Finite generation of canonical rings II

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Outline of the talk

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1. The structure of the proof
2. The directed MMP
3. Existence of models
4. Finiteness of models

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  $A_{n-1}+B_{n-1}+C_{n-1}$ imply A_n;
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  $B_{n-1}+C_{n-1}+A_n$ imply B_n;
  $C_{n-1}+A_n+B_n$ imply C_n.
- We will use the fact that by a result of Hacon-M[c]Kernan, $A_{n-1}+B_{n-1}+C_{n-1}$ imply that PL-flips exist in dimension n.

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- **Thm A.** (Existence of log terminal models.) *Suppose that $K_X + \Delta$ is KLT and Δ is big over U. If $K_X + \Delta \sim_{\mathbb{R},U} D \geq 0$, then $K_X + \Delta$ has a log terminal model over U.*

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  If $K_X + \Delta \sim_{\mathbb{R},U} D \geq 0$, then $K_X + \Delta$ has a log terminal model over U.
- **Thm B.** (Non-vanishing.) *Suppose that $K_X + \Delta$ is KLT and Δ is big over U.*
  If $K_X + \Delta$ is π-pseudo-effective over U, then there is an effective $\mathbb{R}$-divisor D such that $K_X + \Delta \sim_{\mathbb{R},U} D$.

The structure of the proof II

- **Thm C. (resp. $\mathbf{C}^-$)** (Finiteness of models.) *Fix A an ample divisor over U. Suppose that there exists a divisor Δ_0 such that $K_X + \Delta_0$ is KLT.*
  Let $\mathcal{C} \subset \{\Delta | K_X + \Delta + A$ is log canonical$\}$ be a rational polytope (resp. a rational polytope s.t. $\mathcal{C} + K_X + A$ is contained in the big cone).
  Then the set of isomorphism classes

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- We work over the field of complex numbers $\mathbb{C}$.
- Throughout the proof I will usually assume for simplicity that $U = \mathrm{Spec}\mathbb{C}$.

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- If $\lambda = 0$ we STOP. Otherwise there is an extremal ray R such that $(K_X + \Delta + \lambda C) \cdot R = 0$ and $(K_X + \Delta + tC) \cdot R < 0$ for all $0 \leq t < \lambda$.

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- If the contraction induced by R is not birational, we have a Mori fiber space and we STOP.
- If the contraction induced by R is birational, we replace X by the corresponding flip/divisorial contraction and repeat the procedure.

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- In terms of our induction, we are only allowed to assume Thm C_{n-1} and this makes the proof more technical. (An analog of special termination will be required.)

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- **Lemma.** Suppose that

$$K_X + \Delta = K_X + S + A + B$$

is DLT where A is ample and $B \geq 0$.
Then any sequence of flips $X = X_1 \dashrightarrow X_2 \dashrightarrow X_3 \cdots$ for the $K_X + \Delta$-MMP with scaling is eventually disjoint from $S = \lfloor \Delta \rfloor$.

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- **Proof.** We may assume that S is irreducible and that the induced maps $S_i \dashrightarrow S_{i+1}$ are isomorphisms in codim. 1.
- We write $(K_{X_i} + \Delta_i)|_{S_i} = K_{S_i} + \Theta_i$.

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- If Σ_i is a flipping curve for $X_i \dashrightarrow X_{i+1}$ that intersects S_i, then either $\Sigma_i \subset S_i$ or $\Sigma_i \cdot S_i > 0$ in which case the flipped curves are contained in S_{i+1}. This means that there is a center ν such that $a(\nu, S_i, \Theta_i) < a(\nu, S_{i+1}, \Theta_{i+1})$.

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- Since $a(\nu, S_{i+1}, \Theta_{i+1}) \leq a(\nu, S_j, \Theta_j)$ then $S_j \not\cong S_i$ for all $j > i$. But there are only finitely many models S_j and so the flips are eventually disjoint from S_j.

