assume that $(W, B_W + cC_W + hH_W)$ is terminal for any $0 \leq c, h \leq 2$.

Claim 13.12. *There exists an integer $N > 0$ and a finite sequence of links of type I, II, III and IV*

$$X = X_0 \dashrightarrow X_1 \dashrightarrow \ldots \dashrightarrow X_N$$

such that if $\phi_i : X_i \to S_i$ are the corresponding contraction morphisms, $p_i : W \dashrightarrow X_i$ the corresponding rational maps and $C_i = (p_i)_* C_W$, $H_i = (p_i)_* H_W$ and $B_{X_i} = (p_i)_* B_W$, then

(1) *there exist positive rational numbers*

$$1 = c_0 \geq c_1 \geq c_2 \geq \ldots \geq c_N = 0 \text{ and}$$

$$0 = h_0 \leq h_1 \leq h_2 \leq \ldots \leq h_N \leq 1$$

such that $K_{X_i} + B_{X_i} + c_i C_i + h_i H_i$ is nef,

(2) $(X_i, B_{X_i} + c_i C_i + h_i H_i) \leq (W, B_W + c_i C_W + h_i H_W)$ *(in particular $p_i : W \dashrightarrow X_i$ extracts no divisors cf.* (13.9)*), and*

(3) *each link is given by a sequence of $K_{X_i} + B_{X_i} + c_i C_i + h_i H_i$ trivial rational maps (in particular $K_{X_i} + B_{X_i} + c_i C_i + h_i H_i$ is ϕ_i-numerically trivial).*

Remark 13.13. *We have that:*

(1) *By* (13.8)*, $(X_i, B_{X_i} + c_i C_i + h_i H_i)$ is kawamata log terminal.*

(2) *Let Σ_i be a very general ϕ_i-vertical curve. By our choices of H and C, we have that $H_i \cdot \Sigma_i > 0$ and $C_i \cdot \Sigma_i > 0$ for all $i \geq 0$.*

Proof of 13.12. We let Σ_i be a very general ϕ_i-vertical curve and we set $r_i = H_i \cdot \Sigma_i / C_i \cdot \Sigma_i$. Let s_{i+1} be the supremum of all numbers $0 \leq \sigma \leq c_i / r_i$ such that:

(1) $K_{X_i} + B_{X_i} + c_i C_i + h_i H_i + \sigma(H_i - r_i C_i)$ is nef, and

(2) $(X_i, B_{X_i} + c_i C_i + h_i H_i + \sigma(H_i - r_i C_i)) \leq (W, B_W + c_i C_W + h_i H_W + \sigma(H_W - r_i C_W))$.

We will define X_j by induction on i. Assume that X_i has already been defined for some $i \geq 0$.

If $s_{i+1} = c_i/r_i$, then we let $N = i+1$ so that $c_N = c_i - r_i s_{i+1} = 0$ and we are done.

If $s_{i+1} < c_i/r_i$, we let $c_{i+1} = c_i - r_i s_{i+1} > 0$ and $h_{i+1} = h_i + s_{i+1} \geq h_i$. Notice that $K_{X_i} + B_{X_i} + c_{i+1}C_i + h_{i+1}H_i$ is nef and kawamata log terminal. Let F be the extremal face defined by $K_{X_i} + B_{X_i} + c_{i+1}C_i + h_{i+1}H_i$. Clearly $R_i \subset F$ where R_i is the ray spanned by Σ_i.

Suppose that there is an X-exceptional divisor $E \subset W$ such that

$$a(E, X_i, B_{X_i} + c'_{i+1}C_i + h'_{i+1}H_i) < a(E, W, B_W + c'_{i+1}C_W + h'_{i+1}H_W)$$

for $c'_{i+1} = c_{i+1} - \epsilon r_i$, $h'_{i+1} = h_{i+1} + \epsilon$ and $0 < \epsilon \ll 1$. It follows that

$$a(E, X_i, B_{X_i} + c_{i+1}C_i + h_{i+1}H_i) = a(E, W, B_W + c_{i+1}C_W + h_{i+1}H_W) \leq 0$$

and so by (13.7), there exists a divisorial extraction $\mu : Z \to X_i$ which extracts E. By (13.8), $\varphi : W \dashrightarrow Z$ extracts no divisors. We let

$$K_Z + B_Z + c_{i+1}C_Z + h_{i+1}H_Z = \mu^*(K_{X_i} + B_{X_i} + c_{i+1}C_i + h_{i+1}H_i).$$

Notice that $B_Z = \varphi_* B_W$. We now run a minimal model program with scaling over S_i for

$$K_Z + \Delta_Z = \mu^*(K_{X_i} + B_{X_i} + (c'_{i+1} - \delta)C_i + h'_{i+1}H_i)$$

for some $0 < \delta \ll \epsilon \ll 1$. Note that $K_Z + \Delta_Z$ is a kawamata log terminal pair numerically trivial over X_i and $K_{X_i} + B_{X_i} + (c'_{i+1} - \delta)C_i + h'_{i+1}H_i$ is negative on Σ_i. Each step of this minimal model program is over S_i, so it is $\mu^*(K_{X_i} + B_i + c'_{i+1}C_i + h'_{i+1}H_i)$ numerically trivial and hence C_i-positive. In particular this is a minimal model program for $\mu^*(K_{X_i} + B_{X_i} + (c'_{i+1} - \delta')C_i + h'_{i+1}H_i)$ for any $0 < \delta' \leq \delta$.

