The birational geometry of algebraic varieties

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Outline of the talk

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Curves

- Curves are topologically determined by their genus.
- $g = 0$: $\mathbb{P}^1$
- $g = 1$: There is a 1-dimensional family of elliptic curves; $K = 0$.
- $g \geq 2$: Curves of general type.
  These form a $3g - 3$-dimensional family.
  K is ample and $3K$ determines an embedding.
- Any two birational curves are isomorphic.

Birational maps

- Given any surface S, one can make a new birational surface S' by blowing up a point $o \in S$. There is a natural map

$$S' \to S$$

  which is an isomorphism over $S - o$ and replaces o by an **exceptional curve** E.
- One sees that $E \cong \mathbb{P}^1$ and $E^2 = K_S \cdot E = -1$. Any curve with these properties can be blown down as above (Castelnuovo's Contractability Criterion).
- If S_1 and S_2 are birational surfaces (i.e. they have isomorphic open subsets), then S_2 can be obtained from S_1 by a sequence of blow ups followed by a sequence of blow downs.
- It is easy to see that any surface is birational to a **minimal surface**: that is a surface with no exceptional curves.

Kodaira dimension

- The **Kodaira dimension** of a variety is $-\infty$ if $h^0(\omega_X^{\otimes m}) = 0$ for all $m > 0$ and otherwise it is the maximum of the dimension of the image of X under the maps determined by the sections of $\omega_X^{\otimes m}$ for $m > 0$.
- If $\kappa(X) = \dim X$ we say that X is of general type.
- If $\dim X = 1$, then $\kappa(X) = -\infty$ for $g = 0$; $\kappa(X) = 0$ for $g = 1$; $\kappa(X) = 1$ for $g \geq 2$.
- If $\dim X = 2$, then $\kappa(X) \in \{-\infty, 0, 1, 2\}$.

Surfaces with $\kappa = -\infty$

- If $\kappa(X) = -\infty$ and $h^0(\Omega^1_X) = 0$ then X is birational to $\mathbb{P}^2$. In this case $-K_X$ is ample.
- If $\kappa(X) = -\infty$ and $h^0(\Omega^1_X) = q > 0$ then X is birational to a ruled surface over a curve of genus q. Note that the fibers F are rational curves and $F \cdot K_X = -2$.
- The minimal surface is not unique. (But it is understood how two different minimal surfaces are related.)
- However, for $\kappa(X) \geq 0$, the minimal surface is unique.

Minimal surfaces with $\kappa = 0$

- There are 4 possibilities:
  Abelian surfaces ($p_g = 1, h^0(\Omega^1_X) = 2$),
  $K3$ surfaces ($p_g = 1, h^0(\Omega^1_X) = 0$),
  bielliptic surfaces ($p_g = 0, h^0(\Omega^1_X) = 1$) and
  Enriques surfaces ($p_g = 0, h^0(\Omega^1_X) = 0$).
- In all cases $12K_X = 0$.

Minimal surfaces with $\kappa = 1$

- X is an elliptic surface, so it admits a fibration $f : X \to B$ whose fibers are elliptic curves.
- $K_X \sim_{\mathbb{Q}} f^*M$ where $\deg M > 0$. Therefore, $K_X \cdot C \geq 0$ for any curve $C \subset X$ and $K_X^2 = 0$.

Minimal surfaces with $\kappa = 2$

- There are too many families to classify.
- Some general features are well understood.
- For example:
  $P_2(X) > 0$,
  $|5K_X|$ defines a birational map,
  $\chi(\mathcal{O}_X) > 0$,
  $\oplus_{m\in\mathbb{N}} H^0(X, \omega_X^m)$ is finitely generated.

Outline of the talk

Strategy I

- One would like to generalize the above approach to 3-folds and higher dimensional varieties.
- Given a smooth projective 3-fold, one expects that $\kappa(X) = -\infty$ when X is ruled;
  If $\kappa(X) \geq 0$, then X is birational to a **minimal** variety (that is a variety with nef canonical class K).
- In order to achieve this one has to allow varieties with mild singularities and the minimal models will not be unique.

Strategy II

- Mild singularities is vague, and the choice is important. We will insist that :
- X is normal, so K_X is well defined;
- K_X is $\mathbb{Q}$-Cartier (so that one can make sense of $K_X \cdot C$ and of f^*K_X)
- some more technical conditions (e.g. X has terminal singularities so that one can compare sections of $\omega_X^{\otimes m}$ with the corresponding pluri-canonical sections on a resolution)

Strategy III

- We plan to use the canonical class to help us find the minimal model.
- If K_X is nef, we are done.
- If K_X is not nef, then the K_X-negative part of the cone of curves on X is rationally polihedral. We pick a K_X-negative extremal ray R and by the Contraction Theorem, there exists a unique morphism

$$\mathrm{cont}_R : X \to Y$$

  such that a curve $C \subset X$ maps to a point if and only if $[C] \in R$.
- If $\dim X > \dim Y$, then X is ruled and we are done. We call this a Fano fibration.

