Which powers of a holomorphic function are integrable?

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The question

- Let $p \in \mathbb{C}^n$ and $f(z_1, \ldots, z_n) \in \mathbb{C}[z_1, \ldots, z_n]$. For which values $s \in \mathbb{R}$ is $|f|^s$ integrable on a neighborhood $p \in U \subset \mathbb{C}^n$?
- It is clear that $|f|^s$ is integrable for all $s \geq 0$, however, when $s = -t < 0$, then $\frac{1}{|f|^t}$ may fail to be integrable near the zeroes of f i.e. the poles of $\frac{1}{f}$.
- It is easy to see that if $\frac{1}{|f|^t}$ is integrable then so is $\frac{1}{|f|^{t'}}$ for $t' \leq t$.

Definition

The **log canonical threshold** of f at p, $c = \mathrm{lct}_p(f)$ is the supremum of the numbers $s \in \mathbb{R}_{\geq 0}$ such that $|f|^{-2s}$ is integrable on a neighborhood $p \in U \subset \mathbb{C}^n$. (It follows that $|f|^{-2s}$ is not integrable near p if $s > c$ and integrable if $s < c$.)

The log canonical threshold

- By convention $\mathrm{lct}_p(0) = 0$.
- Note that if $u(p) \neq 0$, then $\frac{1}{|f|^t}$ is integrable on a neighborhood $p \in U \subset \mathbb{C}^n$ iff so is $\frac{1}{|uf|^t}$.
- Thus the log canonical threshold $\mathrm{lct}_p(f)$ is determined by the zero set $Z = \{f = 0\} \subset \mathbb{C}^n$ (must count zeroes with multiplicity).
- We then let $\mathrm{lct}_p(Z) = \mathrm{lct}_p(f)$.
- Note $\mathrm{lct}_p(Z) < +\infty$ iff $p \in Z$.
- The log canonical threshold is a natural sophisticated invariant of the singularities of Z at p which appears in a variety of contexts (Kahler Einstein metrics, Bernstein-Sato polynomial, Arnold's complex singular index,...).
- It also naturally generalizes to pairs (X, B) (more about this later) and hence plays an important role in the minimal model program.

Some examples

- $\mathrm{lct}_O(1/z) = 1$. To see this we work in polar coordinates ρ, θ and note that

$$\int \frac{1}{|z|^{2t}} dVol = \int_0^{\epsilon} \int_0^{2\pi} \frac{1}{\rho^{2t}} \rho d\theta d\rho = \int_0^{\epsilon} \frac{2\pi}{\rho^{2t-1}} d\rho$$

  which is integrable iff $t < 1$.
- Similarly if $p \in Z$ and Z is smooth at p, then $\mathrm{lct}_p(Z) = 1$.
- More generally, if $Z = \sum b_i Z_i$ where Z_i are smooth codimension one subvarieties meeting transversely (i.e. $\sum Z_i$ has **simple normal crossings**) and $b_i \in \mathbb{Z}_{>0}$ (i.e. Z is an **effective $\mathbb{Z}$-divisor**), then $\mathrm{lct}_p(Z) = \min\{\frac{1}{b_i}\}$.
- Let $\mathcal{HT}_n$ be the set of all possible n-dimensional log canonical ($\mathcal{H}$ypersurface) $\mathcal{T}$hresholds, then $\mathcal{HT}_1 = \{0, 1, \frac{1}{2}, \frac{1}{3}, \dots\}$ as $f = 0, z, z^2, z^3, \dots$ (we always assume $f(p) = 0$).
- We would like to understand $\mathcal{HT}_n$ for $n \geq 2$.

$\mathcal{HT}_n$

- $\mathcal{HT}_2$ is completely understood by work of Varchenko.
- It seems too difficult (and maybe not so important) to completely determine $\mathcal{HT}_n$ for $n \geq 3$.
- It is known that $\mathcal{HT}_n \subset [0,1] \cap \mathbb{Q}$ (we assume $p \in Z$) and $\cup_{n \geq 1} \mathcal{HT}_n = [0,1] \cap \mathbb{Q}$.
- Never-the-less the sets $\mathcal{HT}_n$ have some remarkable structure (first investigated by Shokurov) which is useful in applications.
- We begin by explaining how LCT's are computed in practice and giving some examples.