Since Z is covered by $K_Z + \Delta_Z$-negative curves (over S_i), it follows that $K_Z + \Delta_Z$ is not pseudo-effective (over S_i). Therefore, after finitely many flips, we either have a $K_Z + \Delta_Z$ Mori fiber space or a $K_Z + \Delta_Z$ divisorial contraction.

Case 1. In the first case, we have a sequence of flips $\eta : Z \dashrightarrow X_{i+1}$ followed by a Mori fiber space $\phi_{i+1} : X_{i+1} \to S_{i+1}$. This is a link of type I.

Case 2. In the second case, we have a sequence of flips $\eta : Z \dashrightarrow Z'$ followed by a divisorial contraction $\nu : Z' \to X_{i+1}$. Since $\rho(X_{i+1}/S_i) = 1$, one sees that there is an induced contraction morphism $\phi_{i+1} : X_{i+1} \to S_{i+1} := S_i$ which is a $K_Z + \Delta_Z$ Mori fiber space (as $K_Z + \Delta_Z$ is not pseudo-effective over S_i). We have obtained a link of type II.

Suppose on the other hand that

$$a(E, X_i, B_{X_i} + c'_{i+1}C_i + h'_{i+1}H_i) \geq a(E, W, B_W + c'_{i+1}C_W + h'_{i+1}H_W)$$

for any X-exceptional divisor $E \subset W$ and for any $0 < \epsilon \ll 1$. In this case $F \neq R_i$, and we may find an extremal ray $P \subset F$ such that P and R_i span a two dimensional face of F. Let $X_i \to T$ be the corresponding contraction which factors through S_i. We now run a minimal model program over T for $K_{X_i} + B_{X_i} + c'_{i+1}C_i + h'_{i+1}H_i$ where as above $c'_{i+1} = c_{i+1} - \epsilon r_i$ and $h'_{i+1} = h_{i+1} + \epsilon$ for some $0 < \epsilon \ll 1$. Notice that R_i is $K_{X_i} + B_{X_i} + c'_{i+1}C_i + h'_{i+1}H_i$-trivial and P is $K_{X_i} + B_{X_i} + c'_{i+1}C_i + h'_{i+1}H_i$-negative. After finitely many flips, we either have a $K_{X_i} + B_{X_i} + c'_{i+1}C_i + h'_{i+1}H_i$ minimal model, divisorial contraction or Mori fiber space (over T).

Case 3. In the first case, we let $X_i \dashrightarrow X_{i+1}$ be the corresponding sequence of flips. We have that $K_{X_{i+1}} + B_{X_{i+1}} + c'_{i+1}C_{i+1} + h'_{i+1}H_{i+1}$ is nef over T and there is a unique $K_{X_{i+1}} + B_{X_{i+1}} + c'_{i+1}C_{i+1} + h'_{i+1}H_{i+1}$ trivial extremal ray which is spanned by Σ_{i+1} the pushforward of Σ_i. Let $\phi_{i+1} : X_{i+1} \to S_{i+1}$ be the corresponding fibration. We have obtained a link of type IV.

Case 4. In the second case, we let $X_i \dashrightarrow Z'$ be the corresponding sequence of flips and let $Z' \to X_{i+1}$ be the corresponding divisorial contraction. Since $\rho(X_{i+1}/T) = 1$, one sees that there is an induced contraction morphism $\phi_{i+1} : X_{i+1} \to S_{i+1} := T$. We have obtained a link of type III.

Case 5. In the third case we let $X_i \dashrightarrow X_{i+1}$ be the corresponding sequence of flips and $\phi_{i+1} : X_{i+1} \to S_{i+1}$ be the corresponding Mori fiber space over T. We have obtained a link of type IV.

Lemma 13.14. *We have that*

$$(X_{i+1}, B_{X_{i+1}} + c_{i+1}C_{i+1} + h_{i+1}H_{i+1}) \leq (W, B_W + c_{i+1}C_W + h_{i+1}H_W).$$

Proof. The rational map $X_{i+1} \dashrightarrow X_i$ extracts a divisor E in Cases 1 and 2. Since $E \subset W$, one sees that $W \dashrightarrow X_{i+1}$ extracts no divisors. By definition of c_{i+1} and h_{i+1}, we have that

$$(X_i, B_{X_i} + c_{i+1}C_i + h_{i+1}H_i) \leq (W, B_W + c_{i+1}C_W + h_{i+1}H_W).$$

Since in Cases 1 and 2 (resp. Cases 3, 4 and 5) the rational map $X_i \dashrightarrow X_{i+1}$ is over S_i (resp. over T) and $K_{X_i} + B_{X_i} + c_{i+1}C_i + h_{i+1}H_i$ is numerically trivial over S_i (resp. over T), it follows that $a(E, X_i, B_{X_i} + c_{i+1}C_i + h_{i+1}H_i) = a(E, X_{i+1}, B_{X_{i+1}} + c_{i+1}C_{i+1} + h_{i+1}H_{i+1})$ for any divisor E and so the inequality

$$a(E, X_{i+1}, B_{X_{i+1}} + c_{i+1}C_{i+1} + h_{i+1}H_{i+1}) \geq a(E, W, B_W + c_{i+1}C_W + h_{i+1}H_W)$$

for all divisors $E \subset W$ also holds. □

Lemma 13.15. *We have that $h_i \leq 1$ for all i and if $h_i = 1$, then $c_i = 0$ and $X_i \dashrightarrow X'$ induces a rational map $S_i \dashrightarrow S'$.*

