MATHEMATICS 6130

Quiz 4

Let $X \subset \mathbb{A}^n$ be an irreducible closed subset. Show that if $\phi \in k(X)$ is regular at all $x \in X$, then $\phi \in k[X]$.

Proof. For all $x \in X$ we can write $\phi = f_x/g_x$ where $f_x, g_x \in k[X]$ and $g_x \neq 0$. Let $J = (g_x | x \in X) \subset k[X]$. As $k[X]$ is Noetherian, $J = (g_{x_1}, \ldots, g_{x_r})$ is finitely generated. Note that $V(J) = \emptyset$ (as for any $x \in X$, $g_x \in J$ and $g_x(x) \neq 0$). Thus $J = (1)$ so that $1 = \sum_{i=1}^r h_i g_{x_i}$. But then

$$\phi = \sum_{i=1}^r h_i g_{x_i} \phi = \sum_{i=1}^r h_i g_{x_i} f_{x_i}/g_{x_i} = \sum_{i=1}^r h_i f_{x_i} \in k[X].$$