An interesting example

- Consider $f = y^2 - x^3$. We use the substitution $x = uv^2$ and $y = uv^3$.
- The integral of $1/|f|^{2t}$ is then computed by integrating

$$\frac{|uv^4|^2}{|u^2v^6 - u^3v^6|^{2t}} = \frac{1}{|u|^{4t-2}} \cdot \frac{1}{|v|^{12t-8}} \cdot \frac{1}{|1-u|^{2t}}$$

- Here, $|uv^4|^2$ is the Jacobian of our change of variables and $\varphi = \frac{1}{|1-u|^{2t}}$ is a unit ($\varphi(O) \neq 0$) and so it can be ignored.
- But $u \cdot v = 0$ has simple normal crossings, so the integrability condition is just $4t - 2 < 2$ and $12t - 8 < 2$ i.e. $t < 5/6$.
- As $\mathrm{lct}(\mathrm{cusp}) = 5/6 < \mathrm{lct}(\mathrm{node}) = 1$, the cusp is more singular than the node.
- Log resolutions give a more geometric interpretation.

Log resolutions

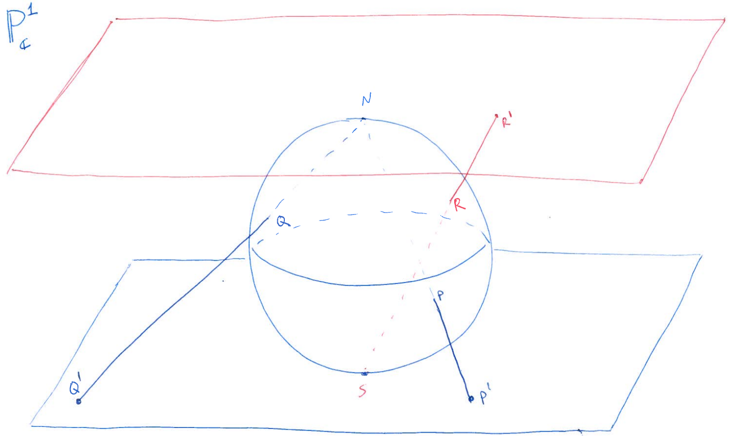
- A **log resolution** is a change of variables such that both the Jacobian and the set $f = 0$ correspond to simple normal crossing divisors (and hence the LCT is easy to compute).
- More precisely we have a map $\nu : X \to \mathbb{C}^n$ s.t. the zeroes of $\mathrm{Jac}(\nu)$ and $\nu^*(f) = f \circ \nu$ have simple normal crossings.
- We denote by $K_{X/\mathbb{C}^n}$ is the divisor corresponding to the (complex) Jacobian.
- In local coordinates

$$\nu^*(dz_1 \wedge \ldots \wedge dz_n) = (\mathrm{unit}) x_1^{k_1} \cdots x_n^{k_n} dx_1 \wedge \ldots \wedge dx_n$$

and so $K_{X/\mathbb{C}^n} = \sum k_i E_i$ where $E_i = \{x_i = 0\}$.

Log resolutions

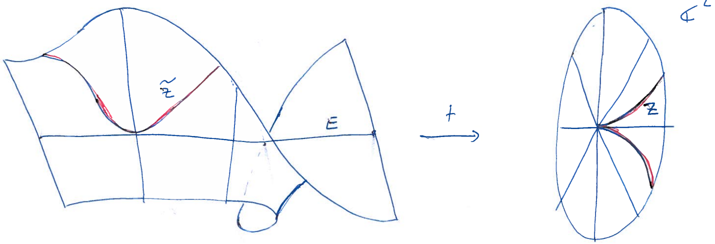
- By a deep result of Hironaka, log resolutions always exist and are given by a finite sequence of **blow ups** along smooth centers.
- Eg. the blow up of $\mathbb{C}^n$ at the O is obtained by replacing O with a copy of $\mathbb{P}^{n-1} = (\mathbb{C}^n \setminus O)/\mathbb{C}^*$ (each point in $\mathbb{P}^{n-1}$ is a tangent direction at $O \in \mathbb{C}^n$).
- $\mathbb{P}^{n-1}$ is covered by charts $\mathbb{A}^{n-1}_i$; given $[y_0 : \ldots : y_{n-1}] \in \mathbb{P}^{n-1}$, if $y_i \neq 0$ then $\phi_i[y_0 : \ldots : y_{n-1}] = (\frac{y_0}{y_i}, \ldots, \frac{\hat{y}_i}{y_i}, \ldots, \frac{y_n}{y_i}) \in \mathbb{A}^{n-1}_i$.
- Eg. $\mathbb{P}^1$ is covered by two copies of $\mathbb{C}$ glued together via $\phi(z) = 1/z$.



