Extension theorems and their applications to birational geometry

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1) Extension Theorems:

The purpose of this talk is to show how appropriate generalizations of the extension theorems of Siu can be applied to problems in birational geometry. We begin by recalling Siu's result:

Theorem 1. *(Siu, 98, 02) Let $f : X \longrightarrow T$ be a smooth projective morphism of smooth quasi-projective varieties. Then for all $m \geq 1$ the plurigenera*

$$P_m(X_t) := h^0(X_t, \mathcal{O}_{X_t}(mK_{X_t}))$$

are independent of t.

By semicontinuity, it suffices to show that for any point $0 \in T$, the sections of $\mathcal{O}_{X_0}(mK_{X_0})$ extend over some neighborhood $0 \in U_m \subset T$.

Theorem 2. *(Kawamata, 99) Let* $f : X \longrightarrow T$ *be a flat morphism from a germ of an algebraic variety to a germ of a smooth curve such that the central fiber* X_0 *has canonical singularities. Then* X *has also canonical singularities and so does any fiber* X_t*.*

Kawamata also showed that if X_η is of general type, then the plurigenera are are constant. Nakayama proved analogous results for terminal varieties.

These results are a direct consequence of an extension theorem which we will now illustrate.

Theorem 3. *Let $S \subset X$ be a smooth divisor in a smooth variety, $f : X \longrightarrow Z$ a projective morphism of quasi-projective varieties. If*

$$K_X + S \sim A + B$$

where A is f-ample and B is f-effective and doesn't vanish along S, then

$$f_* \mathcal{O}_X(m(K_X + S)) \longrightarrow f_* \mathcal{O}_S(mK_S)$$

is surjective for all $m > 0$. (I.e. "sections of mK_S extend.")

To prove such a result, roughly speaking, one first shows that sections of $\mathcal{O}_S(mK_S + H)$ extend (for some fixed sufficiently ample H) and then one applies a limiting procedure.

One reason that makes extending sections from a subvariety very useful is that it simplifies the problem at hand by allowing one to do an induction on the dimension of the ambient variety (eg. Basepoint-free Theorem). Unluckily the previous extension result is not sufficiently flexible. From the point of view of birational algebraic geometry, one would hope for a statement that applies to log pairs. Usually this is done by using the Kawamata-Viehweg vanishing theorem:

If X is smooth, (X,B) is klt and A is nef and big and $L \sim A+B$ is integral then $H^1(X, \mathcal{O}_X(K_X+L)) = 0$ and so the map

$$H^0(X, \mathcal{O}_X(K_X+S+L)) \longrightarrow H^0(X, \mathcal{O}_S(K_S+L))$$

is surjective.

So one hopes that for all $m > 0$, the maps

$$H^0(X, \mathcal{O}_X(m(K_X+S+L))) \longrightarrow H^0(X, \mathcal{O}_S(m(K_S+L)))$$

are surjective (when mL is Cartier). If $K_X + S + L$ is nef, this easily follows, but in general there is a problem!

Let

$$f : X \longrightarrow \mathbb{P}^2$$

be the blow up at a point $0 \in \mathbb{P}^2$, E the exceptional divisor and S (resp H) the strict transforms of lines containing 0 (resp. not containing 0). Consider

$$m(K_X + S + \frac{3}{2}H + \frac{1}{2}E)|_S \sim -\frac{m}{2}S|_S \sim 0$$

One sees that sections do not extend.

Never-the-less, Tsuji's work, shows that such a generalization is indeed possible and very useful.

Theorem 4. *(Tsuji, Takayama) Let $S \subset X$ be a smooth divisor in in a smooth variety, L an integral divisor on X such that $L \sim_{\mathbb{Q}} A + B$ where A is ample and B is effective B does not contain S and $(S, B|_S)$ is klt. Then*

$$H^0(X, \mathcal{O}_X(m(K_X+S+L))) \longrightarrow H^0(S, \mathcal{O}_S(m(K_S+L)))$$

is surjective for all $m > 0$.

N.B. 1) In the example above, $K_X + S + L$ is not Cartier.

2) From the point of view of the MMP the requirement that $K_X + S + L$ be Cartier is too restrictive.

