MATHEMATICS 6130

Quiz 1

(1) Show that $x^2 + 2xy = 3$ is rational (over an algebraically closed field of characteristic $\neq 2, 3$).

Solution: Project from the point $P = (1, 1)$. So $y - 1 = t(x - 1)$ and hence

$$0 = x^2 + 2x(t(x-1)+1) - 3 = (1+2t)x^2 + 2(1-t)x - 3.$$

Since one solution is $x = 1$, the other solution satisfies $-x - 1 = \frac{2-2t}{1+2t}$ and hence $x = \frac{-3}{1+2t}$. Finally $y = t(\frac{-3}{1+2t} - 1) + 1 = -\frac{2t^2+2t-1}{2t-1}$. (Alternatively, let $x = t$ and $y = \frac{3-x^2}{2x} = \frac{3-t^2}{2t}$.)

We must exclude $p = 2, 3$ because if this is the case then the curve is reducible (Thanks to Todd Harry Reeb for pointing out a typo in the previous version.)

(2) Define the field of rational functions $k(X)$ of a plane curve X. When is a rational function regular at a point $P \in X$?

Solution: Suppose $X = \{f(x, y) = 0\}$, where $f = f(x, y) \in k[x, y]$ is irreducible. If $a, b, c, d \in k[x, y]$ and $b, d \notin (f)$, then we will say that $\frac{a}{b} \equiv \frac{c}{d}$ if $ad - bc \in (f)$ (i.e. $ad - bc$ is divisible by f). Then

$$k(X) := \{\frac{p}{q} | p, q \in k[x, y], q \notin (f)\}/ \equiv$$

is the field of rational functions $k(X)$.

Let $P \in X$ and $u \in k(X)$, then we say that u is regular at p if there is a representative $u = \frac{p}{q}$ such that $q(P) \neq 0$.