MATHEMATICS 6130

Homework 1

(1) Define curve, irreducible curve and rational curve.
(2) Show that if k is algebraically closed, then any curve has infinitely many points.
(3) Show that $y^2 = x^2(x+1)$, $x^2+2xy = 3$ and $y^2 = x^3$ are rational (over an algebraically closed field).
(4) Show that (if k is algebraically closed) an irreducible curve determines its equations (up to units). (You may assume the following lemma: If $f \in k[x, y]$ is irreducible and does not divide $g \in k[x, y]$, then the set of common solutions is finite.)
(5) State Luroth's Theorem.
(6) Define the field of rational functions $k(X)$ of a plane curve X. When is a rational function regular at a point $p \in X$?
(7) Show that a plane curve X is rational if and only if $k(X) \cong k(t)$ ($k = \bar{k}$).
(8) Define rational map between two plane curves.

Homework 2

(1) Define nonsingular points, point of multiplicity r, cusps, node of a plane curve.
(2) What can you say about a plane curve of degree d with a point of multiplicity $d, d-1, d-2$?
(3) Show that irreducible plane curves have finitely many singular points.
(4) Define local parameter of an irreducible plane curve.
(5) Define the tangent line and the flex of a curve at a nonsingular point.
(6) Define $\mathbb{P}^n$. Explain why polynomial functions are not defined on $\mathbb{P}^n$, but the zeroes of homogeneous polynomials are defined.
(7) Show that all circles contain the point at infinity $(1, i, 0)$ and $(1, -i, 0)$.
(8) Show that a parabola is tangent to the line at infinity (eg. $y = x^2$ is tangent at the point $[0 : 1 : 0]$).
(9) Which rational functions in $k(x)$ are regular at the point at infinity $[1 : 0] \in \mathbb{P}^1$?
(10) Show that if $f \in k(x)$ is regular at every point in $\mathbb{P}^1$, then f is constant.
(11) Prove that an irreducible cubic has at most 1 singular point.
(12) Show that if the characteristic of k is $p > 0$, then every line through the origin is tangent to $y = x^{p+1}$.

Homework 3

(1) Define closed subsets of $\mathbb{A}^n$ and show that they define a topology.
(2) Define the ring of rational functions $k[X]$ of a closed subset $X \subset \mathbb{A}^n$ and a regular map between closed subsets. When are two closed subsets isomorphic?
(3) Show how to recover a closed subset from its ring of regular function and the rational map $X \to Y$ between two closed subsets from a homomorphism between their rings of rational functions $k[Y] \to k[X]$.
(4) State the Nullstellensatz.
(5) Define the Frobenius map $\phi : \mathbb{A}^n \to \mathbb{A}^n$ and show that if X $subset$ $\mathbb{A}^n$ is defined over $\mathbb{F}_p$ (i.e. it's ideal is generated by functions in $\mathbb{F}_p[x_1, \dots, x_n]$) then ϕ defines a rational map from X to X.
(6) Let I_X be the ideal of a closed subset $X \subset \mathbb{A}^n$. Show that $k[X] = k[\mathbb{A}^n]/I_X$ has no nilpotents.

Homework 4

(1) Let $X \subset \mathbb{A}^n$ be an irreducible closed subset.Define the field of rational functions $k(X)$. When is $\phi \in k(X)$ regular at $p \in X$? Show that the domain of definition of $\phi \in k(X)$ is a non-empty open subset.
(2) Let $X \subset \mathbb{A}^n$ be an irreducible closed subset. Show that if $\phi \in k(X)$ is regular at all $x \in X$, then $\phi \in k[X]$.
(3) Define rational maps $\phi : X \to Y$ where $X \subset \mathbb{A}^n$ and $Y \subset \mathbb{A}^m$ are irreducible closed subsets. When is ϕ dominant? Prove that ϕ is dominant iff $\phi^* : k[Y] \to k(X)$ is injective.
(4) Define birational map and show that a map is birational iff it induces an isomorphism of function fields.
(5) Show that if $I \subset k[x_0, \dots, x_n]$ is a homogeneous ideal defining the empty subset of $\mathbb{P}^n$, then $I \supset m^s$ where $m = (x_0, \dots, x_n)$.

Homework 5

(1) Define quasi-projective variety, regular functions on quasi-projective varieties and rational map / regular map between quasi-projective varieties.
(2) Define principal affine open subsets and show that any quasi-projective variety can be covered by principal affine open subsets.
(3) Explain why a regular map of quasi-projective varieties is continuous.
(4) Define $\mathcal{O}_X$ and $k(X)$ (the function field of X) for an irreducible quasi-projective variety X.

(5) Show that if X is an irreducible quasi-projective variety then $k(X) = k(\bar{X})$ (however $k[X] \neq k[\bar{X}]$ usually) and if X is an irreducible affine variety then . $k(X)$ is the field of rational functions of X.
(6) Show that $\mathbb{A}^2 \setminus (0,0)$ is not affine.
(7) Define the projection from a linear subspace $E \subset \mathbb{P}^n$ and the Veronese embedding $\nu_m : \mathbb{P}^n \to \mathbb{P}^N$ where $N = \binom{n+m}{n} - 1$.