Proof. We will proceed by induction. Since $h_0 = 0$, it suffices to prove that if $h_i \leq 1$ then $h_{i+1} \leq 1$. Let $\nu : W' \to W$ be a proper birational morphism such that $p'_i = p_i \circ \nu$ and $q' = q \circ \nu$ are appropriate log resolutions. We may write $K_{W'} + B_{W'} = \nu^*(K_W + B_W) + E_W$ where $E_W \geq 0$ is exceptional and $B_{W'} = (\nu^{-1})_* B_W$. We let $C_{W'} = (\nu^{-1})_* C_W$ and $H_{W'} = (\nu^{-1})_* H_W$ Assume that $h_i \leq 1$ and $h_{i+1} > 1$. Then there is a number $1 < h < \min\{h_{i+1}, 2\}$ and we let $c = c_i - r_i(h - h_i) > c_{i+1}$. We have

$$K_{W'} + B_{W'} + cC_{W'} + hH_{W'} = (p'_i)^*(K_{X_i} + B_{X_i} + cC_i + hH_i) + E,$$

$$K_{W'} + B_{W'} + hH_{W'} = (q')^*(K_{X'} + B_{X'} + hH') + E'.$$

Note that since

$$(W', B_{W'} + cC_{W'} + hH_{W'}) \geq (W, B_W + cC_W + hH_W) \geq (X_i, B_{X_i} + cC_i + hH_i),$$

by (13.8), E is effective. Since H' is very general, E' is also effective. Since Σ_i is a general ϕ_i-trivial curve, we may identify Σ_i with its inverse image in W'. We let $q'_*\Sigma_i$ be its image in X'. We have

$$0 = (K_{X_i} + B_{X_i} + cC_i + hH_i) \cdot \Sigma_i = (K_{W'} + B_{W'} + cC_{W'} + hH_{W'}) \cdot \Sigma_i$$
$$\geq (K_{X'} + B_{X'} + hH') \cdot q'_*\Sigma_i > 0.$$

This is a contradiction and so we may assume that $h_{i+1} \leq 1$. If we have $h_{i+1} = 1$, then one sees that $c_{i+1} = 0$ and $q'_*\Sigma_i$ is ϕ' vertical so that $X_i \dashrightarrow X'$ induces a rational map $S_i \dashrightarrow S'$. □

Note that $K_{X_i} + B_{X_i} + c_{i+1}C_i + h_{i+1}H_i$ is pulled back from S_i in cases 1 and 2 and is pulled back from T in cases 3, 4 and 5. By (13.14) and (13.15), we have that the link given by $X_i \dashrightarrow X_{i+1}$, satisfies all of the conditions of (13.12). To prove Claim 13.12, it suffices to show that there exists an integer $N > 0$ such that $c_N = 0$. To this end we prove the following lemmas which will allow us to show that there is no infinite sequence of links.

Lemma 13.16. *For links of type III and IV, we have that the inequality*

$$a(E, X_{i+1}, B_{X_{i+1}} + c'_{i+1}C_{i+1} + h'_{i+1}H_{i+1}) \geq a(E, W, B_W + c'_{i+1}C_W + h'_{i+1}H_W)$$

for all divisors $E \subset W$ for any $0 < \epsilon \ll 1$ also holds.

Proof. This follows from the fact that in cases 3, 4 and 5 we have the inequality

$$a(E, X_i, B_{X_i} + c'_{i+1}C_i + h'_{i+1}H_i) \geq a(E, W, B_W + c'_{i+1}C_W + h'_{i+1}H_W)$$

and the fact that we are running a $K_{X_i} + B_{X_i} + c'_{i+1}C_i + h'_{i+1}H_i$ minimal model program (over T). □

Lemma 13.17. *We have that $r_{i+1} \geq r_i$. Moreover, in Case 5, we have that $r_{i+1} > r_i$.*

Proof. In Cases 3, 4 and 5, we let $\mu : Z \to X_i$ be the identity and $E = 0$. Since $Z \dashrightarrow X_{i+1}$ is an isomorphism over a big open set of X_{i+1}, we may identify Σ_{i+1} and its preimage Σ_{i+1} in Z. We claim that

$$\Sigma_{i+1} \cdot \mu^*(K_{X_i} + B_{X_i} + c'_{i+1}C_i + h'_{i+1}H_i) \leq 0$$

and the above inequality is strict in Case 5. In Cases 1 and 2, this follows as $Z \dashrightarrow X_{i+1}$ is an isomorphism in a neighborhood of Σ_{i+1} and

$$\Sigma_{i+1} \cdot \mu^*(K_{X_i} + B_{X_i} + (c'_{i+1} - \delta)C_i + h'_{i+1}H_i)0$$
$$\Sigma_{i+1} \cdot (K_{i+1} + B_{X_{i+1}} + (c'_{i+1} - \delta)C_{i+1} + h'_{i+1}H_{i+1}) < 0$$

for $0 \ll \delta \ll \epsilon \ll 1$. In Cases 3 and 4, this is clear as $-(K_{X_i} + B_{X_i} + c'_{i+1}C_i + h'_{i+1}H_i)$ is nef over S_i and $X_i \dashrightarrow X_{i+1}$ is an isomorphism in a neighborhood of Σ_{i+1}. In Case 5, we moreover obtain a strict inequality as $-(K_{X_{i+1}} + B_{X_{i+1}} + c'_{i+1}C_{i+1} + h'_{i+1}H_{i+1})$ is ample over S_{i+1}

