ON THE DEGREE OF THE CANONICAL MAPS OF 3-FOLDS

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ABSTRACT. We prove the following result that answers a question of M. Chen: *Let X be a Gorenstein minimal complex projective 3-fold of general type with locally factorial terminal singularities. If $|K_X|$ defines a generically finite map $\phi : X \dashrightarrow \mathbb{P}^{p_g-1}$, then* $\deg(\phi) \leq 576$. For any positive integer $m > 0$, we give infinitely many examples of (non-Gorenstein) 3-folds of general type with canonical map of degree m.

Keywords: Canonical maps, threefolds.

1. Introduction

The study of the canonical maps of projective varieties of general type is one of the central problems in algebraic geometry. The case of surfaces has attracted a great deal of attention. By work of Beauville, it is known that the degree of the canonical maps of surfaces are bounded. In dimension at least three, the situation seems much less clear. The purpose of this brief note is to address this question. We prove the following theorem which answers a question of M. Chen (cf. Open problem 2.10 [Ch]).

Theorem 1.1. *Let X be a Gorenstein minimal complex projective 3-fold of general type with locally factorial terminal singularities. Suppose that $|K_X|$ defines a generically finite map $\phi : X \dashrightarrow \mathbb{P}^{p_g-1}$. Then* $\deg(\phi) \leq 576$.

When X is not Gorenstein, the situation is however quite different. For any positive integer $m > 0$, we give infinitely many families of 3-folds of general type with canonical map of degree m (see Example 2.1).

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2. Proofs

Proof. (of Theorem 1.1) Since ϕ is generically finite, one has that $p_g := h^0(X, \omega_X) \geq 4$. Let d be the generic degree of ϕ. Since $\phi(X)$ is a non-degenerate variety in $\mathbb{P}^{p_g-1}$, we have $\deg(\phi(X)) \geq p_g(X) - 3$. Recall that by the Miyaoka-Yau inequality (cf. [Mi]), we have

$$\frac{1}{4} d p_g(X) \leq d(p_g(X) - 3) \leq K_X^3 \leq 72\chi(\omega_X).$$

It suffices therefore to show that $\chi(\omega_X) \leq 2p_g(X)$. If $q(X) := h^0(X, \Omega^1_X) \leq 5$, this is clear as then

$$\chi(\omega_X) = p_g - h^0(\Omega^2_X) + q(X) - 1 \leq p_g + 4 \leq 2p_g.$$

So, we assume that $q(X) \geq 6$ and let $a : X \longrightarrow Y$ be the Stein factorization of the Albanese morphism of X. In particular a has connected fibers, Y is normal and the induced morphism $Y \longrightarrow \mathrm{A}(X)$ is finite. By semicontinuity and [Ha] Corollary 4.2, for general $P \in \mathrm{Pic}^0(Y)$,

$$(1) \qquad p_g(X) = h^0(a_*\omega_X) \geq h^0(a_*\omega_X \otimes P) = \chi(a_*\omega_X \otimes P) = \chi(a_*\omega_X).$$

If $\dim(Y) \geq 2$, then $R^i a_*\omega_X = 0$ for $i > 1$ (cf. [Ko]). By [Ha] Corollary 4.2, $\chi(R^i a_*\omega_X) \geq 0$, therefore

$$\chi(\omega_X) = \chi(a_*\omega_X) - \chi(R^1 a_*\omega_X) \leq \chi(a_*\omega_X).$$

Combining this with (1), one has $\chi(\omega_X) \leq p_g(X)$ as required.

If $\dim(Y) = 1$, then $g(Y) = q$ and proceeding as above

$$(2) \qquad \chi(\omega_X) \leq \chi(a_*\omega_X) + \chi(R^2 a_*\omega_X).$$

One has that $a_*\omega_X = a_*(\omega_{X/Y}) \otimes \omega_Y$ and $R^2 a_*\omega_X = \omega_Y$ (cf. [Ko] Proposition 7.6). Let F be a general fiber of $X \longrightarrow Y$, then the rank of $a_*(\omega_{X/Y})$ is just $h^0(\omega_{X/Y}|_F) = h^0(\omega_F) = p_g(F)$. By a result of Fujita, $\deg(a_*(\omega_{X/Y})) \geq 0$, and so by Riemann-Roch on Y,

$$\chi(a_*\omega_X) = \chi(a_*(\omega_{X/Y}) \otimes \omega_Y) \geq (q-1)p_g(F) = \chi(\omega_Y)p_g(F).$$

Since $p_g(X) > 0$, one has $p_g(F) > 0$ and so, by the above equation, $\chi(\omega_Y) \leq \chi(a_*\omega_X)$. It follows that

$$(3) \qquad \chi(a_*\omega_X) + \chi(R^2 a_*\omega_X) \leq 2\chi(a_*\omega_X).$$

Combining equations (1), (2) and (3), one sees that $\chi(\omega_X) \leq 2p_g(X)$ as required. □

Example 2.1. *For any integer $m > 0$, there exists infinitely many families of minimal 3-folds of general type with index 2 terminal singularities such that the canonical map has degree m.*

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Proof. Let $c : C \longrightarrow E$ be the $\mathbb{Z}_2$ cover of an elliptic curve defined by

$$c_*(\omega_C) = \mathcal{O}_E \oplus L$$

and a reduced divisor $B \in |2L|$. We denote the corresponding involution by σ. Let $\tilde{X} = C \times C \times C$ and $X := \tilde{X}/\mathbb{Z}_2$ where the $\mathbb{Z}_2$ action on $\tilde{X}$ is generated by $\sigma \times \sigma \times \sigma$. One sees that X has terminal and hence rational singularities and X is minimal of index 2 (cf. [EL] Example 1.13). Let $a : X \longrightarrow E \times E \times E$ be the induced map of degree 4. One has that

$$a_*(\omega_X) \cong \mathcal{O}_{E\times E\times E} \oplus (L \boxtimes L \boxtimes \mathcal{O}_E) \oplus$$
$$(L \boxtimes \mathcal{O}_E \boxtimes L) \oplus (\mathcal{O}_E \boxtimes L \boxtimes L).$$

Assume that $\deg L \geq 3$ and so L is very ample. It is easy to see that the linear series $|K_X|$ is birational. Now let $P \in \mathrm{Pic}^0(E)$ be a point of order m and $f : Y \longrightarrow X$ be the étale $\mathbb{Z}_m$-cover corresponding to $P \boxtimes P \boxtimes P$. One sees that $|K_Y| = f^*|K_X|$ and so the degree of ϕ_m is m. By varying the degree of L, one obtains infinitely many examples with distinct values of K_X^3. □

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