4.3 Undetermined Coefficients

The **method of undetermined coefficients** applies to solve differential equations

$$ay'' + by' + cy = f(x). \tag{1}$$

The method has **restrictions**: a, b, c are constant, $a \neq 0$, and $f(x)$ is a sum of terms of the general form

$$p(x)e^{kx}\cos(mx) \quad \text{or} \quad p(x)e^{kx}\sin(mx) \tag{2}$$

with $p(x)$ a polynomial and k, m constants. The method's importance is argued from its direct applicability to equations from mechanics and circuit theory.

Included as possible functions f in (1) are $\sinh x$ and $\cos^3 x$, due to identities from algebra and trigonometry. *Specifically excluded* are $\ln|x|$, $|x|$, e^{x^2} and fractions like $x/(1+x^2)$.

Happily, solving equation (1) for $y = y_h + y_p$ is a routine application of the linear equation *recipe* for y_h plus an *algorithm* to find y_p, called *the method of undetermined coefficients*.

The **library of special methods** for finding y_p (also called Kümmer's method) is presented on page 171. It uses only college algebra and polynomial calculus. The trademark of this method is the *absence of linear algebra, tables or special cases*, that can be found in other literature on the subject. The alternative **trial solution shortcut** method, which requires linear algebra, is presented on page 175.

The Algorithm for Undetermined Coefficients

A particular solution y_p of (1) will be expressed as a sum

$$y_p = y_1 + \cdots + y_n$$

where each y_k solves a related easily-solved differential equation.

The idea can be quickly communicated for $n = 3$. The superposition principle applied to the three equations

$$\begin{aligned} ay_1'' + by_1' + cy_1 &= f_1(x), \\ ay_2'' + by_2' + cy_2 &= f_2(x), \\ ay_3'' + by_3' + cy_3 &= f_3(x) \end{aligned} \tag{3}$$

shows that $y = y_1 + y_2 + y_3$ is a solution of

$$ay'' + by' + cy = f_1 + f_2 + f_3. \tag{4}$$

If each equation in (3) is easily solved, then solving equation (4) is also easy: *add the three answers for the easily solved problems.*

To use the idea, it is necessary to start with $f(x)$ and determine a decomposition $f = f_1 + f_2 + f_3$ so that equations (3) are easily solved.

The process is called the **method of undetermined coefficients**. This method consists of decomposing (1) into a number of **easy-to-solve equations**, each of which is ultimately solved by determining a polynomial **trial solution**

$$y = c_0 + c_1 x + \cdots + c_m \frac{x^m}{m!}$$

with **undetermined coefficients** $c_0, \ldots, c_m$. Values for the undetermined coefficients are found by college algebra back–substitution.

The Easily Solved Equations. Each easy-to-solve equation is engineered to fit one of the solution methods described below in the *library of special methods*. The objective is to isolate those terms in the right side of the differential equation having one of the four forms below, each of which is called an **atom**:

(5)

$p(x)$	**polynomial**,
$p(x)e^{kx}$	**polynomial** $\times$ **exponential**,
$p(x)e^{kx}\cos mx$	**polynomial** $\times$ **exponential** $\times$ **cosine**,
$p(x)e^{kx}\sin mx$	**polynomial** $\times$ **exponential** $\times$ **sine**.

To illustrate, consider

$$ay'' + by' + cy = x + xe^x + x^2\sin x - \pi e^{2x}\cos x + x^3. \tag{6}$$

The right side is decomposed as follows, in order to define the easily solved equations (also called the **atomic equations**):

$ay'' + by' + cy = x + x^3$	Polynomial.
$ay'' + by' + cy = xe^x$	Polynomial $\times$ exponential.
$ay'' + by' + cy = x^2\sin x$	Polynomial $\times$ exponential $\times$ sine.
$ay'' + by' + cy = -\pi e^{2x}\cos x$	Polynomial $\times$ exponential $\times$ cosine.

There are $n = 4$ equations. In the illustration, x^3 is included with x, but it could have caused creation of a fifth equation. To decrease effort, minimize the number n of easily solved equations. *One final checkpoint*: the right sides of the n equations must add to the right side of (6).

Library of Special Methods

Recorded here are special methods for efficiently solving the easy-to-solve equations. It is emphasized that a given problem may already be in easy-to-solve form, making the application direct. It is equally likely that the problem requires a decomposition into easy-to-solve problems, each solvable by the present methods; the desired solution is then the sum of these answers.

Equilibrium and Quadrature Methods. The special case of $ay'' + by' + cy = k$ where k is a constant occurs so often that an efficient method has been isolated to find y_p. It is called the **equilibrium method**, because in the simplest case y_p is a constant solution or an *equilibrium solution*. The method in words:

> Verify that the right side of the differential equation is constant. Cancel on the left side all derivative terms except for the lowest order and then solve for y by quadrature.