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- Recall that $K_X + \Delta \sim_{\mathbb{R}} D \geq 0$ and write $D = M + F$ where $\mathrm{Supp}(F) \subset \mathrm{SBs}(K_X + \Delta)$ and M is mobile.

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- For simplicity assume that M is irreducible (this avoids some linear algebra).
- Set $\Theta = \mathrm{Supp}(M + F) \vee \Delta = S + A + B'$ where $0 \leq B' \leq B$. (If $G = \sum g_i G_i$, $G' = \sum g_i' G_i$ then $G \vee G' := \sum \max\{g_i, g_i'\} G_i$.)

Proof of Claim 1, Cont.

- we will peform a sequence of flips/contractions (for various divisors $\Theta_i \geq \Delta$) such that:
  1) the induced rational map $\phi : X \dashrightarrow Y$ only contracts divisors in $\mathrm{SBs}(K_X + \Delta)$, and
  2) $K_Y + \Gamma$ is nef for some divisor $\Gamma = \phi_*(\Delta + \Phi)$ where $\mathrm{Supp}(\Phi) \subset \mathrm{SBs}(K_X + \Delta)$.

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- It is then not hard to show that Y is a log terminal model for (X, Δ).

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  and so the flipping locus is contained in S.
- Therefore, this is a sequence of PL-flips that must terminate and we may assume that $K_X + \Theta$ is nef.

Proof of Claim 1, Step 2

- We wish to remove M, so we write

$$K_X + S + A + B' = K_X + F_{\rm red} + A + B' + M_{\rm red}$$

and we run a MMP with scaling of $M_{\rm red}$.

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- By a similar argument to the one above, the flipping locus is always contaned in F_{red} so that this is a sequence of PL-flips that must terminate.
- Therefore, we have a model where $K_X + S + A + B' = K_X + \Delta + \Phi$ is nef and $\Phi \subset \mathrm{SBs}(K_X + \Delta)$. As remarked above, this is a log terminal model of (X, Δ).

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Proof of Claims 2 and 4

- In proving Thm C (resp. C-), we need to know that for any pseudo-effective class $K_X + \Delta$ that we consider, there is an effective divisor $D \geq 0$ such that $K_X + \Delta \sim_{\mathbb{R}} D$.

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- This will hold as we are assuming Thm B (resp. we are assuming that $K_X + \Delta$ is big).
- We will just show that there exists an integer $k > 0$ and rational maps $\phi_i : X \dashrightarrow Y_i$ such that if $\Delta \in \mathcal{C}$ is pseudo-effective then there is an integer i with $1 \leq i \leq k$ such that ϕ_i is a log terminal model for (X, Δ).

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- ϕ is $K_X + \Delta_0$ negative in the sense that for any exceptional divisor $E \subset X$, $a(E, X, \Delta_0) < a(E, Y, \phi_* \Delta_0)$.

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- So after shrinking U, we have that ϕ is $K_X + \Delta$ negative for all $\Delta \in U$.
- So we may replace X by Y and assume that $K_X + \Delta_0$ is nef.
- In particular, as Δ_0 is big, by the base point free theorem, $K_X + \Delta_0 = f^* H$ for some morphism $f : X \to Z$ and some ample $\mathbb{R}$-divisor H on Z.

Proof of Claims 2 and 4 cont.

- Since $\mathcal{C}$ is a rational polytope, we may assume that U is a rational polytope and so by induction on the dimension (of the affine space spanned by the polytope) there exist finitely many rational maps $\phi_i : X \dashrightarrow Y_i$ over Z such that for all $\Delta \in \partial\mathcal{C} \cap \mathrm{PSEF}(X/Z)$ then ϕ_i is a log terminal model of (X, Δ) for some $1 \leq i \leq k$.

Proof of Claims 2 and 4 cont.

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- Let $\Delta_0 \in \partial U$. For any $\Theta \in [\Delta_0, \Delta]$, we have

$$\Theta - \Delta_0 = (K_X + \Theta) - (K_X + \Delta_0) \sim_{\mathbb{R},Z} (K_X + \Theta)$$

so that $\Theta - \Delta_0$ is PSEF/Z iff $K_X + \Theta$ is PSEF/Z iff $K_X + \Delta$ is PSEF/Z.

