1) Let $K \subset F$ be fields and $V$ be a $K$ vector space, then $K$ is also a $F$ vector space. Show that if $V$ is finite dimensional over $K$ and $K$ is finite dimensional over $F$, then $\dim_F V = \dim_K V \cdot \dim_F K$.

**Proof.** Let $k_1, \ldots, k_n$ be a basis for $K$ over $F$ and $v_1, \ldots, v_m$ be a basis for $V$ over $K$. We claim that $k_i v_j$ is a basis for $V$ over $F$.

Suppose that $v \in V$, since $v_1, \ldots, v_m$ generate $V$ over $K$, then we may write $v = \sum_{i=1}^{m} a_i v_i$ for some $a_i \in K$. Since, $k_1, \ldots, k_n$ generate $K$ over $F$ we can write $a_i = \sum_{j=1}^{n} b_{i,j} k_j$ for some $b_{i,j} \in F$. Putting all of this together we obtain

$$v = \sum_{i=1}^{m} a_i v_i = \sum_{i=1}^{m} \left( \sum_{j=1}^{n} b_{i,j} k_j \right) v_i = \sum_{i,j} b_{i,j} k_j v_i$$

and hence $k_i v_j$ generate $V$ over $F$.

Suppose that $\sum_{i,j} b_{i,j} k_j v_i = 0$ for some $b_{i,j} \in F$, then $0 = \sum_{i=1}^{m} \left( \sum_{j=1}^{n} b_{i,j} k_j \right) v_i$ where $\sum_{j=1}^{n} b_{i,j} k_j \in K$ and since $v_1, \ldots, v_m$ are linearly independent over $K$, we have $\sum_{j=1}^{n} b_{i,j} k_j = 0$ (for all $1 \leq i \leq m$). Since $k_1, \ldots, k_n$ are linearly independent over $F$, we have $b_{i,j} = 0$ for all $i,j$. Hence $k_j v_i$ are linearly independent over $F$. \(\square\)

2) Give an example of two fields $K \subset F$ and two elements $a, b \in K$ such that $[F(a, b) : F] < [F(a) : F][F(b) : F]$.

**Proof.** Let $F = \mathbb{Q}$ and $K = \mathbb{C}$. Pick $a = \sqrt{2}$ and $b = 2^{1/4}$. Then $a \in \mathbb{Q}(b)$ and so $\mathbb{Q}(a, b) = \mathbb{Q}(b)$. Note that by Eisenstein, the polynomials $x^2 - 2$ and $x^4 - 2$ are irreducible so that $[\mathbb{Q}(a) : \mathbb{Q}] = 2$ and $[\mathbb{Q}(b) : \mathbb{Q}] = 4$. Thus

$$4 = [\mathbb{Q}(a, b) : \mathbb{Q}] < [\mathbb{Q}(a) : \mathbb{Q}][\mathbb{Q}(b) : \mathbb{Q}] = 2 \cdot 4 = 8.$$ \(\square\)

3) Define a field of order $5^3$.

**Proof.** It suffices to find an irreducible polynomial $P(x) \in \mathbb{Z}_5[x]$ of degree 3. Take $P(x) = x^3 + x + 1$. Then $P(0) = 1$, $P(1) = 3$, $P(2) = 1$, $P(3) = 1$ and $P(4) = 4$ so that $P(x)$ has no roots and hence $P(x)$ is irreducible. \(\square\)

4) Show that $[\mathbb{Q}(\sqrt[3]{2} + \sqrt[3]{3}) : \mathbb{Q}] = 4$. 

1
Proof. Let $a = \sqrt{2} + \sqrt{3}$, then $a^2 = 5 + 2\sqrt{6}$ and $a^3 = 5a + 4\sqrt{3} + 3\sqrt{2} = 8a + \sqrt{3}$. But then $\sqrt{3} \in \mathbb{Q}(\sqrt{2} + \sqrt{3})$ and hence $\sqrt{2} \in \mathbb{Q}(\sqrt{2} + \sqrt{3})$ so that $\mathbb{Q}(\sqrt{2} + \sqrt{3}) = \mathbb{Q}(\sqrt{2}, \sqrt{3})$. It follows that $[\mathbb{Q}(\sqrt{2} + \sqrt{3}) : \mathbb{Q}] = [\mathbb{Q}(\sqrt{2} + \sqrt{3}) : \mathbb{Q}(\sqrt{3})][\mathbb{Q}(\sqrt{3}) : \mathbb{Q}] \leq 4$. To show equality, it suffices to show that $\sqrt{2} \not\in \mathbb{Q}(\sqrt{3})$. Suppose $\sqrt{2} = a + b\sqrt{3}$ where $a, b \in \mathbb{Q}$, then $2 = a^2 + 3b^2 + 2\sqrt{3}$ and so $\sqrt{3} = \frac{2a^2 - 3b^2}{2} \in \mathbb{Q}$ which is impossible.  

5) Let $K \supset F$ be an extension of field and $E$ be the algebraic closure of $F$ in $K$. Show that $E$ is a field.

Proof. We must show that if $a, b \in E$, then $a \pm b, a \times b$ and $a/b$ (if $b \neq 0$) are all in $E$. As $a, b$ are algebraic over $F$, we have that both $[F(a) : F]$ and $[F(b) : F]$ are finite. Thus, we know that $[F(a,b) : F]$ is finite (and in fact $\leq [F(a) : F][F(b) : F]$). It follows that any element of $F(a,b)$ is algebraic over $F$. We are done since $a \pm b, a \times b$ and $a/b$ (if $b \neq 0$) are all in $F(a,b)$.  

\[\]