Since

$$\Sigma_{i+1} \cdot \mu^*(K_{X_i} + B_{X_i} + c_{i+1}C_i + h_{i+1}H_i) = 0,$$

we have that $\Sigma_{i+1} \cdot \mu^*(H_i - r_iC_i) \leq 0$. Since $\mu^*(H_i - r_iC_i) = (\mu^{-1})_*(H_i - r_iC_i) + eE$ for some $e \geq 0$, we have that

$$\Sigma_{i+1} \cdot (H_{i+1} - r_iC_{i+1}) = \Sigma_{i+1} \cdot (\mu^{-1})_*(H_i - r_iC_i) = \Sigma_{i+1} \cdot (\mu^*(H_i - r_iC_i) - eE) \leq 0,$$

or equivalently that

$$r_{i+1} = \frac{\Sigma_{i+1} \cdot H_{i+1}}{\Sigma_{i+1} \cdot C_{i+1}} \leq r_i.$$

Moreover, in Case 5, the above inequality is strict. □

Lemma 13.18. *There are only finitely many possibilities for $\phi_i : X \to S_i$.*

Proof. Notice that the pairs $(X_i, B_{X_i} + c_iC_i + h_iH_i)$ are minimal models for $(W, B_W + c_iC_W + h_iH_W)$ where $0 \leq c_i, h_i \leq 1$. Suppose that $h_i = 0$ for all i, then $s_i = h_{i+1} - h_i = 0$ for all i and so $c_i = c_0 > 0$ for all i and by (13.6), the claim follows.

If $h_i > 0$ for some i, then $h_j \geq h_i$ for all $j \geq i$ and again by (13.6) the claim follows. □

Lemma 13.19. *The sequence of links $X_i \dashrightarrow X_{i+1}$ is finite.*

Proof. Assume that the given sequence of links is infinite. By (13.18), we may assume that $\phi_i : X_i \to S_i$ is isomorphic to $\phi_k : X_k \to S_k$ for some $i < k$. It then follows from the definitions of c_j and h_j, that $c_{k+1} = c_{i+1}$ and $h_{k+1} = h_{i+1}$ and as these sequences are monotone, then $c_j = c_i$ and $h_j = h_i$ for all $j \geq i$. By (13.17), we have that $r_i = r_j$ for all $j \geq i$.

Assume that we have a link of type III or IV, then for all divisors $E \subset W$ we have

$$a(E, X_i, B_{X_i} + c'_{i+1}C_i + h'_{i+1}H_i) \geq a(E, W, B_W + c'_{i+1}C_W + h'_{i+1}H_W).$$

By (13.16), this property continues to hold on X_{i+1}. Since $c_{i+1} = c_i$ and $r_{i+1} = r_i$ it then follows that for all $j \geq i$, all links $X_j \dashrightarrow X_{j+1}$ are of type III or IV. Since links of type III (resp. IV) decrease (do not change) $\rho(X_i)$, there are only links of type IV (corresponding to Cases 3 and 5). The links of Cases 3 and 5 increase the discrepancies with respect to $K_{X_i} + B_{X_i} + c'_{i+1}C_i + h'_{i+1}H_i$ and in Case 3 strictly increase at least one discrepancy (as in this case it is easy to see that X_i and X_{i+1} are not isomorphic). Therefore, we may assume that there are no links as in Case 3. For any link of Case 5, by (13.17), we have that $r_{i+1} < r_i$ and hence a contradiction.

Therefore, we may assume that we only have links of type I and II. Since $\rho(X_{i+1}) > \rho(X_i)$ (resp. $\rho(X_{i+1}) = \rho(X_i)$) for links of type I (resp. of type II), it follows that there are no links of type I. But links of type II increase the discrepancies with respect to $K_{X_i} + B_{X_i} + (c'_{i+1} - \delta)C_i + h'_{i+1}H_i$ (for $0 < \delta \ll \epsilon \ll 1$) and strictly increase at least some discrepancy. This is the required contradiction. □

Since the sequence of links $X_i \dashrightarrow X_{i+1}$ is finite, it follows that $c_N = 0$ for some $N > 0$ and hence (13.12) is proven. □

We may therefore assume that $(X_N, B_{X_N} + h_N H_N) \leq (W, B_W + h_N H_W)$ and that $K_{X_N} + B_{X_N} + h_N H_N$ is nef and ϕ_N numerically trivial.

Claim 13.20. *$h := h_N = 1$ and $X_N \dashrightarrow X'$ is an isomorphism inducing an isomorphism $S_N \to S'$.*

Proof. Let $\nu : W' \to W$ be a proper birational morphism such that $p' = p_N \circ \nu$ and $q' = q \circ \nu$ are appropriate log resolutions. We may write $K_{W'} + B_{W'} = \nu^*(K_W + B_W) + E_W$ where $E_W \geq 0$ is exceptional and $B_{W'} = (\nu^{-1})_* B_W$. We have

$$K_{W'} + B_{W'} + hH_{W'} = (p')^*(K_{X_N} + B_{X_N} + hH_N) + E,$$

$$K_{W'} + B_{W'} + hH_{W'} = (q')^*(K_{X'} + B_{X'} + hH') + E'$$

where E is p' exceptional and E' is q'-exceptional. Note that since $(W, B_W + hH_W) \geq (X, B_{X_N} + hH_{X_N})$, by (13.8), E is effective. By (13.15), we have that $h \leq 1$ and if $h = 1$, then $X \dashrightarrow X'$ induces a rational map $S_N \dashrightarrow S'$. Now let Σ' be a general curve contracted by

