Finite generation of canonical rings II

Christopher Hacon

University of Utah

February, 2007
Outline of the talk

1. Introduction
Outline of the talk

1. Introduction
2. The MMP with scaling
Outline of the talk

1. Introduction
2. The MMP with scaling
3. The structure of the proof
Outline of the talk

1. Introduction
2. The MMP with scaling
3. The structure of the proof
Notation

- $X \subset \mathbb{P}^N$ is defined by homogeneous polynomials $P_1, \ldots, P_t \in \mathbb{C}[x_0, \ldots, x_N]$.
- $X, X'$ are birational if they have isomorphic open subsets $U \cong U'$.
- If $X$ is smooth of dimension $n$, then $T_X$ denotes the tangent bundle of $X$ and $\omega_X = \wedge^n T_X^\vee$ is the canonical line bundle.
- $H^0(X, \omega_X^\otimes m)$ denotes the space of global pluricanonical sections (locally given by $f(z)(dz_1 \wedge \ldots \wedge dz_n)^\otimes m$).
- $K_X$ denotes (a choice of) the canonical divisor i.e. a formal linear combination of codimension 1 subvarieties of $X$ given by the zeroes of a meromorphic section $s \neq 0$ of $\omega_X$. 
If $X$ and $X'$ are birational, then $\forall m \geq 0$

$$H^0(X, \omega_X^\otimes m) \cong H^0(X', \omega_{X'}^\otimes m).$$

The **canonical ring** of $X$ is given by

$$R(K_X) = \bigoplus_{m \in \mathbb{N}} H^0(X, \omega_X^\otimes m).$$

One would like to understand the structure of this ring and to use its features to classify complex projective varieties.
Today I will illustrate the techniques involved in the proof of the following result

**Theorem (Birkar-Cascini-Hacon-McKernan / Siu)**

*Let $X$ be a smooth projective variety, then $R(K_X)$ is finitely generated.*

The two proofs are independent. Our proof is algebraic and uses the ideas of the MMP (minimal model program). Siu’s proof is analytic and requires $X$ to be of general type (i.e. $K_X$ is big).
KLT Log Pairs

- We consider log pairs \((X, \Delta)\) where \(X\) is a normal \(\mathbb{Q}\)-factorial variety and \(\Delta = \sum \delta_i \Delta_i\) is a divisor with \(\delta_i \in \mathbb{R}\) and \(0 \leq \delta_i < 1\).

- Then \(K_X + \Delta\) is \(\mathbb{R}\)-Cartier and so \((K_X + \Delta) \cdot C\) and \(f^*(K_X + \Delta)\) make sense.

- Consider a map \(f : X' \to X\) such that \(X'\) is smooth and \(f^*\Delta + \text{Exc}(f)\) has simple normal crossings support. Write

\[
K_{X'} + \Delta' = f^*(K_X + \Delta).
\]

- \((X, \Delta)\) is KLT if each coefficient of \(\Delta'\) is less than 1.

- A KLT pair \((X, \Delta)\) is a minimal model if \(K_X + \Delta\) is nef (i.e. \((K_X + \Delta) \cdot C \geq 0\) for any curve \(C \subset X\)).
A divisor $D$ is big if $D \sim_{\mathbb{R}} A + B$ where $A$ is ample and $B \geq 0$.

A divisor $D$ is pseudo-effective if it is a limit of big divisors.

Note that any nef divisor is pseudo-effective.

A $\mathbb{Q}$-divisor $D$ is semiample if for some $m > 0$ the divisor $mD$ is base point free.

If $D$ is semiample, then $R(D) = \bigoplus_{i \geq 0} H^0(\mathcal{O}_X(iD))$ is finitely generated.

Any semiample divisor is nef but the converse is not true.

By the Base Point Free Theorem, if $K_X + \Delta$ is KLT and $\Delta$ is big, then $K_X + \Delta$ is semiample.
Recall that Mori and Fujino have shown:

**Theorem**

Let $(X, \Delta)$ be any KLT $\mathbb{Q}$-factorial pair with $\Delta \in \text{Div}_\mathbb{Q}(X)$. Then there exists a big KLT $\mathbb{Q}$-factorial pair $(Y, \Gamma)$ with $\Gamma \in \text{Div}_\mathbb{Q}(Y)$ such that

$$R(K_X + \Delta)^{(m)} \cong R(K_Y + \Gamma)^{(m)}$$

for any sufficiently divisible integer $m > 0$.