Proof of Claims 2 and 4 cont.

- We have seen that $K_X+\Delta_0=f^*H$ and $K_X+\Delta_0\sim_{\mathbb{R},Z}0$
- Further shrinking U, we may assume that for all $\Delta\in U$ $K_X+\Delta$ is nef if and only if $K_X+\Delta$ is nef/Z.
- Let $\Delta_0\in\partial U$. For any $\Theta\in[\Delta_0,\Delta]$, we have

$$\Theta-\Delta_0=(K_X+\Theta)-(K_X+\Delta_0)\sim_{\mathbb{R},Z}(K_X+\Theta)$$

  so that $\Theta-\Delta_0$ is PSEF/Z iff $K_X+\Theta$ is PSEF/Z iff $K_X+\Delta$ is PSEF/Z.
- If $K_X+\Theta$ is PSEF then it is PSEF/Z and so $K_X+\Delta$ is PSEF/Z. A log terminal model/Z of $K_X+\Delta$ is a log terminal model/Z of $K_X+\Theta$ and hence it is a log terminal model of $K_X+\Theta$.

Outline of the talk

Proof of Claim 3

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- If for any fixed $k \gg 0$ sufficiently divisible,

$$h^0(\mathcal{O}_X([mk(K_X + \Delta)] + kA))$$

is a bounded function of m, then by a result of Nakayama, we have $K_X + \Delta \equiv N_\sigma \geq 0$. Recall that $N_\sigma = N_\sigma(K_X + \Delta)$ is the limit as $\epsilon \to 0$ of divisors in $\mathrm{SBs}(K_X + \Delta + \epsilon A)$.

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- So $A' \sim_{\mathbb{R}} A + N_\sigma - (K_X + \Delta)$ is ample and $K_X + A' + B \sim_{\mathbb{R}} N_\sigma \geq 0$ and hence $K_X + A' + B$ has a log terminal model which is also a log terminal model for $K_X + A + B$.

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- By the base point free theorem $K_X + A + B \sim_{\mathbb{R}} D \geq 0$.

Proof of Claim 3 cont.

- **Claim.** If $h^0(\mathcal{O}_X([mk(K_X + \Delta)] + kA))$ is not bounded, then we may assume that $\Delta = S + A + B$ where $[\Delta] = S$, A is ample, $S \not\subset N_\sigma(K_X + \Delta)$.

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- **Proof.** Pick $H \sim_{\mathbb{R}} m(K_X+\Delta)+A$ with $\mathrm{mult}_x(H) > n$ for some general $x \in X$.

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- **Proof.** Pick $H \sim_{\mathbb{R}} m(K_X+\Delta)+A$ with $\mathrm{mult}_x(H) > n$ for some general $x \in X$.
- For $t \in [0, m]$ consider

$$(t+1)(K_X+\Delta) = K_X + \frac{m-t}{m}A + B + t(K_X+\Delta+\frac{1}{m}A) \sim_{\mathbb{R}}$$

$$K_X + \frac{m-t}{m}A + B + \frac{t}{m}H = K_X + \Delta_t.$$

Proof of Claim 3 cont.

- Then for some $0 < \epsilon \ll 1$ we have $K_X + \Delta_0$ is KLT; $\Delta_t \geq A' = (\epsilon/m)A$ for $t \in [0, m - \epsilon]$, and $K_X + \Delta_{m-\epsilon}$ has a log canonical center not contained in $\mathrm{SBs}(K_X + \Delta)$.

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- We pass to a log resolution $\pi : Y \to X$ and write

$$K_Y + \Gamma_t = \pi^*(K_X + \Delta_t) + E_t$$

and we then cancel common components of Γ_t and $\mathrm{SBs}(\pi^*(K_X + \Delta_t))$. The claim now follows easily.

