MATHEMATICS 6130

Quiz 6

a) Define finite maps of affine varieties.

A dominant regular map of affine varieties $f : X \to Y$ is finite if $k[X]$ is a finitely generated module over $f^*k[Y] \subset k[X]$.

b) Show that all fibers of any such map have finite cardinality and

Assume $Y \subset \mathbb{A}^n$ and let y_i be the images of the coordinate functions on $\mathbb{A}^n$. As $k[X]$ is a finitely generated module over $f^*k[Y] \subset k[X]$, there exists a monic polynomial $p_i(z) = z^n + a_1 z^{n-1} + \ldots + a_{n-1}z + a_n \in k[Y][z]$ such that $p_i(y_i) = 0$. For any $\bar{x} \in X$ and $\bar{y} = f(\bar{x}) \in Y$, the coordinate $\bar{y}_i$ is a solution of $z^n + a_1(\bar{y})z^{n-1} + \ldots + a_{n-1}(\bar{y})z + a_n(\bar{y}) = 0$. This is a polynomial of degree n in $k[z]$, and hence has at most n solutions. Thus for any $\bar{y} =\in Y$, the set $f^{-1}(\bar{y})$ is finite.

c) show by example that the converse does not hold.

Let $f : \mathbb{A}^1 \setminus 0 \to \mathbb{A}^1$ be the dominant map corresponding to the inclusion $k[x] \to k[x, x^{-1}]$. It is easy to see that x^{-1} does not satisfy a **monic** polynomial with coefficients in $k[x]$ and hence f is not finite.