ϕ'. We may identify this curve with its inverse image in W' and we let $p'_*\Sigma'$ be its image in X_N. If $h < 1$, we then have

$$0 > (K_{X'} + B_{X'} + hH') \cdot \Sigma' = (K_{W'} + B_{W'} + hH_{W'}) \cdot \Sigma'$$
$$= (K_{X_N} + B_{X_N} + hH_N) \cdot q'_*\Sigma' + E \cdot \Sigma'.$$

Since $E \cdot \Sigma' \geq 0$, we have $(K_{X_N} + B_{X_N} + hH_N) \cdot q'_*\Sigma' < 0$ contradicting the fact that $K_{X_N} + B_{X_N} + hH_N$ is nef. Therefore $h = 1$. Now, let D be an ample divisor on X_N and D' be its strict transform on X'. Note that D' is relatively ample. Then $K_{X_N} + B_{X_N} + H_N + \delta D$ and $K_{X'} + B_{X'} + H' + \delta D'$ are ample and kawamata log terminal for any $0 < \delta \ll 1$ and hence X_N and X' are isomorphic (by uniqueness of log canonical models). It then follows that the rational map $S_N \dashrightarrow S' = \mathrm{Proj}(R(K_{X'} + B_{X'} + H'))$ is a morphism and in fact an isomorphism as $\rho(X_i/S_i) = \rho(X'/S') = 1$. □

□

14. Further results

14.1. **Fujita's Approximation Theorem.**

Theorem 14.1. *Let $L \in \mathrm{Div}_{\mathbb{Q}}(X)$ be a big divisor on a normal irreducible projective variety of dimension n. Then, for any $\epsilon > 0$, there exist a projective birational morphism $f : Y \to X$ and effective $\mathbb{Q}$-divisors A and B such that A is ample, $f^*L = A + B$ and*

$$\mathrm{vol}(A) \geq \mathrm{vol}(L) - \epsilon.$$

Proof. We follow [12]. After resolving the singularities of X, we may assume that X is smooth. It is enough to show that there is a nef $\mathbb{Q}$ divisor A with the above properties (recall that if A is nef and big, then $A \sim_{\mathbb{Q}} H + F$ where H is ample and F is effective, but then $A - \delta F = (1-\delta)A + \delta H$ is ample for any $0 < \delta < 1$).

Let $B \in \mathrm{Div}(X)$ be very ample such that $K_X + (n+1)B$ is very ample. For any $p \geq 0$, let $M_p = pL - (K_X + (n+1)B)$ and $\mathcal{J}_p = \mathcal{J}(||M_p||)$. One sees that $M_p \otimes \mathcal{J}_p$ is generated by global sections cf. (3.8) and that

$$H^0(\mathcal{O}_X(lM_p)) \subset H^0(\mathcal{O}_X(lpL \otimes \mathcal{J}_p^l)) \qquad \forall\ l > 0.$$

To see the last assertion note that clearly

$$H^0(\mathcal{O}_X(lM_p) \otimes \mathcal{J}(||M_p||)^l) \subset H^0(\mathcal{O}_X(lpL) \otimes \mathcal{J}(||M_p||)^l)$$

and that since by (3.22) $\mathcal{J}(||lM_p||) \subset \mathcal{J}(||M_p||)^l$, then by (3.32), we have

$$H^0(\mathcal{O}_X(lM_p)) = H^0(\mathcal{O}_X(lM_p) \otimes \mathcal{J}(||M_p||)^l).$$

Let $f : Y \to X$ be a log resolution of $\mathcal{J}_p$ so that $\mathcal{J}_p \cdot \mathcal{O}_Y = \mathcal{O}_Y(-E_p)$ where $E_p \geq 0$. Then $f^*(pL) - E_p$ is generated by global sections and hence nef. Notice that

$$H^0(\mathcal{O}_Y(l(f^*(pL) - E_p))) \supset H^0(\mathcal{O}_X(plL) \otimes \mathcal{J}_p^l) \supset H^0(\mathcal{O}_X(lM_p)).$$

Therefore,

$$(f^*(pL) - E_p)^n = \mathrm{vol}(f^*(pL) - E_p) \geq \mathrm{vol}(M_p) \geq p^n(\mathrm{vol}(L) - \epsilon)$$

and the result follows letting $A = (1/p)(f^*(pL) - E_p)$ and $B = (1/p)E_p$. □

Definition 14.2. *Let $L \in \mathrm{Div}_{\mathbb{Q}}(X)$ be a big divisor on a normal projective variety. For any $m \gg 0$ sufficiently divisible, $\phi_{|mL|}$ is birational. The moving self-intersection number $(mL)^{[n]}$ of $|mL|$ is given by*

$$(mL)^{[n]} = \sharp(D_1 \cap \ldots \cap D_n \cap (X - \mathrm{Bs}(|mL|)))$$

where $D_i \in |mL|$ are general.

