MATHEMATICS 6130

Quiz 5

(1) Define principal affine open subsets and show that any quasi-projective variety X can be covered by principal affine open subsets.

Proof. A principal affine subset is a subset of the form $U \setminus V(f)$ where U is an affine variety and $f \in k[U]$. Since X is quasi-projective, $X = X_0 \setminus X_1$ where $X_i \subset \mathbb{P}^N$ are closed. Let $x \in X$ and pick $i \in 0, \ldots, N$ such that $x \in X \cap \mathbb{A}^N_i$. Then $X \cap \mathbb{A}^N_i = Y_0 \setminus Y_1$ where $Y_j = X_j \cap \mathbb{A}^N_i$ are affine closed sets. Since $x \in Y_0 \setminus Y_1$ and Y_1 is closed, there exists $f \in k[Y_0]$ such that $Y_1 \subset V(f)$ but $x \notin V(f)$. Thus $x \in V_0 \setminus V(f)$ which is a principal affine open subset.

(2) Show that $\mathbb{A}^2 \setminus (0, 0)$ is not affine.

Proof. We have seen that $k[\mathbb{A}^2 \setminus (0, 0)] = k[\mathbb{A}^2]$. (This follows from the fact that if $f \in k[\mathbb{A}^2]$ is not constant, then $V(f)$ is infinite and hence not contained in $(0, 0)$.) By the Nullstellensatz, if $\mathbb{A}^2 \setminus (0, 0)$ is affine, since $(1) \neq (x, y) \subset k[x, y] = k[\mathbb{A}^2 \setminus (0, 0)]$, we must have $V(x, y) \neq \emptyset$. This is a contradiction.