It follows that to show that $R(K_X)$ is finitely generated, it suffices to prove the corresponding statement for big KLT log pairs.
The main Theorem

Theorem (Birkar-Cascini-Hacon-McKernan)

Let \((X, \Delta)\) be any KLT \(\mathbb{Q}\)-factorial pair with \(\Delta \in \text{Div}_R(X)\). If \(\Delta\) is big and \(K_X + \Delta\) is pseudo-effective, then \((X, \Delta)\) has a minimal model.

More precisely, there is a sequence of well understood operations (flips and divisorial contractions) \(X \dashrightarrow X_1 \dashrightarrow \ldots \dashrightarrow X_n\) such that \(R(K_X + \Delta) \cong R(K_{X_n} + \Delta_n)\) and \(K_{X_n} + \Delta_n\) is nef and KLT. By the base-point-free theorem, \(K_{X_n} + \Delta_n\) is semiample and so \(R(K_X + \Delta)\) is finitely generated.
Note that our theorem also applies when $K_X + \Delta$ is big and KLT.

Since $K_X + \Delta$ is big, we have $K_X + \Delta \sim_{\mathbb{R}} D \geq 0$.

Therefore for any rational number $0 < \epsilon \ll 1$, we have

$$K_X + \Delta' = K_X + \Delta + \epsilon D \sim_{\mathbb{R}} (1 + \epsilon)(K_X + \Delta)$$

where $(X, \Delta')$ is KLT and $\Delta'$ is big.

It follows that if $K_X + \Delta$ is a big and KLT $\mathbb{Q}$-divisor, then $R(K_X + \Delta)$ is finitely generated.

By the Theorem of Mori-Fujino, if $X$ is a smooth projective variety, then $R(K_X)$ is finitely generated.
Outline of the talk

1. Introduction
2. The MMP with scaling
3. The structure of the proof
We begin with a $\mathbb{Q}$-factorial KLT pair $(X, \Delta)$ such that $\Delta$ is big and $K_X + \Delta$ is pseudo-effective. We allow $\Delta \in \text{Div}_\mathbb{R}(X)$.

These conditions will be preserved by the birational maps that we will consider.

We wish to find a birational model $X \dasharrow X'$ such that $K_{X'} + \Delta'$ is nef.

Assume that $A$ is ample and $K_X + \Delta + A$ is nef.

Let $\lambda = \inf\{t \geq 0 | K_X + \Delta + tA\}$ is nef.

If $\lambda = 0$ we STOP ($K_X + \Delta$ is nef).
Otherwise, by the cone theorem, there is an extremal ray $R$ such that $(K_X + \Delta + \lambda A) \cdot R = 0$ and $(K_X + \Delta + tA) \cdot R < 0$ for $0 \leq t < \lambda$.

Let $\text{cont}_R : X \to Z$ be the corresponding morphism which contracts a curve $C$ iff $[C] = R$.

If $\text{cont}_R$ is not birational, we STOP. (We have a Mori-Fano fiber space and $R(K_X + \Delta) \cong \mathbb{C}$. This only happens when $K_X + \Delta$ is not pseudo-effective.)

If $\text{cont}_R$ is birational and divisorial (i.e. it contracts a divisor), then we replace $X$ by the image of the corresponding contraction and we may continue this procedure.
If $\text{cont}_R$ is birational and small, (i.e. $\text{Exc}(\text{cont}_R)$ has codimension $\geq 2$) then the image of the corresponding contraction is not KLT. Therefore, we can NOT replace $X$ by the corresponding contraction!

Instead we replace $(X, \Delta)$ by the corresponding flip $\phi : X \to X^+$ (assuming it exists).

Let $\Delta^+ = \phi_* \Delta$. The flip is a codimension 2 surgery which replaces $K_X + \Delta$ negative curves by $K_{X^+} + \Delta^+$ positive curves.
Flips

- More precisely, $f = \text{cont}_R : X \to Z$ is a flipping contraction i.e. a small birational morphism with $\rho(X/Z) = 1$ and $-(K_X + \Delta)$ ample over $Z$.