We were able to prove the following result:

Theorem 5. *(Hacon-M^{c}Kernan) Let $S \subset X$ be a smooth divisor in a smooth projective variety, $B = \sum b_i B_i$ a $\mathbb{Q}$-divisor with with $0 < b_i < 1$ such that $S + B$ is a divisor with simple normal crossings. Assume that*

$$B \sim_{\mathbb{Q}} A + D$$

where A is ample and D is effective not containing S and that there is a divisor

$$G \sim_{\mathbb{Q}} K_X + S + B$$

not containing any of the log canonical centers of $(X, \ulcorner S + B \urcorner)$, then

$$H^0(X, \mathcal{O}_X(m(K_X+S+B))) \longrightarrow H^0(S, \mathcal{O}_S(m(K_S+B)))$$

is surjective for any $m > 0$ such that $m(K_X + S + B)$ is Cartier.

The proof follows the ideas of Siu. One proceeds inductively comparing the multiplier ideals of $|t(K_X+S+B)|_S|$ and $\frac{1}{k}|tk(K_X+S+B)|$ for $k \gg 0$. These ideals measure the singularities of the base loci of the corresponding linear series. The condition that "$G \sim_{\mathbb{Q}} K_X+S+B$ does not contain any of the log canonical centers of $\ulcorner S+B \urcorner$" allows us to avoid technical problems that occur when the singularities of $(X, S+B)$ and of the base locus of $|tk(K_X+S+B)|$ are not disjoint.

There are 3 main applications that I would like to discuss:

- "Boundedness of pluricanonical maps"
- "Rational curves on varieties with mild singularities"
- "Existence of flips"

2) Boundedness of pluricanonical maps

Theorem 6. *(Tsuji, Takayama, Hacon-M^cKernan) For any positive integer n, there exists a positive integer r_n such that if X is a smooth variety of general type and dimension n, then $\phi_{rK_X} : X \dashrightarrow \mathbb{P}(H^0(\mathcal{O}_X(rK_X)))$ is birational for all $r \geq r_n$.*

Tsuji's idea: It suffices to show that there exist A, B are positive constants depending only on $n = \dim(X)$ such that for any

$$r \geq \frac{A}{(\mathrm{vol}(K_X))^{1/n}} + B$$

then, the map ϕ_{rK_X} is birational. If $\mathrm{vol}(K_X) \geq 1$, the theorem is clear. If $\mathrm{vol}(K_X) < 1$, one then shows that X belongs to a birationally bounded family and hence there is a uniform lower bound for the volume of K_X.

In fact, let $\lambda = A/(\text{vol}(K_X))^{1/n} + B$ and let Z be the image of $\phi_{\lambda K_X}$, one sees that

$$\deg(Z) \leq \text{vol}(\lambda K_X) = \lambda^n \text{vol}(K_X) \leq (An+B)^n.$$

In order to show that the maps ϕ_{rK_X} are birational, it suffices to show that there is an open set $U \subset X$ such that sections of rK_X separate arbitrary points $x, y \in U$. Therefore, it suffices to define a $\mathbb{Q}$-divisor $G \sim \lambda K_X$ with isolated log canonical centers at x, y. This is a technically condition which means that $\mathcal{J}(G) = \mathcal{J}_{x,y}$ in a neighborhood of x, y. (Roughly speaking this means that G has high multiplicity at x, y and low multiplicity in a neighborhood of x, y.) Then one has that $H^1(\mathcal{O}_X((\lambda+1)K_X \otimes \mathcal{J}(G))) = 0$ and hence the map

$$H^0(\mathcal{O}_X((\lambda+1)K_X)) \longrightarrow \mathbb{C}_{x,y}$$

is surjective, as required.

It is straightforward to produce a $\mathbb{Q}$-divisor $G \sim \lambda K_X$ with nontrivial log canonical center V_x at x. The point now is to cut down this center to a point. This can be achieved if there are many sections in the image of

$$H^0(X, \mathcal{O}_X(mK_X)) \longrightarrow H^0(V_x, \mathcal{O}_{V_x}(mK_X)).$$

Since X is of gen. type and $x \in X$ is general, we may assume that V_x is of general type. By Kawamata's Subadjunction, one expects that $(1+\lambda)K_X|_{V_x} \geq K_{V_x}$. By induction on the dimension, mK_{V_x} has enough sections. The main difficulty is then to lift these sections to X. If K_X is ample, this is immediate. In general, it is a very delicate statement. Using the extension results explained above, we are able to achieve this on an appropriate log resolution $Y \longrightarrow X$.

3) Rational curves on varieties with mild singularities.

Here "mild singularities" is to be interpreted from the point of view of the MMP.