Strategy IV

- If $\dim X = \dim Y$ and cont_R is **divisorial** (i.e. it contracts some divisor), then Y has mild singularities and we may replace X by Y.
- If $\dim X = \dim Y$ and cont_R is **small** (i.e. the codimension of the exceptional locus is at least 2), then Y does not have mild singularities and we need to introduce a new operation called **flipping**.
- The flip is a surgery that replaces K-negative curves by K-positive curves.

Flipping

- Constructing the flip of a small birational K-negative map $X \to Y$ is not easy.
- If it exists, it is unique and given by

$$X^+ = \mathrm{Proj}_{\mathcal{O}_Y} \oplus_{m \in \mathbb{N}} \pi_* \mathcal{O}_X(mK_X).$$

- Therefore constructing flips, is equivalent to showing that certain algebras are finitely generated.

Flipping II

- Even if one can show that flips exists, for this operation to be of any use, we must show that it terminates.
- That is, we must show that there is no infinite sequence of flips.
- Note that it is easy to see that there are only finitely many divisorial contractions, as each time the Picard number decreases by 1.
- In dimension 3, this program was succesfully completed in 1988 when Mori proved the existence of flips.

Log Pairs

- The minimal Model Program is expected to work in a more general setting.
- We consider log pairs (X, Δ) where X is a normal $\mathbb{Q}$-factorial variety and Δ is a divisor with positive rational coefficients.
- Then $K_X + \Delta$ is $\mathbb{Q}$-Cartier and so $(K_X + \Delta) \cdot C$ and $f^*(K_X + \Delta)$ still make sense.
- The goal is once again, to find a birational map $\phi : X \dashrightarrow Y$ so that $K_Y + \phi_*\Delta$ is nef or that Y has Fano fibration (i.e. a map $Y \to S$ with $\dim Y > \dim S$ induced by a $K_Y + \phi_*\Delta$ negative contraction).

Mild singularities

- One does not expect the MMP to work unless the pair (X, Δ) has mild singularities. One possible definition is the following:
- Consider a map $f : X' \to X$ such that X' is smooth and $f^*\Delta + \mathrm{Exc}(f)$ has simple normal crossings support. Write

$$K_{X'} + \Delta' = f^*(K_X + \Delta).$$

- (X, Δ) is KLT if each coefficient of Δ' is less than 1.
- It is known that the Cone and Contraction Theorems still hold for KLT pairs (X, Δ).

Conjectures of the MMP

Let (X, Δ) be a $\mathbb{Q}$-factorial KLT pair then one expects:

Conjecture

Let $g : X \to Z$ be a flipping contraction (g is small, $\rho(X/Z) = 1$ and $-(K_X + \Delta)$ is relatiely ample) then the flip of g exists ($g^+ : X^+ \to Z$ is small, $\rho(X^+/Z) = 1$ and $(K_{X^+} + \Delta)$ is relatiely ample).

Conjecture

There is no infinite chain of flips. (AKA flips terminate).

Conjectures of the MMP II

To complete the analogy with surface, one also expects

Conjecture

If $K_X + \Delta$ is nef, then it is semiample.

The above conjectures would immediately imply

Conjecture

$R(K_X + \Delta) = \oplus H^0(\mathcal{O}_X(m(K_X + \Delta)))$ is finitely generated.

Existence of flips is a local version of this last conjecture.

Conjectures of the MMP III

In dimension 3 everything works well and the above conjectures were settled by Kawamata, Kollár, Mori, Shokurov and others in the late 80's, and the early 90's.
Considering the number of technical difficulties encountered allready in dimension 3, it seems hard to extend this to higher dimensions.

Outline of the talk

Reduction to PL Flips

- The strategy is to proceed by induction on the dimension.
- Assuming Existence and termination of flips in dimension $n-1$, one hopes to first prove existence and then termination in dimension n.
- Shokurov has shown that to construct flips it suffices then to construct PL-flips.
- That is, we assume that Δ has a component S of multiplicity 1, $-S$ is relatively ample and $K_X + \Delta$ is "almost KLT".
- By adjunction $(K_X + \Delta)|_S = K_S + \Delta'$ and (S, Δ') is KLT.