- The flip $f^+ : X^+ \to Z$ (if it exists) is a small birational morphism such that $\rho(X^+/Z) = 1$ and $K_{X^+} + \Delta^+$ is ample over $Z$.

- The flip $f^+ : X^+ \to Z$ (if it exists) is given by

$$X^+ = \text{Proj}_Z \bigoplus_{m \in \mathbb{N}} f_* \mathcal{O}_X([m(K_X + \Delta)]).$$

- Note that after perturbing $\Delta$, we may assume that $\Delta \in \text{Div}_\mathbb{Q}(X)$. 
The problem of existence of flips is local over $\mathbb{Z}$ and hence we may assume that $\mathbb{Z}$ is affine. It suffices then to show that $R(K_X + \Delta)$ is finitely generated. So this is a local version of the original problem.

To run this MMP with scaling, we must therefore show that such flips exist and terminate (i.e. there is no infinite sequence of such flips).

Note that after each divisorial contraction, the Picard number drops by 1 and so there are only finitely many divisorial contractions.
Outline of the talk

1. Introduction
2. The MMP with scaling
3. The structure of the proof
   - Existence of minimal models
   - Non-vanishing
   - Finiteness of models
The structure of the proof

We proceed by induction on $n = \dim X$. We show that for an $n$-dimensional KLT pair $(X, \Delta)$:

**Claim 1.** Flips exist.

**Claim 2.** If $K_X + \Delta \sim_{\mathbb{R}} D \geq 0$ and $\Delta$ is big, then $(X, \Delta)$ has a minimal model.

**Claim 3.** If $\Delta$ is big and $K_X + \Delta$ is pseudo-effective, then there is an $\mathbb{R}$-divisor $D$ such that $K_X + \Delta \sim_{\mathbb{R}} D \geq 0$.

**Claim 4.** The set of minimal models for KLT pairs with $A \leq \Delta \leq \Delta_0$ where $A$ is ample and $(X, \Delta_0)$ is KLT, is finite.

All statements should be relative over a base.
The structure of the proof II

- Roughly speaking, the inductive structure of the proof is:

\[(1 - 4)^{n-1} \Rightarrow 1_n \Rightarrow 2_n \Rightarrow 3_n \Rightarrow 4_n.\]

- Note that that \((1 - 4)^{n-1} \Rightarrow 1_n\) was obtained by Hacon M\(^c\)Kernan in 2005 using ideas of Shokurov and an extension result based on work of Siu, Kawamata and Tsuji. (This was discussed yesterday by James.)

- The remaining implications heavily use the ideas of the MMP established by Kawamata, Kollár, Mori, Reid, Shokurov and others.
Termination of MMP with scaling

- Since we are unable to show that an arbitrary sequence of flips must terminate, we run a MMP with scaling.
- The idea is that if $A$ is sufficiently ample, then $K_X + \Delta + A$ is ample and so it is a minimal model. We then consider $K_X + \Delta + tA$ and let $t$ go to 0.
- Proceeding this way we get a sequence of pairs $(X_i, \Delta_i + \lambda_i A_i)$ such that $K_{X_i} + \Delta_i + \lambda_i A_i$ is nef and trivial on some $K_{X_i} + \Delta_i$ extremal ray. We then flip/contract.
- Each $(X_i, \Delta_i + \lambda_i A_i)$ is a minimal model for $(X, \Delta + \lambda_i A)$. 
If we could assume Claim 4\(_n\), we would be done. (As \(\Delta\) is big, we may assume that
\[
A' \leq \Delta \leq \Delta + tA \leq \Delta + A
\]
for some ample divisor \(A'\).)

Since we can only assume \(4_{n-1}\), we replace \((X, \Delta)\) by an appropriate DLT pair \((X', \Delta')\) and we run a MMP with scaling such that all flips/contractions are contained in \([\Delta']\).

This step is similar to Shokurov’s reduction to PL-flips.
We must show that if $\Delta$ is big and $K_X + \Delta$ is pseudo-effective, then $K_X + \Delta \sim_{\mathbb{R}} D \geq 0$.

If $h^0(\mathcal{O}_X(m(K_X + \Delta) + A))$ is bounded (for $A$ ample), then the result follows from work of Nakayama.