Theorem 7. *Let (X, Δ) be a log pair, $f : X \longrightarrow S$ a projective morphism such that $-K_X$ is relatively big and $\mathcal{O}_X(-m(K_X + \Delta))$ is relatively generated for some $m > 0$. Let $g : Y \longrightarrow X$ be any birational morphisms. Then every fiber of $\pi := f \circ g$ is rationally chain connected modulo $g^{-1}LCS(X, \Delta)$.*

That is, for any two points of any fiber, there is a chain of curves connecting these points such that each curve is either rational or contained in $g^{-1}LCS(X, \Delta)$.

When $S = Spec\mathbb{C}$ we get the following result of Q. Zhang:

Theorem 8. *Let (X, Δ) be a KLT pair such that $-(K_X + \Delta)$ is nef and big, then X is rationally connected (i.e. two general points can be joined by a rational curve).*

When $X = S$ we have:

Theorem Let (X, Δ) be a KLT pair and $g : Y \longrightarrow X$ a birational morphism, then the fibers of g are rationally chain connected.

In particular if (X, Δ) is a KLT pair, then:

- if $g : X \dashrightarrow Z$ is a rational map to a proper variety which is not everywhere defined, then Z contains a rational curve.

- X is rationally chain connected if and only if it is rationally connected.

The statement is sharp:

Let $f : S \longrightarrow C$ be a $\mathbb{P}^1$ bundle over an elliptic curve and E a section of minimal self intersection $E^2 < 0$. Contracting E we get a surface T which is rationally chain connected but not rationally connected. Notice that

$$K_S + tE$$

is KLT for $0 \leq t < 1$ and LC for $t = 1$ but $-(K_S + tE)$ is nef for $1 \leq t \leq 2$ and ample for $1 < t < 2$.

N.B. T is RCC but not RC, so RCC is not a birational property (the point is that $(T, \emptyset)$ is not KLT).

$S \to C$ is the MRC fibration: a surjective map with connected fibers which are RC, and the base is not uniruled (see the result of Graber-Harris-Starr).

Further consequences: Corollary: Let (X, Δ) be a projective log pair such that $-(K_X+\Delta)$ is semiample and $-(K_X+\Delta)$ is big. Then $\Pi_1(X)$ is a quotient of $\Pi_1(LCS(X,\Delta))$.

E.G. (Zhang): If (X, Δ) is KLT, $-(K_X + \Delta)$ is nef and big then X is simply connected.

Theorem Let (X, Δ) be a KLT pair, $f : X \longrightarrow S$ a projective morphism with connected fibers such that $-K_X$ is relatively big and $-(K_X+\Delta)$ is relatively nef for some $m > 0$. Let $g : Y \longrightarrow X$ be any birational morphisms.Then
1) the natural map

$$\pi_* := (f \circ g)_* : CH^0(Y) \longrightarrow CH^0(S)$$

is an isomorphism.
2) π has a section over any curve.

Idea of the proof Suppose that:

1) Y is smooth and the fiber of $\pi : Y \longrightarrow X \longrightarrow S$ is a smooth divisor E.

2) We may write $K_Y + E + D \sim F$ where D, F are effective with no common components, D contains an ample divisor, F is g-exceptional $\llcorner D \lrcorner = 0$, $D + E + F$ has simple normal crossings.

3) The MRC fibration $E \longrightarrow Z$ is a morphism.

Recall that the fibers of $E \longrightarrow Z$ are rationally connected and Z is not uniruled. By a result of Boucksom, Demailly, Paun and Peternell, K_Z is pseudo effective. I.e. given an ample divisor H on Z, for all $\epsilon > 0$

$$K_Z + \epsilon H \qquad \textit{is big.}$$

By log additivity of the Kodaira dimensions, since $K_E + D$ is effective and D contains an ample divisor, sections of

$$m(K_Z + \epsilon H)$$

lift to sections of

$$m(K_E + D|_E).$$

By the extension result, these sections lift to sections of

$$m(K_Y + E + D) \sim mF$$

which is exceptional and therefore has $\kappa(F) = 0$. But then as $K_Z + \epsilon H$ is big, one sees that $\dim Z = 0$ and so, by definition of MRC-fibration, E is rationally connected.

4) Existence of flips.

There are two main problems in the MMP: A) existence and B) termination of flips. Shokurov has shown that assuming the MMP in dimension $n-1$, to prove A) in dimension n, it suffices to construct all **pl flips**.

Definition 1. *A morphism $f : X \longrightarrow Z$ is a pl flipping contraction if*

1) f is a small birational contraction with relative Picard number 1,

2) X is $\mathbb{Q}$-factorial,

3) $K_X + S + B$ is purely log terminal, where S is irreducible, and

4) $-(K_X + S + B)$ and $-S$ are f-ample.