$\mathbb{P}^1_{\mathbb{C}}$
N
R'
R
Q
P
Q'
S
P'

Blow ups

$$X = Bl_0\mathbb{C}^2 \subset \mathbb{C}^2 \times \mathbb{P}^1$$



$$\{(x_1, x_2)[y_1 : y_2] \mid x_1 y_2 = x_2 y_1\}$$

Cusp $x_1^2 - x_2^3 = 0$

In the local chart $y_2 \neq 0$ set $t = y_1/y_2$ so $x_1 = t x_2$

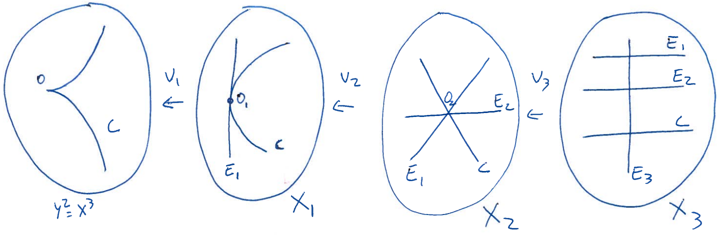
$$f^*(x_1^2 - x_2^3) = t^2 x_2^2 - x_2^3 = \underbrace{x_2^2}_{2E}\,\underbrace{(t^2 - x_2)}_{\tilde{Z}}$$

$$f^*(dx_1 \wedge dx_2) = x_2\, dt \wedge dx_2 \implies K_{X/\mathbb{C}^2} = E$$ is the zero set of the Jacobian

Log resolutions and LCT's

- $1/|f|^{2t}$ is locally integrable at p iff $|\mathrm{Jac}(f)|^2/|\nu^*(f)|^{2t}$ is integrable on a neighborhood of $\nu^{-1}(p)$.
- We assume that $\nu^{-1}(p)$ is compact and so it is enough to check that $|\mathrm{Jac}(f)|^2/|\nu^*(f)|^{2t}$ is locally integrable at every point of $\nu^{-1}(p)$.
- If we denote $K_{X/\mathbb{C}^n} = \sum k_i E_i$ the divisor given by the zeroes of $\mathrm{Jac}(\nu)$ and $\mu^* Z = \sum a_i E_i$, then as $\sum E_i$ is simple normal crossings, $|\mathrm{Jac}(\nu)|^2/|\nu^*(f)|^{2t}$ is locally integrable iff $ta_i - k_i < 1$ for all i (assuming $p \in \mu(E_i)$, $\forall i$). Thus

$$\mathrm{lct}_p(f) = \min\{\frac{k_i + 1}{a_i}\} \in \mathbb{Q}.$$



$$\nu_1^* C = 2E_1 + C$$

$$\nu_2^* C = 3E_2 + 2E_1 + C$$

$$\nu_3^* C = 6E_3 + 3E_2 + 2E_1 + C$$

$$\tfrac{5}{6}\nu_3^* C - K_{X_3/\mathbb{C}^2} = E_3 + \tfrac{1}{2}E_2 + \tfrac{2}{3}E_1 + \tfrac{5}{6}C$$

