**MATHEMATICS 6130**

Quiz 3

(1) Show how to recover a closed subset from its ring of regular function and the rational map $X \to Y$ between two closed subsets from a homomorphism between their rings of rational functions $\Psi : k[Y] \to k[X]$.

**Solution:** If $A$ is a finitely generated commutative ring, then there is a surjective homomorphism $\phi : k[x_1, \ldots, x_N] \to A$ (which sends $x_i \to f_i$ where $f_1, \ldots, f_N$ generate $A$ over $k$). Let $I$ be the kernel of $\phi$, then $X$ is the zero set of the ideal $I$ and $k[X] = A = k[x_1, \ldots, x_N]/I$. Assume that $X \subset \mathbb{A}^n$ and $Y \subset \mathbb{A}^m$ so that we have surjective homomorphisms $\phi : k[x_1, \ldots, x_n] \to k[X]$ and $\psi : k[y_1, \ldots, y_m] \to k[X]$. Given $\Psi : k[Y] \to k[X]$, let $f_i = \Psi(\psi(y_i)) \in k[X]$, then $X \to Y$ is defined by $(f_1, \ldots, f_m)$. .

(2) Let $I_X$ be the ideal of a closed subset $X \subset \mathbb{A}^n$. Show that $k[X] = k[\mathbb{A}^n]/I_X$ has no nilpotents.

**Solution:** Let $f \in k[x_1, \ldots x_n]$ such that $f^m = 0 \in k[X]$ for some $m > 0$, then $f^m(P) = 0$ for all $P \in X$ and so $f(P) = 0$ (note that $f(P) \in k$ which is a field). But then $f \in I_X = \{g \in k[x_1, \ldots x_n] | g(P) = 0 \ \forall P \in X\}$.