The flip of f (if it exists) is a small birational contraction $f' : X' \longrightarrow Z$, such that $K_{X'} + S' + B'$ is f'-ample (where $S' + B'$ is the strict transform of $S + B$).

If it exists, it is given by

$$f' : X' = \mathrm{Proj}_Z \bigoplus_{n \in \mathbb{N}} f_* \mathcal{O}_X(nD) \longrightarrow Z$$

where D is any positive rational multiple of $K_X + S + B$.

Therefore, to prove existence of flips, it suffices to show that the $\mathcal{O}_Z$-algebra $\bigoplus f_* \mathcal{O}_X(nD)$ is finitely generated.

The advantage of considering a pl-flip is that Shokurov has shown that $\mathcal{R} = \bigoplus f_*\mathcal{O}_X(nD)$ is finitely generated if and only if its image $\mathcal{R}|_S \subset \bigoplus f_*\mathcal{O}_S(nD|_S)$ is finitely generated.

The advantage is 2-fold:
– First of all this is now a problem in dimension $n-1$ and therefore one can use results from the MMP.
– Secondly, $-(K_X+S+B)|_S = -(K_S+Diff_S(B))$ is ample i.e. $(S, Diff_S(B))$ is a log Fano (relative to Z) and one expects these varieties to be more tractable. In particular Shokurov conjectures that a very general class of algebras on log-Fano varieties are finitely generated. This conjecture would in particular imply that $\mathcal{R}|_S$ is finitely generated and that flips exist.

The main technical difficulty, is that $\mathcal{R}|_S$ is not a divisorial algebra that is one does not know that there is a divisor $D' \leq D|_S$ on S such that $\mathcal{R}|_S = \bigoplus f_*\mathcal{O}_S(nD')$. Using the extension result, we are however able to show that (on an appropriate resolution of S), there exists a positive integer $k > 0$ and sequence of divisors D_m such that

$$\mathcal{R}|_S = \bigoplus f_*\mathcal{O}_S(D_m)$$

where (replacing S by an appropriate birational model)

1) $D_m = km(K_S + B_m)$ is an integral additive sequence, that is $B_i + B_j \leq B_{i+j}$

2) The limit $B = \lim B_m$ exists and $K_S + B$ is kawamata log terminal.

If one assumes the real MMP in dimension $n-1$, then we may assume that the moving part of each D_m is free and the moving part of K_S+B is semiample. By Kawamata-Viehweg vanishing, since $-(K_X+S+B)$ is ample, it is easy to see that the restricted algebra is saturated (i.e. there exists a $\mathbb{Q}$-divisor $F = K_Y + T - g^*(K_X + S + B)$ with $\ulcorner F \urcorner \geq 0$ such that for all $i, j > 0$, one has

$$\mathrm{Mov}(\ulcorner jD_i + F \urcorner) \leq D_j.$$

Following Shokhurov's ideas (Diophantine Approximation), it is then easy to sees that this algebra is finitely generated.

It suffices therefore to find an appropriate birational model T of S on which the restricted algebra is of the form $\mathcal{R}|_T = \bigoplus f_*\mathcal{O}_T(D_m)$.

The main point is the following: The algebra $\mathcal{R}$ is determined by the moving parts of $|m(K_X + S + B)|$. Let $g : Y \longrightarrow X$ be a log resolution so that we may write

$$K_Y + T + B' = g^*(K_X + S + B) + E$$

where $E, T + B'$ are effective, have no common component, E is exceptional and $T = (g^{-1})_* S$. Then, the moving part of $|m(K_Y + T + B')|$ is just the pull-back of the moving part of $|m(K_X + S + B)|$. In order to apply the extension result, one has to ensure that the log canonical centers of $(Y, \ulcorner T + B' \urcorner)$ are not contained in the base locus of $km(K_Y + T + B')$. We may assume that all components of B' are disjoint. Since T is not in the base locus, we must only worry about components of B' and of $B' \cap T$.

Canceling common components we may write the decomposition into mobile and fixed parts as

$$|mk(K_Y + T + B_m)| = |M_m| + G_m$$

where $T + B_m$ and G_m have no common components. Blowing up along components of $B_m \cap T$, we may assume that the moving part does not vanish along any of these components. Note that these components are codimension 1 in T and so this does not affect T! So all sections of $mk(K_T + B_m)$ extend to Y as required.