MATHEMATICS 6130

Quiz 2

(1) Let X be a plane curve. When is $P \in X$ a point of multiplicity r? What can you say about curves of degree d with points of multiplicity d or $d-1$?

Solution: Suppose $P = (\alpha, \beta)$ and X is defined by $f(x, y) = 0$ where $f \in k[x, y]$ is a non-zero polynomial. The multiplicity of $P \in X$ is the lowest degree of a monomial with non-zero coefficient in the polynomial $f(x+\alpha, y+\beta)$. (Equivalently, expand f as a polynomial in $(x-\alpha)$ and $(y-\beta)$, the multiplicity is then given by the degree of the leading term. In characteristic 0, the degree is given by the smallest order of a derivative not vanishing at P.)

If $r = d$, then $f(x+\alpha, y+\beta)$ is homogeneous of degree d and hence (assuming $k = \bar{k}$) it splits as a product of linear terms. Thus X is the union of d lines counted with multiplicity.

If $r = d-1$ and X is irreducible, then X is rational. To see this project from the point P by letting $y - \beta = t(x-\alpha)$. $f(x, \beta + t(x-\alpha)) = c(x-\alpha)^{d-1}(x - x_1(t))$ so that $t \to (x_i(t), \beta + t(x_1 - \alpha))$ is the required parametrization.

(2) Show that if $f \in k(X)$ is regular at every point of $\mathbb{P}^1$ then f is constant. (Assume $k = \bar{k}$.)

Solution: Let $f = a(x)/b(x)$ where $a(x), b(x) \in k[x]$ are coprime. It follows that if $b(\alpha) = 0$, then $a(\alpha) \neq 0$ (or else $(x-\alpha)$ divides both polynomials). Suppose that $b(\alpha) = 0$ for some $\alpha \in k$. Since f is regular at α, we have $f = c(x)/d(x)$ where $d(\alpha) \neq 0$ and $c(x), d(x) \in k[x]$ are coprime. But then $a(x)d(x) = b(x)c(x)$. Substituting $x = \alpha$ we have $0 \neq a(\alpha)d(\alpha) = b(\alpha)c(\alpha) = 0$. Therefore, $b(\alpha) \neq 0$ for all $\alpha \in k$ and so $b(x) \in k$ and hence we may assume that $f = a(x) \in k[x]$. We have also assumed that f is regular at infinity. This is equivalent to $a(1/y)$ being regular at 0. Let d be the degree of $a(x)$, then $a(1/y) = a'(y)/y^d$ where $a'(y) \in k[y]$ has degree d. By what we have seen above, $d = 0$ and hence $f = a(x)$ is constant.