Theorem 14.3. *Let $L \in \mathrm{Div}_{\mathbb{Q}}(X)$ be a big divisor on a normal projective variety. Then*

$$\mathrm{vol}(L) = \lim\sup \frac{(mL)^{[n]}}{m^n}.$$

Proof. Let $f_m : Y_m \to X$ be a log resolution of $|mL|$ so that $f_m^*|mL| = |P_m| + F_m$ where $|P_m|$ is base point free and $\mathrm{Bs}(|mL|) = (f_m)_*(F_m)$. It is easy to see that that since $|P_m|$ is base point free, we have

$$(mL)^{[n]} = \sharp(M_1 \cap \ldots \cap M_n) = P_m^n$$

where $M_i \in |P_m|$ are general. It then follows that

$$\mathrm{vol}(L) = \mathrm{vol}(mL)/m^n \geq \mathrm{vol}(P_m)/m^n \geq P_m^n = (mL)^{[n]}/m^n.$$

To see the reverse implication, consider ϵ, $f : Y \to X$ and A, B as in (14.1). Pick $k > 0$ such that kA is very ample. Replacing Y by a common resolution of Y and Y_k, we can write

$$f^*(kL) \sim_{\mathbb{Q}} A_k + E_k$$

where $A_k \sim_{\mathbb{Q}} kA$ is generated by global sections and $E_k \geq 0$ and

$$A_k^n \geq k^n(\mathrm{vol}(L) - \epsilon).$$

As $|A_k|$ is base point free, we have $E_k \geq F_k$ and so

$$A_k^n \leq P_k^n = (kL)^{[n]}.$$

Therefore

$$(kL)^{[n]}/k^n \geq \mathrm{vol}(L) - \epsilon$$

and the Theorem follows by taking the limit. □

14.2. **The Pseudo-effective Cone.** We now recall a result of Boucksom-Demailly-Paun-Peternell.

Theorem 14.4. *Let $L \in \mathrm{Div}_{\mathbb{Q}}(X)$ be a big divisor on a normal irreducible projective variety of dimension n. Let $f : Y \to X$ be a projective birational morphism, A and B effective $\mathbb{Q}$-divisors such that A is ample and $f^*L = A + B$. Let $H \in \mathrm{Div}_{\mathbb{Q}}(X)$, $H \pm L$ be ample, then there is a constant C so that*

$$(A^{n-1} \cdot B)^2 \leq C \cdot H^n \cdot (\mathrm{vol}(L) - \mathrm{vol}(A)).$$

Proof. See [12, §11]. □

Theorem 14.5. *Let X be a normal irreducible projective variety of dimension n. Then the cones $\overline{\mathrm{Mov}}(X)$ and $\overline{\mathrm{Eff}}(X)$ are dual.*

Recall that $\overline{\mathrm{Eff}}(X)$ is the closure of the big cone i.e. the cone of pseudo-effective divisors. $\overline{\mathrm{Mov}}(X) \subset N_1(X)_{\mathbb{R}}$ is the cone of movable (or mobile) curves i.e. the closed convex cone spanned by all curves of the form

$$f_*(A_1 \cdot \ldots \cdot A_{n-1})$$

where $f : Y \to X$ is a projective birational morphism and A_i are ample divisors in $\mathrm{Div}_{\mathbb{R}}(Y)$.

Notive that if $D \in \mathrm{Div}(X)$ is effective and $\gamma \in \overline{\mathrm{Mov}}(X)$, then $D \cdot \gamma \geq 0$. It follows that

$$\overline{\mathrm{Mov}}(X) \subset \overline{\mathrm{Eff}}(X)^{\vee}.$$

Corollary 14.6. *Let X be a smooth projective variety, the X is uniruled if and only if K_X is not pseudo-effective.*

Proof. By (14.5), there is a $\gamma \in \overline{\mathrm{Mov}}(X)$ such that $K_X \cdot \gamma < 0$. This implies that there is a covering family of curves C_t with $K_X \cdot C_t < 0$. But by a result of Miyaoka and Mori, this is equivalent to X being uniruled. □

Remark 14.7. *Note that* (14.6) *also follows from the fact that as K_X is not pseudo-effective, then there is an MMP $X \dashrightarrow X'$ that ends with a Mori-fiber space $g : X' \to S$. The fibers of g are known to be uniruled (and in fact rationally connected).*

Proof of (14.5). (See [12, §11].) Since $\overline{\mathrm{Mov}}(X) \subset \overline{\mathrm{Eff}}(X)^{\vee}$, it suffices to prove the reverse inclusion i.e. that $\overline{\mathrm{Mov}}(X)^{\vee} \subset \overline{\mathrm{Eff}}(X)$. Suppose by way of contradiction that there is a class ξ on the boundary of $\overline{\mathrm{Eff}}(X)$ but in the interior of $\overline{\mathrm{Mov}}(X)^{\vee}$. In particular $\mathrm{vol}(\xi) = 0$. Let

h be an ample class such that $h+\xi$ and $h-\xi$ are ample. Note that $\xi - \epsilon h \in \overline{\mathrm{Mov}}(X)^\vee$ for $0<\epsilon \ll 1$ and so

$$\frac{\xi\cdot\gamma}{h\cdot\gamma}\geq \epsilon$$

for any mobile class γ. Notice that $\xi+\delta h$ is big for any $1\gg\delta>0$ and by (14.1), we may find

$$f_\delta : Y_\delta \to X, \qquad f_\delta^*(\xi+\delta h) = A_\delta + B_\delta$$

where A_δ is ample, $B_\delta \geq 0$,

$$(\star) \qquad \mathrm{vol}(A_\delta) \geq \mathrm{vol}(\xi+\delta h) - \delta^{2n} \geq \frac{1}{2}\mathrm{vol}(\xi+\delta h) \geq \frac{\delta^n}{2}\cdot h^n.$$