$$K_{X_1/\mathbb{C}^2} = E_1$$

$$K_{X_2/\mathbb{C}^2} = E_1 + 2E_2$$

$$K_{X_3/\mathbb{C}^2} = E_1 + 2E_2 + 4E_3$$

Multiplicity vs LCT

- Note that blowing up $p \in \mathbb{C}^n$ yields an exceptional divisor E with $k = n-1$ and $a = \mathrm{mult}_p(f)$, thus $\mathrm{lct}_p(f) \leq n/\mathrm{mult}_p(f)$.
- On the other hand it is not too hard to show that if $f(p) = 0$, then $\mathrm{lct}_p(f) \geq 1/\mathrm{mult}_p(f)$.
- We have $\mathrm{lct}_O(z_1^{a_1} + \cdots + z_n^{a_n}) = \min\{1, \frac{1}{a_1} + \cdots + \frac{1}{a_n}\}$ (generalizes $\mathrm{lct}_O(z_1^2 + z_2^3) = \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$).
- In particular $\cup_{n \geq 1} \mathcal{HT}_n = [0, 1] \cap \mathbb{Q}$.
- More generally, $\mathrm{lct}_p(f(x) \oplus g(y)) = \min\{1, \mathrm{lct}_p(f(x)) + \mathrm{lct}_p(g(y))\}$.
- So $\mathrm{lct}_p(f(x_1, \ldots x_{n-1}) + y^m) = \mathrm{lct}_p(f(x_1, \ldots x_{n-1})) + \frac{1}{m}$.
- Thus the set of accumulation points of $\mathcal{HT}_n$ contains $\mathcal{HT}_{n-1}$.

The structure of $\mathcal{HT}_n$

The structure of $\mathcal{HT}_n$ was understood by de Fernex-Mustata, Kollár and de Fernex-Ein-Mustata. This gives a positive answer to a conjecture of Shokurov.

Theorem (dFEMK)

- *The set $\mathcal{HT}_n$ satisfies the ACC (ascending chain condition) so that any non-decreasing sequence is eventually constant.*
- *The accumulation points of $\mathcal{HT}_n$ are given by $\mathcal{HT}_{n-1} \setminus \{1\}$.*

- In particular there is a biggest element $1-\epsilon_n$ in $[0,1) \cap \mathcal{HT}_n$. Conjecturally this is computed as follows.
- Let $c_1 = 1$, $c_{n+1} = c_1 \cdots c_n + 1$ so $c_i = 2, 3, 7, 43, 1807, 3263443, \ldots$. Then

$$1-\epsilon_n = 1 - \frac{1}{c_1 \cdots c_n} = \sum_{i=1}^{n} \frac{1}{c_i} = \mathrm{lct}_O(z_1^{c_1} + \cdots + z_n^{c_n}).$$

Proof of the theorem of dFEMK

- The proof relies on two main ingredients: **generic limits** and $\mathfrak{m}$-**adic approximation**.
- Suppose that we have a sequence of f_i s.t. $\mathrm{lct}_p(f_i)$ is non decreasing, then we must show it is eventually constant.
- The main idea is to find an accumulation point $f_i \to f_\infty$, such that (passing to a subsequence) we may assume that
  1) $\lim \mathrm{lct}_p(f_i) = \mathrm{lct}_p(f_\infty)$ (generic limits) and
  2) $\mathrm{lct}_p(f_i) = \mathrm{lct}_p(f_\infty)$ for all $m \gg 0$ ($\mathfrak{m}$-adic approximation).
- Since $\mathbb{C}[z_1, \ldots, z_n]$ is infinite dimensional, it is not clear that these accumulation points/limits exist.

Proof of the theorem of dFEMK

- Assume that $p = O = (0, \ldots, 0) \in \mathbb{C}^n$ and let $\tau_{<m}(f)$ denote the truncation of f (Taylor polynomial of degree $\leq m-1$).
- It is not hard to see that $|\mathrm{lct}_O(f_i) - \mathrm{lct}_O(\tau_{<m}(f_i))| \leq \frac{n}{m}$ (**continuity of LCT's**).
- $\mathbb{C}[z_1, \ldots, z_n]/(z_1, \ldots, z_n)^m$ is finite dimensional and the zeroes of $\tau_{<m}(f_i)$ are determined by a point in the compact space $\mathbb{P}^{N_m} = \mathbb{P}(\mathbb{C}[z_1, \ldots, z_n]/(z_1, \ldots, z_n)^m)$.
- Therefore, after passing to a subsequence, we may assume that $\tau_{<m}(f_i)$ converges to "$\tau_{<m}(f_\infty)$".
- Repeating this for bigger and bigger m and passing to further subsequences, we obtain the required f_∞.
- Unluckily, this choice of f_∞ is not good because we may have