Homework 6

(1) Define a closed embedding $\phi : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^N$ and show that $\phi(\mathbb{P}^n \times \mathbb{P}^m) \subset \mathbb{P}^N$ is closed. Explain why ϕ is closed.
(2) Explain why a regular map from a projective variety to an affine variety is constant (you may use the fact that the image of a projective variety under a regular map is closed).
(3) Explain why the set of reducible homogeneous polynomials of degree m in $n+1$ variables is a proper closed subset of the set of all homogeneous polynomials of degree m parametrized by $\mathbb{P}^N$ where $N = \binom{m+m}{m} - 1$.
(4) Define finite map of affine varieties. Show that all fibers of any such map have finite cardinality but the converse does not hold.
(5) Show that a finite map is surjective (you may use the fact that if B is a finite A module, $1_B \in A \subset B$ a subring, then for any proper ideal $(1) \neq a \subset A$ we have that $aB \neq B$).
(6) Prove that $\mathbb{A}^2 \setminus x$, $\mathbb{P}^2 \setminus x$, and $\mathbb{P}^1 \times \mathbb{A}^1$ are neither affine nor projective varieties.

Homework 7

(1) Prove that if $X \subset \mathbb{P}^N$ is a projective variety and $E \subset \mathbb{P}^N$ is a d-dimensional linear subspace such that $E \cap X = \emptyset$, then the projection $\pi : \mathbb{P}^N \to \mathbb{P}^{N-d-1}$ is finite on to its image.
(2) Deduce from the previous exercise that if $F_0, \ldots, F_s$ are homogeneous of degree m and $X \subset \mathbb{P}^N$ is a projective variety such that $X \cap V(F_0, \ldots, F_s) = \emptyset$, then the rational map $\phi : X \dashrightarrow \mathbb{P}^s$ defined by $(F_0, \ldots, F_s)$ is finite.
(3) Let X be a quasi-projective variety. Define $\dim X$ and show that if X is irreducible then $\dim X$ is a birational invariant.
(4) Show that if $X \subset Y$ is an inclusion of quasi projective varieties, then $\dim X \leq \dim Y$.
(5) Show that every irreducible component of a hyprsurface in $\mathbb{A}^N$ has codimension 1.
(6) Give alternative definitions of $\dim X$ in terms of the Noether Normalization theorem and by sequences of irreducible proper closed subsets.

(7) Let $X, Y \subset \mathbb{P}^N$ be irreducible quasi-projective varieties. Show that $codim(X \cap Y) \leq codim(X) + codim(Y)$.
(8) Let $f : X \to Y$ be a regular map of irreducible quasi-projective varieties. Show that each (non-empty) fiber has dimension $\geq \dim X - \dim Y$. What can you say about the function $\dim(f^{-1}(y)$ where $y \in Y$?

Homework 8+9

(1) Let $x \in X$ be a point on an irreducible affine variety. Define $\mathcal{O}_{x,X}$.
(2) Let $x \in X$ be a point on an irreducible affine variety. Define $\Theta_{x,X}$ and show that $\Theta_{x,X} \cong (\mathfrak{m}_x/\mathfrak{m}_x^2)^\vee$.
(3) Prove that the curve $C \subset \mathbb{A}^N$ given by $t \to (t^N, t^{N+1}, \ldots, t^{2N-1})$ can not be embedded in $\mathbb{A}^n$ for any $n < N$.
(4) Explain why there exists an open subset $X_{sm} \subset X$ such that $\dim \Theta_{x,X} = \dim X$ for all $x \in U$ and $\dim \Theta_{x,X} > \dim X$ for all $x \notin U$.
(5) Define local parameters, dimension of a local ring, regular local ring and Taylor series expansion.
(6) Let X be a quasi-projective variety and x a nonsingular point of X. Show that X is irreducibe at x.
(7) Explain why the set of singular points on a quasi projective variety is closed.
(8) Explain why if $x \in X$ is a nonsingular point, then X is locally a complete intersection at x.
(9) Define local equations of a subvariety $Y \subset X$ at a point $x \in Y$.
(10) Explain why a rational map from a non-singular quasi projective variey to projective space is regular in codimension 1.

Homework 10

(1) Define the blow up $\pi : X \to \mathbb{P}^n$ centered at the point $p = (1 : 0 : ... : 0)$ and show that X is smooth and irreducible, $\pi^{-1}(p) \cong \mathbb{P}^{n-1}$ and $X \setminus \pi^{-1}(p) \cong \mathbb{P}^n \setminus p$.
(2) Define the normalization of an irreducible quasi projectve variety and compute an example where the induced map is not an isomorphism.
(3) Show that an irreducible quasi-projective variety X is normal if and only if $\mathcal{O}_x$ is normal for all points $x \in X$.
(4) Show that smooth varieties are normal.
(5) Show that a normal variety is non-singular in codimension 1.