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- For $0 < t \ll 1$ we get log terminal models $\phi_t : X \dashrightarrow Y_t$ of $K_X + S + A + B + tA$ and $S \dashrightarrow T_t$.

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- S is not contained in $\mathrm{SBs}(K_X + S + A + B)$ and so it is not contracted.

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- S is not contained in $\mathrm{SBs}(K_X + S + A + B)$ and so it is not contracted.
- We would like to say that for $0 < t \ll 1$, we have $Y_t \cong Y$ a log terminal model of $K_X + S + A + B$. Then by the base point free theorem, $K_X + S + A + B \sim_{\mathbb{R}} D \geq 0$.

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- S is not contained in $\mathrm{SBs}(K_X+S+A+B)$ and so it is not contracted.
- We would like to say that for $0<t\ll 1$, we have $Y_t \cong Y$ a log terminal model of $K_X+S+A+B$. Then by the base point free theorem, $K_X+S+A+B \sim_{\mathbb{R}} D \geq 0$.
- We can only conclude that $T_t \subset Y_t \cong T \subset Y$ in a neighborhood of T_t.
- So $K_T+\Theta=(K_{Y_t}+\Gamma_t)|_T$ is nef, Θ is big and hence $K_T+\Theta$ is semiample.

Proof of Claim 3 cont.

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- For simplicity assume $K_X + \Delta$ is $\mathbb{Q}$-Cartier (this avoids a diophantine approximation argument).
- We may then assume that for some integer $m > 0$, we have $m(K_X + \Delta)$ is integral Weil, $m(K_Y + \Gamma)$ is Cartier in a neighborhood of T and $h^0(m(K_T + \Theta)) > 0$,

Proof of Claim 3 cont.

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- For simplicity assume $K_X + \Delta$ is $\mathbb{Q}$-Cartier (this avoids a diophantine approximation argument).
- We may then assume that for some integer $m > 0$, we have $m(K_X + \Delta)$ is integral Weil, $m(K_Y + \Gamma)$ is Cartier in a neighborhood of T and $h^0(m(K_T + \Theta)) > 0$,
- Then we look at the corresponding short exact sequence and apply Kawamata-Viehweg vanishing:

Proof of Claim 3 cont.

- We have

$$0 \to \mathcal{O}_{Y_t}(m(K_{Y_t} + \Gamma) - T) \to \mathcal{O}_{Y_t}(m(K_{Y_t} + \Gamma))$$

$$\to \mathcal{O}_T(m(K_T + \Theta)) \to 0.$$

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$$\to \mathcal{O}_T(m(K_T + \Theta)) \to 0.$$

- Then $K_{Y_t} + \Gamma - T - (m-1)tA$ is KLT (for $0 < t \ll 1$) and $(m-1)(K_{Y_t} + \Gamma + tA)$ is nef and big.

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$$\to \mathcal{O}_T(m(K_T + \Theta)) \to 0.$$

- Then $K_{Y_t} + \Gamma - T - (m-1)tA$ is KLT (for $0 < t \ll 1$) and $(m-1)(K_{Y_t} + \Gamma + tA)$ is nef and big.
- By Kawamata-Viehweg vanishing, we may lift sections to $m(K_{Y_t} + \Gamma)$.

Proof of Claim 3 cont.

- We have

$$0 \to \mathcal{O}_{Y_t}(m(K_{Y_t}+\Gamma)-T) \to \mathcal{O}_{Y_t}(m(K_{Y_t}+\Gamma))$$
$$\to \mathcal{O}_T(m(K_T+\Theta)) \to 0.$$

- Then $K_{Y_t}+\Gamma - T-(m-1)tA$ is KLT (for $0<t\ll 1$) and $(m-1)(K_{Y_t}+\Gamma+tA)$ is nef and big.
- By Kawamata-Viehweg vanishing, we may lift sections to $m(K_{Y_t}+\Gamma)$.
- It is easy to see that these sections lift to sections of $K_X+\Delta$.