$$\lim_{i \to \infty} (\mathrm{lct}_O(f_i)) \neq \mathrm{lct}_O(f_\infty).$$

- For example if $f_i(x) = x^2 + \frac{1}{i}x$, then $f_\infty(x) = x^2$ and so $\mathrm{lct}_O(f_i) = 1$ but $\mathrm{lct}_O(f_\infty) = 1/2$.
- The problem is that LCT's are are semicontinuous in the **Zariski toplology**: they are constant on open subsets and jump down on closed subvarieties defined by polynomial equations (i.e. acquire worse singularities).
- Another example $f_t(x, y) = y^2 - x^3 + tx^2$. For $t \neq 0$ we have a node and hence $\mathrm{lct}_O(f_t) = 1$ but for $t = 0$ we have a cusp and hence $\mathrm{lct}_O(f_t) = 5/6$.
- The proper closed subsets in the Zariski toplology are given by zeroes of polynomial equations (and hence are finite unions of hypersurfaces).
- In particular, the proper closed subsets of $\mathbb{C}$ consist of finitely many points and hence $\overline{\{1 - \frac{1}{i} | i > 0\}} = \mathbb{C}$.

- We instead consider Z_m an irreducible component of the Zariski closure of $\{\tau_{<m}(f_i)\}_{i>0} \in \mathbb{P}(\mathbb{C}[z_1, \ldots, z_n]/\mathfrak{m}^m)$ and g_m a very general point of Z_m. (Here $\mathfrak{m} = (z_1, \ldots, z_n)$.)
- Passing to a subsequence, we have $\tau_{<m}(f_i) \in Z_m$ for all $i \geq m$.
- We then choose Z_{m+1} so that its image is dense in Z_m and hence g_{m+1} a very general point of Z_{m+1} s.t. $g_m = \tau_{<m}(g_{m+1})$.
- We thus obtain the compatible sequence $\{g_m\}_{m>0}$ and hence the **generic limit** $g_\infty \in \mathbb{C}[[z_1, \ldots, z_n]]$ s.t. $g_m = \tau_{<m}(g_\infty)$ for any $m > 0$.
- By construction, we have $\mathrm{lct}_O(g_m) = \mathrm{lct}_O(\tau_{<m}(f_m))$ for all m.
- Then by continuity of LCT's and construction of g_m, we have

$$\mathrm{lct}_O(g_\infty) = \lim_{m\to\infty} \mathrm{lct}_O(g_m) = \lim_{m\to\infty} \mathrm{lct}_O(\tau_{<m}(f_m)) = \lim_{m\to\infty} \mathrm{lct}_O(f_m).$$

$\mathfrak{m}$-adic approximation

- We must show that we can remove the limits in the sequence of equalities

$$\mathrm{lct}_O(g_\infty) = \lim_{m\to\infty} \mathrm{lct}_O(g_m) = \lim_{m\to\infty} \mathrm{lct}_O(\tau_{<m}(f_m)) = \lim_{t\to\infty} \mathrm{lct}_O(f_m).$$

- It turns out that if we assume that the sequence $\mathrm{lct}_O(f_m)$ is non decreasing, then $\mathrm{lct}_O(g_\infty)$ is computed by an exceptional divisor E over O and so we can apply the following.

Theorem ($\mathfrak{m}$-adic approximation)

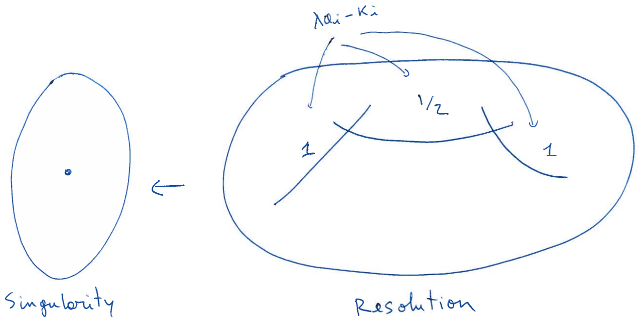
If $\mathrm{lct}_O(g)$ is computed by a divisor E over O then $\mathrm{lct}_O(\tau_{<m}(g)) = \mathrm{lct}_O(g)$ for all $m \gg 0$ (in fact $m \geq \mathrm{val}_E(g)$).

Thus $\mathrm{lct}_O(g_\infty) =^{m\gg 0} \mathrm{lct}_O(g_m) = \mathrm{lct}_O(\tau_{<m}(f_m)) =^{m\gg 0} \mathrm{lct}_O(f_m)$.