The class $\gamma_\delta = (f_\delta)_*(A_\delta^{n-1})$ is movable and we have

$$(\sharp) \qquad h\cdot\gamma_\delta = f_\delta^* h\cdot A_\delta^{n-1} \geq (h^n)^{1/n}\cdot (A_\delta^n)^{(n-1)/n}$$

by the generalized Hodge inequalities. One sees that

$$\xi\cdot\gamma_\delta \leq (\xi+\delta h)\cdot\gamma_\delta = f_\delta^*(\xi+\delta h)\cdot A_\delta^{n-1} = A_\delta^n + B_\delta\cdot A_\delta^{n-1}.$$

By (14.4) and the first inequality in $(\star)$, we have that

$$B_\delta\cdot A_\delta^{n-1} \leq (C_1\cdot h^n\cdot \mathrm{vol}(\xi+\delta h) - \mathrm{vol}(A_\delta))^{1/2} \leq C_2\cdot\delta^n$$

where C_i are constants independent of δ. The above inequality, $(\star)$ and $(\sharp)$ toghether imply that

$$(\flat) \qquad \frac{\xi\cdot\gamma_\delta}{h\cdot\gamma_\delta} \leq \frac{A_\delta^n + C_2\cdot\delta^n}{(h^n)^{1/n}\cdot(A_\delta^n)^{(n-1)/n}} \leq C_3\cdot(A_\delta^n)^{1/n} + C_4\cdot\delta$$

where C_i are constants independent of δ. Now $\mathrm{vol}(\xi)=0$ so that for $\delta\to 0$, we have

$$\lim A_\delta^n = \lim \mathrm{vol}(A_\delta^n) = \lim \mathrm{vol}(\xi+\delta h) = 0.$$

By $(\flat)$ we then have

$$\lim\frac{\xi\cdot\gamma_\delta}{h\cdot\gamma_\delta} = 0$$

which is the required contradiction. □

15. Rationally Connected Fibrations

Recall the following.

Definition 15.1. *d-long Let X be a smooth complex projective variety, then*

1. *X is* **rational** *if it is birational to $\mathbb{P}^n_{\mathbb{C}}$ i.e. $\mathbb{C}(X)\cong\mathbb{C}(x_1,\ldots,x_n)$.*
2. *X is* **unirational** *if there is a dominant rational map $P^m_{\mathbb{C}} \dashrightarrow X$ i.e. if there are inclusions $\mathbb{C}\subset\mathbb{C}(X)\subset\mathbb{C}(x_1,\ldots,x_m)$.*

(3) X *is* **rationally connected** *if for any two general points p and q there is a rational curve C passing through p and q.*

(4) X *is* **uni-rationally** *if for any general point $p \in X$ there is a rational curve C passing through p (i.e. if K_X is not pseudo-effective cf.* (14.5)*).*

Remark 15.2. *Clearly rational implies unirational which implies rationally connected which implies unirational. Note that if X is unirational, then $\kappa(X) < 0$ and if X is rationally connected, then $h^0(\Omega^1_X) = 0$.*

Theorem 15.3. *If* $\dim X \leq 2$*, then X is rational if and only if it is unirational or rationally connected.*

Proof. By (15.2), it suffices to show that if X is rationally connected then it is rational. In this case, we have $P_2(X) = h^0(\omega_X^{\otimes 2}) = 0$ and $h^0(\Omega^1_X) = 0$. If $\dim X = 1$ this means that the genus of X is 0 and so X is $\mathbb{P}^1_{\mathbb{C}}$. If $\dim X = 2$, then by a theorem of Castelnuovo, X is rational. □

Remark 15.4. *In* dim ≥ 3 *there are uniruled varieties that are not rational. It is not known if rationally connected are allways unirational. Note that by a Theorem of Campana and Kollár-Mori-Myiaoka, any Fano manifold ($-K_X$ is ample) is rationally connected. Eg. a general hypersurface in $\mathbb{P}^{n+1}$ of degree $n+1$ with $n \gg 0$ and a $2-1$ cover of $\mathbb{P}^n$ branched along a general divisor of degree $2n$ with $n \gg 0$. We don't know if these varieties are unirational.*

Theorem 15.5. *If* $\dim X = 2$*, then X is uniruled if and only if $\kappa(X) < 0$.*

Corollary 15.6 (Lüroth's problem)**.** *Let $\mathbb{C} \subset L \subset \mathbb{C}(x, y)$ be any field. Then $L \cong \mathbb{C}$ or $L \cong \mathbb{C}(t)$ or $L \cong \mathbb{C}(s, t)$.*

Remark 15.7. *It is known that X is rationally connected if and only if there is a morphism $f : \mathbb{P}^1_{\mathbb{C}} \to X$ such that f^*T_X is ample (if $n \geq 3$ this is equivalent to the existence of a smooth rational curve with ample normal bundle).*

Lemma 15.8. *Given any two points p, q on a smooth rationally connected variety X, there is a chain of rational curves containing p and q.*

It is conjectured that

Conjecture 15.9. *Let X be a smooth complex projective variety, then*

(1) *X is rationally connected if and only if $h^0(X, (\Omega^1_X)^{\otimes m}) = 0$ for all $m > 0$.*

(2) *X is uniruled if and only if $h^0(\omega_X^{\otimes m}) = 0$ for all $m > 0$.*

Note that the second part is known in dimension ≤ 3 by the MMP. The second part is known to imply the first part cf. (15.12)

Definition 15.10. *Let X be a smooth complex projective variety. The* **maximally rationally connected fibration** *(MRC fibration) is a morphism $\pi : X' \to Z$ such that X' is birational to X, the general fiber of π is rationally connected and for any $z \in Z$ very general, then any rational curve on X' which meets the fiber X'_z is contained in X_z.*

The existence of MRC fibrations is guaranteed by a result of Campana and Kollár-Mori-Myiaoka. It is unique up to birational equivalence. Recall the following fundamental result of Graber-Harris-Starr:

Theorem 15.11. *Let $\pi : X \to B$ be a proper morphism of smooth complex varieties where $\dim B = 1$. If the general fiber of π is rationally connected then there is a section of π.*

Corollary 15.12. *The image Z of the MRC fibration $\pi : X' \to Z$ is not uniruled.*

15.1. **singular varieties.** The situation is more complicated for singular varities. The following definitions are useful.

