MATHEMATICS 6130

Quiz 9

1) Define the normalization of an irreducible quasi projective variety and compute an example where the induced map is not an isomorphism.

Answer: $\nu : X^\nu \to X$ is the normalization of an irreducible quasi projective variety if there exists $X = \cup U_i$ a finite cover by affine open subsets such that if $A_i = k[U_i]$ and $B_i = k[V_i]$ where $V_i = \nu^{-1}(U_i)$, then each B_i is the integral closure of A_i in $k(X)$.

If $X = V(y^2 - x^3) \subset \mathbb{A}^2$, then $k[X] = k[x,y]/(y^2 - x^3)$ is not integrally closed as $y/x \in k(X)$ satisfies the monic polynomial $t^2 - x \in k[X][t]$.

2) Show that an irreducible quasi-projective variety X is normal if and only if $\mathcal{O}_x$ is normal for all points $x \in X$.

Answer: We may assume that X is affine. Assume that $\mathcal{O}_x$ is normal for all points $x \in X$. Let $g \in k(X)$ satisfy a monic polynomial $f(t) \in k[X][t]$, then we must show that $g \in k[X]$. Since $k[X] = \cap_{x \in X} \mathcal{O}_x$, for any $x \in X$, $g \in k(X)$ satisfies monic polynomial $f(t) \in \mathcal{O}_x[t]$ and hence $g \in \mathcal{O}_x$ (as $\mathcal{O}_x$ is normal for all points $x \in X$). But then $g \in \cap_{x \in X} \mathcal{O}_x = k[X]$ as required.

Assume that X is normal. Let $g \in\in k(X)$ satisfy a monic polynomial $f(t) \in \mathcal{O}_x[t]$. Write $f(t) = t^n + \frac{a_1}{b_1} t^{n-1} + \ldots + \frac{a_n}{b_n}$ and let $b = b_1 \cdots b_n$. Multyplying by b^n and letting $T = bt$ we obtain a polynomial $F(T) = T^n + \frac{a_1 b}{b_1} T^{n-1} + \ldots + \frac{b^n a_n}{b_n} \in k[X][T]$ such that $F(bg) = 0$. Thus $bg \in k[X]$ (as X is normal) and so $b \in \mathcal{O}_x$ (as $b(x) \neq 0$, i.e. $b \notin \mathfrak{m}_x$).