$\mathfrak{m}$-adic approximation

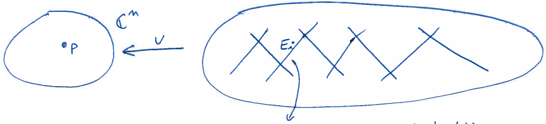
- The proof of $\mathfrak{m}$-adic approximation relies only of resolution of singularities and the Kollár-Shokurov Connectedness Theorem which dates back to 1992 and is (by now) a standard consequence of Kodaira-Kawamata-Viehweg vanishing.
- The Connectedness Theorem implies that if $\nu : X \to \mathbb{C}^n$ is a log resolution of $(\mathbb{C}^n, f)$, and $\lambda > 0$ then the *non log canonical* set $\sum_{\lambda a_i - k_i \geq 1} E_i$ is connected over O.

Kollár-Shokurov Connectedness



$\mathfrak{m}$-adic approximation

- If $\mathrm{lct}_p(f)$ is computed by $E_i \to p$ & $m > a_i$ $\exists$ $\mathrm{lct}_p(f) = \mathrm{lct}_p(\tau_{<m}(f))$



$c\cdot a_i - k_i = 1$ $\quad$ $c = \mathrm{lct}_p(f)$ $\quad$ $c \cdot a_j - k_j \le 1$

- $a_i = \mathrm{ord}_{E_i}(f)$ $\quad$ $b_j = \mathrm{ord}_{E_j}(\tau_{<m}(f))$ $\quad$ $k_j = \mathrm{ord}_{E_j}(K_{X/\mathbb{C}^n})$ $\quad$ $E_i \leftrightarrow (y_i = 0)$
- Since $f - \tau_{<m}(f) \in m^{a_i} \Rightarrow a_i = b_i$ & $\left(\frac{\nu^* f - \nu^* \tau_{<m}(f)}{y_i^{a_i}}\right)\Big|_{E_i} = 0$
- If $\mathrm{lct}_p(\tau_{<m}(f)) < c \Rightarrow \exists\, j$ s.t. $c\, b_j - k_j > 1$
- By Kollár-Shokurov connectedness, we may assume $E_i \cap E_j \neq 0$
- If $E_j = \{y_j = 0\}$ then $\nu^* f = y_i^{a_i} y_j^{a_j} \cdot \alpha$ $\quad$ $\nu^*(\tau_{<m}(f)) = y_i^{a_i} y_j^{b_j} \cdot \beta$
- since $\left(\frac{\nu^* f - \nu^* \tau_{<m}(f)}{y_i^{a_i}}\right)\Big|_{E_i} = 0 \Rightarrow a_j = b_j$
- But $1 \geq c a_j - k_j = c\, b_j - k_j > 1$ ↯

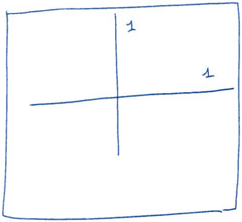
Further results

- The method of de Fernex, Ein, Mustata, Kollár generalizes to studying LCT's on varieties with bounded singularities (defined by equations of bounded degree).
- This is still too restrictive for most applications. However using the full strength of the MMP, Hacon-M^{c}Kernan-Xu prove the analog for log canonical pairs.

Log pairs

- The easiest example of **log canonical pair** (X, B) is given by X a smooth manifold (defined by polynomial eqn's) and $B = \sum b_i B_i$ a simple normal crossings divisor with coefficients $0 \leq b_i \leq 1$.
- In general we allow X to be mildly singular in codimension ≥ 2 (X is normal) and we let K_X be the canonical divisor. It corresponds to the zeroes of a section of ω_X (the line bundle locally defined on the smooth locus by $dx_1 \wedge \ldots \wedge dx_n$).
- We also require that $K_X + B$ is $\mathbb{R}$-**Cartier** i.e. is an $\mathbb{R}$-combination of divisors defined by rational functions (in particular $K_X + B$ pulls-back to log resolutions).
- Then (X, B) is **log canonical** if for any resolution $\nu : Y \to X$, with $K_Y + B_Y = \nu^*(K_X + B)$, the coefficients of B_Y are ≤ 1.
- The Jacobian (or $K_{Y/X}$) corresponds to the difference between $dx_1 \wedge \ldots \wedge dx_n$ and $dy_1 \wedge \ldots \wedge dy_n$, i.e. to $K_Y - \nu^* K_X$.

