(1) Define $A(S)$ and show that it is a group. If $|S| = n$, then show that $|A(S)| = n!$.

(2) State the well ordering principle. State and prove Euclid’s algorithm.

(3) Define the greatest common divisor $c = (a, b)$ of two integers $a, b$ (not both 0). Show that it exists and is unique. Prove that $c = ma + nb$ for some integers $m, n$. Compute $(104, 189)$.

(4) Prove that there are infinitely many prime numbers.

(5) Let $p$ be a prime number. Prove that $\sqrt{p}$ is irrational.

(6) If $a, b$ are non-zero integers, show that $ab$ equals the product of their least common multiple and their greatest common divisor.

(7) Prove by induction that $\sum_{i=1}^{n} i^3 = \frac{1}{4}n^2(n + 1)^2$.

(8) Prove by induction that if $a \neq 1$, then $\sum_{i=1}^{n} a^i = (a^{n+1} - 1)/(a - 1)$.

(9) Let $z, w$ be complex numbers. Define $\bar{z}$, $|z|$, give a formula for $z^{-1}$, show that $\bar{zw} = \bar{z}\bar{w}$. When is $|z + w| = |z| + |w|$? State the triangle inequality.

(10) find the solutions to $x^5 = 1 + i$.

(11) write $\cos 4\theta$ in terms of $\sin \theta$ and $\cos \theta$.

(12) Define a group; an abelian group; a cyclic group. Give examples of non-commutative groups of order 10 and of infinite order; give an example of an infinite group all of whose elements have finite order.

(13) Show that units and inverses are unique. Prove that $ab = ac$ implies $b = c$.

(14) If all elements of $G$ have order 2, show that $G$ is abelian.

(15) If $|G|$ is even, show that there is an element of order 2.

(16) Define subgroups. Show that if $H$ is a non-empty finite subset closed under multiplication, then it is a subgroup.

(17) Define the center $Z(G)$ and the centralizer $C(g)$ and show that they are subgroups. Compute these in $S_3$ and $D_4$.

(18) If $G$ is cyclic, show that it is abelian. What are the generators of $G$. Show that if a group has no proper subgroups, then it is cyclic.

(19) Show that if $G$ is abelian then $H = \{x \in G | x^2 = e \}$ is a subgroup, but this is not the case for all non-abelian groups.

(20) Define equivalence relations. Show that if $H$ is a subgroup of $G$ then the relation $x \sim y$ iff $xy^{-1} \in H$ is an equivalence relation.
(21) State and prove Lagrange’s Theorem.
(22) Define index of a subgroup.
(23) If $|G| = m$ show that $x^m = e$ for all $x \in G$.
(24) If $|G| = p$ and $p$ is prime, show that $G$ is cyclic.
(25) Define $Z_m$ and $U_m$. Prove that $U_m$ is a group. Find the orders of elements in $U_{18}, U_{20}$.
(26) Prove Euler’s Theorem: If $(a, n) = 1$, then $a^{\phi(n)} \equiv 1 \pmod{n}$.
(27) Define left and right cosets. When is a subgroup normal? Show that subgroups of index 2 are normal.
(28) Define homomorphisms, injective, surjective, isomorphism and Kernel. Show that if $\phi : G \to G'$ is a homomorphism, then $\phi(e) = e'$, $\phi(g^{-1}) = \phi(g)^{-1}$ and $\phi^{-1}(\phi(g)) = \text{Ker } \phi \cdot g$.
(29) Show that $\phi$ is injective iff $\text{Ker } \phi = \{ e \}$.
(30) Show that $\phi(G)$ is a subgroup of $G'$ which may not be normal. If $\phi$ is surjective and $G$ is abelian, then $G'$ is abelian. If $G$ is cyclic then so is $G'$. Inverse images of (normal) subgroups are (normal) subgroups.
(31) State and prove Cayley’s Theorem.
(32) Define normal subgroups and show that $\text{Ker } \phi$ is a normal subgroup.
(33) Find all normal subgroups of $D_4$ and $S_3$.
(34) If $N, M$ are normal and $N \cap M = \{ e \}$, show that $nm = mn$ for all $n \in N$ and $m \in M$.
(35) If $N$ is normal in $G$ define $G/N$ and define a surjective homomorphism $\phi : G \to G/N$ (prove it).
(36) State Cauchy’s theorem and prove it for abelian groups.
(37) Show that if $N$ is normal in $G$ then $G/N$ is abelian iff $aba^{-1}b^{-1} \in N$ for all $a, b \in G$.
(38) If $G$ is abelian, $g, h \in G$ have orders $n$ and $m$, then show that there is an element of $G$ of order the least common multiple of $n$ and $m$.
(39) Find a surjective homomorphism from $(\mathbb{R}, \cdot)$ to $(\mathbb{Z}_2, +)$.
(40) Show that if $G/Z(G)$ is cyclic, then $G$ is abelian. (Recall that $Z(G)$ is the center of a group.)
(41) If $m, n$ are relatively prime numbers, show that if an abelian group $G$, contains elements $x$ and $y$ of order $n$ and $m$ then it contains an element of order $nm$.
(42) Do the previous exercise for non-abelian groups. (Hint: first show that $(x) \cap (y) = e$, then show that $xy = yx$ and conclude.)
(43) State and prove the first homomorphism theorem.
Let $\phi : G \to G'$ be a surjective homomorphism. Show that the (normal) subgroups of $G'$ are in 1-1 correspondence with the (normal) subgroups of $G$.

Show that there are no surjective homomorphisms $\mathbb{Z} \to \mathbb{Z}_4 \times \mathbb{Z}_6$, $\mathbb{Z}_5 \times \mathbb{Z}_3 \to \mathbb{Z}_4 \times \mathbb{Z}_6$, $\mathbb{Z}_4 \times \mathbb{Z}_6 \to \mathbb{Z}_5 \times \mathbb{Z}_3$.

Use Cauchy's theorem to show that a group with 99 elements has a normal subgroup.

Define the direct product of 2 groups. Prove that $\mathbb{Z}_4 \times \mathbb{Z}_3 \cong \mathbb{Z}_{12}$; show that $\mathbb{Z}_4 \times \mathbb{Z}_6 \cong \mathbb{Z}_{12} \times \mathbb{Z}_2$.

Let $M, N$ be normal subgroups of a group $G$ such that $M \cap N = \{e\}$ and $MN = G$. Show that $G \cong M \times N$.

Use Cauchy's theorem to show that a group with 99 elements has a normal subgroup.

State the theorem on the classification of finite abelian groups. Write all groups of order 10,000. How many distinct ones are there?

Find an element of order 70 in $\mathbb{Z}_{42} \times \mathbb{Z}_{45}$. What is the maximum order of an element in this group?

State the class equation and use it to show that if $|G| = p^n$ where $p$ is a prime, then $Z(G) \neq \{e\}$.

Use the previous exercise to show that any group of order $p^2$ is abelian. Give an example of a group of order $p^3$ which is not abelian. What is $|Z(G)|$ for this group?

State Sylow’s theorem.

Find all Sylow subgroups in $S_3, D_4, S_4$.

Define $N(H)$ the normalizer of a subgroup $H$. Show that $N(H)$ is a subgroup of $G$ and that $H$ is a normal subgroup of $N(H)$.