Reduction to PL Flips II

- As we have remarked above, the existence of flips is equivalent to showing that

$$\mathfrak{R} = \oplus_{m \in \mathbb{N}} H^0(\mathcal{O}_X(m(K_X + \Delta)))$$

  is finitely generated. (Here we assume Z is affine.)
- Shokurov shows that this is equivalent to showing that the algebra $\mathfrak{R}_S$ given by the image of the restriction map

$$\oplus_{m \in \mathbb{N}} H^0(\mathcal{O}_X(m(K_X + \Delta))) \to \oplus_{m \in \mathbb{N}} H^0(\mathcal{O}_S(m(K_S + \Delta')))$$

  is finitely generated.

pbd-algebras

- If $\mathfrak{R}_S = \oplus_{m\in\mathbb{N}} H^0(\mathcal{O}_S(m(K_S+\Delta')))$ we would be done by induction.
- This is too much to hope for.
- It is easy to see that $\mathfrak{R}_S$ satisfies certain natural hypothesis (eg. "boundedness, convexity and saturation").
- Shokurov calls such an algebra a pbd-algebra, and conjectures that if $-(K_S+\Delta')$ is ample, then any pbd-algebra is finitely generated.
- Shokurov's Conjecture then implies the existence of flips.

pbd-algebras II

- Shokurov's conjecture holds if $\dim S = 2$ and this gives a conceptually satisfying proof of the existence of 3-fold flips that could possibly generalize to higher dimensions.
- Shokurov's conjecture seems very difficult.
- To prove the existence of 4-fold flips, it seems neccessary to use the ambient variety X to gather extra information about $\mathfrak{R}_S$.
- This is what Shokurov does in dimension 4.

Existence of flips

In joint work with J. M^{c}Kernan, we prove the following:

Theorem

Assume that flips terminate in dimension $n-1$, then flips exist in dimension n.

Corollary

Flips exist in dimension 4 and 5.

(By results of Alexeev, Fujino and Kawamata, "almost" all flips terminate in dimension 4.)

Existence of flips II

- Our strategy is to show that $\mathfrak{R}_S$ is a pbd algebra with very special properties.
- Roughly speaking we show that there exists a divisor $0 \leq \Theta \leq \Delta'$ on S such that

$$\mathfrak{R}_S = \oplus_{m \in \mathbb{N}} H^0(m(K_S + \Theta)).$$

- Actually, we must replace S by a higher birational model.
- Also, the coefficients of Θ are real numbers.

Existence of flips III

- The equality $\mathfrak{R}_S = \oplus_{m\in\mathbb{N}} H^0(m(K_S+\Theta))$, is proved using some extension theorems that were inspired by work of Siu and Kawamata.
- These extension theorems determine Θ.
- The above strategy is promising for finite generation when $K_X + \Delta$ is of general type.
- This would strongly suggest that flips "should" terminate.

The mobile sequence

- Let $M_i = \mathrm{Mov}|i(K_S + \Theta)|$ so that

$$\mathfrak{R}_S = \oplus_{m \in \mathbb{N}} H^0(S, \mathcal{O}_S(M_i)).$$

- This is a very special example of pbd-algebra.
- Convexity means that $M_i + M_j \leq M_{i+j}$
- Boundedness means that setting $D_i = M_i/i$, then there is an effective divisor G with $D_i \leq G$ for all i. Therefore $D = \lim D_i$ exists.
- Saturation is a more subtle property.

Saturation I

- $M_\bullet$ is saturated if there is a divisor F such that $\ulcorner F \urcorner \geq 0$ and for all i, j we have
$$\mathrm{Mov}\ulcorner (j/i) M_i + F \urcorner \leq M_j.$$
- Suppose that $d_j = m_j/j$ is a convex bounded sequence of rational numbers with limit d, and $f > -1$ is a rational number such that
$$\ulcorner j d_i + f \urcorner \leq j d_j,$$
then using diophantine approximation it is easy to see that $d \in \mathbb{Q}$ and that $d = d_j$ for all j sufficiently divisible.

Saturation II

- In order for the same trick to work for the sequence $M_\bullet$ it is necessary to show that $D = \lim M_i/i$ is semiample.
- This is not too hard to arrange as

  $$D = (1/i) \lim \mathrm{Mov}(i(K_S + \Theta)),$$

  where (S, Θ) is KLT and since $\dim S = n - 1$ we may assume that the MMP holds.
- There are some technical issues to worry about: Θ is a $\mathbb{R}$-divisor!

Defining Θ

- Most of the work is in showing that one can define Θ as a divisor on some fixed model of S.
- This is done by a careful use of extension theorems, which in turn are an application of the theory of multiplier ideals.
- Assume for simplicity that the components Δ_i of $\Delta - S$ are all disjoint and $k(K_X + \Delta)$ is integral. Then it turns out that sections of $ik(K_S + \Delta_S)$ extend to X **if** S, Δ_i and $\Delta_i \cap S$ are not contained in $\mathrm{Bs}|ik(K_X + \Delta)|$.
- S is mobile so we may ignore the first condition.

Defining Θ

- If Δ_i is contained in $\mathrm{Bs}|ik(K_X+\Delta)|$, then we can remedy the situation by replacing Δ by

$$\max\{\Delta-(1/ik)\mathrm{Fix}(ik(K_X+\Delta)),0\}.$$

- To to avoid components of $\Delta_i \cap S$ being contained in $\mathrm{Bs}|ik(K_X+\Delta)|$, one can repeatedly blow up along these components and subtract common components as above.
- Note that we replace Y by higher and higher models, but blowing up S along the divisor $\Delta_i \cap S$ does not affect S.