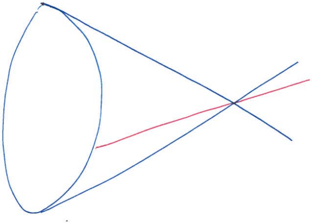
Examples of log canonical pairs



X smooth

B SNC

coefficients $0 \le b_i \le 1$



X = cone over $\mathbb{P}^1$

B = line on the cone

Generalization to log pairs

- Fix a DCC set $I \subset [0,1]$ (DCC means that any non increasing sequence is eventually constant; eg. $I = \{1 - \frac{1}{n} | n \in \mathbb{N}\}$).
- If (X, B) is log canonical and $M = \sum m_i B_i$ is $\mathbb{R}$-Cartier with $m_i \in \mathbb{N}$, then define

$$\mathrm{lct}(X, B; M) = \sup\{t \in \mathbb{R} | (X, B + tM) \text{ is log canonical}\}.$$

- In the SNC case this just means that $b_i + tm_i \leq 1$.
- Let $\mathcal{T}_n(I)$ be the set of all such $\mathrm{lct}(X, B; M)$. Then:

Theorem (Shokurov's ACC for LCTs Conjecture (H-M-X))

- *For any positive integer n, the set $\mathcal{T}_n(I)$ satisfies the ACC.*
- *If $\bar{I} = 1$, then $\overline{\mathcal{T}_n(I)} = \mathcal{T}_n(I) \setminus 1$.*

In the SNC case $\mathcal{T}_n(I) = \{\frac{1-b_i}{m_i} | b_i \in I,\ m_i \in \mathbb{N}\}$.

Shokurov's ACC for LCTs Conjecture

- Unluckily the proof uses all of the recent result of the minimal model program due to Birkar, Cascini, Hacon, M$^{\mathrm{c}}$Kernan, Siu, Xu and others.
- The main idea is to rephrase the local problem as a global one.
- Suppose that the the singularity $p \in (X, B)$ is a cone over a 1-dimensional pair $(C, P = \sum p_i P_i)$, then (X, B) is log canonical iff $\mathrm{vol}(K_C + P) := 2g - 2 + \sum p_i \leq 0$.
- Not suprisingly, for bigger g and p_i we get worse singularities.
- Similarly, questions about the singularities (LCT's) of (X, B) can be rephrased in terms of global properties of (C, P).

Global to local

- Using techniques from the minimal model program [BCHM], we show that the LCT's at $p \in (X, B)$ are measured by an exceptional divisor $E \subset Y \xrightarrow{\nu} X$ (this is the analog of blowing up the vertex of a cone).
- If $\lambda_i = \mathrm{lct}_{p_i}(X_i, B_i; M_i)$ is an increasing sequence with limit λ, then by adjunction one considers (E_i, P_i) where

$$K_{E_i} + P_i = (K_{Y_i} + E_i + \nu_{i,*}^{-1}(B_i + \lambda M_i))|_{E_i}.$$

- If $n = \dim E$, then

$$\mathrm{vol}(K_E + P) = \lim \frac{\dim H^0(m(K_E + P))}{m^n/n!} > 0.$$

- We then prove the corresponding result for volumes (instead of LCT's).

Log boundedness

Theorem (Hacon-M$^{\mathrm{c}}$Kernan-Xu)

Fix $n \in \mathbb{N}$, $I \subset [0,1]$ a DCC set, then there is a constant $M > 0$ such that if $V = \{\mathrm{vol}(K_X + B)\}$ where (X, B) are log canonical pairs, $\dim X = n$ and the coefficients of B are in I, then V satisfies the DCC (and in particular has a positive minimum).

Eg. if $\dim X = 1$ and $B = \sum b_i B_i$, $b_i \in \{1 - \frac{1}{m} | m \in \mathbb{N}\}$, then $V = \{\mathrm{vol}(K_X + B) = 2g - 2 + \sum b_i\}$ is a DCC set with positive minimum $\frac{1}{42}$.

Refs

- Thanks to NSF and Simons Foundation for generous support.
- Useful references:

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