If $h^0(\mathcal{O}_X(m(K_X + \Delta) + A))$ is unbounded, then we can produce a log canonical center not contained in the stable base locus of $K_X + \Delta + \epsilon A$ for $0 < \epsilon \ll 1$.

After replacing $X$ by an appropriate birational model, we may then assume that $(X, \Delta)$ is PLT ("almost KLT") and $S = [\Delta']$ is irreducible and not contained in the stable base locus of $K_X + \Delta + \epsilon A$ for any $0 < \epsilon \ll 1$. 
Since $K_X + \Delta + \epsilon A$ is big, one sees that we may still run a $K_X + \Delta$ MMP with scaling of $A$.

However, since we do not know that $K_X + \Delta$ is effective, this MMP might not terminate.

By induction on the dimension, the flipping locus is eventually disjoint from $S$.

Then $(K_X + \Delta + \epsilon A)|_S$ is nef for all $0 < \epsilon \ll 1$. Therefore, $K_S + \Delta_S = (K_X + \Delta)|_S$ is pseudo-effective, $\Delta_S$ is big and so $h^0(O_S(m(K_S + \Delta_S))) > 0$ for some $m > 0$.

Using Kawamata-Viehweg vanishing, we can lift this section to $m(K_X + \Delta)$. 

Christopher Hacon
Finite generation of canonical rings II
**Theorem**

Assume that $(X, \Delta_0)$ is KLT and $A$ is an ample $\mathbb{Q}$-divisor. Then the set of isomorphism classes

$$\{Y | Y \text{ is a min. model for } (X, \Delta); A \leq \Delta \leq \Delta_0\}$$

is finite.

The proof is based on ideas of Shokurov (1996). It requires a compactness argument and so it is necessary to work with $\mathbb{R}$-divisors.

We will only show that there exist finitely many minimal models $\phi_i : X \dashrightarrow X_i$ such that for any $A \leq \Delta \leq \Delta_0$ there exists $i$ such that $X_i$ is a minimal model for $(X, \Delta)$. 

Christopher Hacon  
Finite generation of canonical rings II
We pick $\Delta \in [A, \Delta_0]$ and work in a sufficiently small neighborhood $U = (\Delta_1, \Delta_2)$ of $\Delta$.

If $K_X + \Delta$ is not pseudo-effective, then (after shrinking $U$), $K_X + \Theta$ is not pseudo-effective for any $\Theta \in U$.

Therefore, we may assume that $K_X + \Delta$ is pseudo-effective.

By Claim 3, $K_X + \Delta \sim_{\mathbb{R}} D \geq 0$ and so, by Claim 2, $(X, \Delta)$ has a minimal model.

One sees that replacing $(X, \Delta)$ by its minimal model, we may assume that $K_X + \Delta$ is nef.

By the base-point free theorem, there is a morphism $f : X \to Z$ such that $K_X + \Delta \sim_{\mathbb{R}} f^*H$ with $H$ ample.
Finiteness of models III

- Working over \( \mathbb{Z} \), we may assume that \( K_X + \Delta \sim_{\mathbb{R},\mathbb{Z}} 0 \).
- Let \( \Theta \in (\Delta, \Delta_i] \) sufficiently close to \( \Delta \). We may also assume that \( K_X + \Theta \) is nef iff it is nef over \( \mathbb{Z} \). Therefore, if \( \phi_i : X \to X_i \) is a minimal model for \((X, \Theta)\) over \( \mathbb{Z} \), then it is a minimal model for \((X, \Theta)\) (over \( \text{Spec}(\mathbb{C}) \)).
- We may write

\[
\Theta - \Delta = (K_X + \Theta) - (K_X + \Delta) \sim_{\mathbb{R},\mathbb{Z}} K_X + \Theta.
\]

- If \( K_X + \Theta \) is pseudo-effective, then it is pseudo-effective over \( \mathbb{Z} \) and hence so is

\[
K_X + \Delta_i \sim_{\mathbb{R},\mathbb{Z}} \Delta_i - \Delta = \lambda(\Theta - \Delta).
\]

- Let \( \phi_i : X \to X_i \) be a minimal model of \((X, \Delta_i)\) over \( \mathbb{Z} \). Then \( X_i \) is a minimal model of \((X, \Theta)\).