Definition 15.13. *Let X be a reduced separated scheme of finite type over $\mathbb{C}$, then X is is* **rationally chain connected** *if for any 2 general points $p, q \in X$, there is a connected chain of rational curves $C = \cup C_i$ containing p and q. If $V \subset X$ is a closed subset, then X is is* **rationally chain connected modulo** *V if for any 2 general points $p, q \in X$, there is a connected chain of curves $C = \cup C_i$ containing p and q such that if C_i is not rational then $C_i \subset V$.*

Remark 15.14. *Note that if X is a smooth variety, then X is rationally chain connected if and only if it is rationally connected. This fails for singular varieties. Eg. let X be a cone over an elliptic curve, then X is rationally chain connected but it is not rationally connected.*

We have the following result of Hacon and M^c^Kernan.

Theorem 15.15. *Let (X, B) be a log pair and $f : X \to S$ be a projective morphism such that $-K_X$ is relatively big and $\mathcal{O}_X(-m(K_X + B))$ is relatively generated for some $m > 0$. Let $g : Y \to X$ be any birational morphism and $\pi = f \circ g$.*

Then, every fiber of π is rationally chain connected modulo $g^{-1}\mathrm{NKLT}(X, B)$.

Corollary 15.16. *If (X, B) is a kawamata log terminal pair and $f : X \to S$ is a projective morphism such that $-(K_X + B)$ is relatively nef and $-K_X$ is relatively big.*

Then every fiber of f is rationally chain connected (and in fact rationally connected).

Corollary 15.17. *If (X, B) is a kawamata log terminal pair and $g : Y \to X$ is any proper birational morphism, then the fibers of g are rationally chain connected.*

Corollary 15.18. *If (X, B) is a kawamata log terminal pair, then (X, B) is rationally connected if and only if it is rationally chain connected*

Corollary 15.19. *If (X, B) is a kawamata log terminal pair and $f : X \dashrightarrow Y$ is a rational map of normal projective varieties, then for any $x \in X$, the indeterminacy locus of x (given by $q(p^{-1}(x))$ where Γ is the graph and $p : \Gamma \to X$ and $q : \Gamma \to Y$) is covered by rational curves.*

Corollary 15.20. *If (X, B) is a kawamata log terminal pair and $f : X \dashrightarrow S$ is a projective morphism with connected fibers such that $-(K_X + B)$ is relatively nef and $-K_X$ is relatively big. Then for any birational morphism $g : Y \to X$, $f \circ g$ has a section over any curve.*

Corollary 15.21. *If (X, B) is a kawamata log terminal pair and $-(K_X + B)$ is big and nef. Then X is rationally connected and in fact simply connected.*

References

[1] V. Alexeev, C. Hacon, and Y. Kawamata, *Termination of (many) 4-dimensional log flips*, arXiv:math.AG/0605137.

[2] A. Corti, *Factoring birational maps of threefolds after Sarkisov*, J. Algebraic Geom. **4** (1995), no. 2, 223–254.

[3] O. Fujino, *Special termination and reduction to pl flips*, Flips for 3-folds and 4-folds (Alessio Corti, ed.), Oxford University Press, 2005, pp. 55–65.

[4] C. Hacon and J. McKernan, *On the existence of flips*, math.AG/0507597.

[5] R. Hartshorne, *Algebraic geometry*, Springer-Verlag, 1977.

[6] Y. Kawamata, *Flops connect minimal models*, arXiv:alg-geom/07041013.

[7] J. Kollár, *Resolution of Singularities – Seattle Lecture*, arXiv:math.AG/0508332.

[8] ———, *Effective base point freeness*, Math. Ann. **296** (1993), no. 4, 595–605.

[9] J. Kollár et al., *Flips and abundance for algebraic threefolds*, Société Mathématique de France, Paris, 1992, Papers from the Second Summer Seminar on Algebraic Geometry held at the University of Utah, Salt Lake City, Utah, August 1991, Astérisque No. 211 (1992).

[10] J. Kollár and S. Mori, *Birational geometry of algebraic varieties*, Cambridge tracts in mathematics, vol. 134, Cambridge University Press, 1998.

[11] R. Lazarsfeld, *Positivity in algebraic geometry. I*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 48, Springer-Verlag, Berlin, 2004, Classical setting: line bundles and linear series.

[12] ———, *Positivity in algebraic geometry. II*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 49, Springer-Verlag, Berlin, 2004, Positivity for vector bundles, and multiplier ideals.

[13] N. Nakayama, *Zariski-decomposition and abundance*, MSJ Memoirs, vol. 14, Mathematical Society of Japan, Tokyo, 2004.

[14] E. Szabó, *Divisorial log terminal singularities*, J. Math. Sci. Univ. Tokyo **1** (1994), no. 3, 631–639.

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