The method works to find a solution, because if a derivative $y^{(n)}$ is constant, then all higher derivatives $y^{(n+1)}$, y^{n+2}, etc., are zero. A precise description follows.

Differential Equation	Cancelled DE	Particular Solution
$ay'' + by' + cy = k$, $c \neq 0$	$cy = k$	$y_p = \frac{k}{c}$
$ay'' + by' = k$, $b \neq 0$	$by' = k$	$y_p = \frac{k}{b}x$
$ay'' = k$, $a \neq 0$	$ay'' = k$	$y_p = \frac{k}{a}\frac{x^2}{2}$

The equilibrium method also applies to nth order linear differential equations $\sum_{i=0}^{n} a_i y^{(i)} = k$ with constant coefficients $a_0, \ldots, a_n$ and constant right side k.

A special case of the equilibrium method is the *simple quadrature method*, illustrated in Example 5, page 177. This method is used in elementary physics courses to solve falling body problems.

The Polynomial Method. The method applies to find a particular solution of $ay'' + by' + cy = p(x)$, where $p(x)$ represents a polynomial of degree $n \geq 1$. Such equations **always have a polynomial solution**.

Let a, b and c be given with $a \neq 0$. Differentiate the differential equation successively until the right side is constant:

$$
\begin{array}{lllllll}
ay'' & + & by' & + & cy & = & p(x), \\
ay''' & + & by'' & + & cy' & = & p'(x), \\
ay^{iv} & + & by''' & + & cy'' & = & p''(x), \\
 & & & & & \vdots & \\
ay^{(n+2)} & + & by^{(n+1)} & + & cy^{(n)} & = & p^{(n)}(x).
\end{array} \tag{7}
$$

Apply the equilibrium method to the *last equation* in order to find a polynomial **trial solution**

$$y(x) = c_m \frac{x^m}{m!} + \cdots + c_0.$$

It will emerge that $y(x)$ always has $n+1$ terms, but its degree can be either n, $n+1$ or $n+2$. The **undetermined coefficients** $c_0, \ldots, c_m$ are resolved by setting $x=0$ in equations (7). The Taylor polynomial relations $c_0 = y(0), \ldots, c_m = y^{(m)}(0)$ give the equations

$$\begin{array}{lllllllll} ac_2 & + & bc_1 & + & cc_0 & = & p(0), \\ ac_3 & + & bc_2 & + & cc_1 & = & p'(0), \\ ac_4 & + & bc_3 & + & cc_2 & = & p''(0), \\ & & & & & \vdots & \\ ac_{n+2} & + & bc_{n+1} & + & cc_n & = & p^{(n)}(0). \end{array} \tag{8}$$

These equations can always be solved by **back-substitution**; linear algebra is **not** required. Three cases arise, according to the number of zero roots of the characteristic equation $ar^2 + br + c = 0$. The values $m = n, n+1, n+2$ correspond to zero, one or two roots $r = 0$.

Case 1: [**No root** $r = 0$]. Then $c \ne 0$. There were n integrations to find the trial solution, so $c_{n+2} = c_{n+1} = 0$. The unknowns are c_0 to c_n. The system can be solved by simple back-substitution to uniquely determine $c_0, \ldots, c_n$. The resulting polynomial $y(x)$ is the desired solution $y_p(x)$.

Case 2: [**One root** $r = 0$]. Then $c = 0$, $b \ne 0$. The unknowns are $c_0, \ldots, c_{n+1}$. There is no condition on c_0; simplify the trial solution by taking $c_0 = 0$. Solve (8) for unknowns c_1 to c_{n+1}, as in **Case 1**.

Case 3: [**Double root** $r = 0$]. Then $c = b = 0$ and $a \ne 0$. The equilibrium method gives a polynomial trial solution $y(x)$ involving $c_0, \ldots, c_{n+2}$. There are no conditions on c_0 and c_1. Simplify y by taking $c_0 = c_1 = 0$. Solve (8) for unknowns c_2 to c_{n+2}, as in **Case 1**.

College algebra back-substitution applied to (8) is illustrated in Example 7, page 178. A complete justification of the polynomial method appears in the proof of Theorem 9, page 184.

Recursive Polynomial Shortcut.

A *recursive method* based upon quadrature appears in Example 9, page 180. This method, independent from the *polynomial method* above, is useful when the number of equations in (7) is two or three.

Some researchers (see [Gupta]) advertise the recursive method as easy to remember, easy to use and faster than other methods. This method is advertised in this textbook as a **shortcut**: equations in (7) are written down, but equations (8) are not. Instead, the undetermined coefficients are found recursively, by repeated quadrature and back-substitution.

Classroom testing of the recursive polynomial method reveals it is best suited to algebraic helmsmen with flawless talents. The method should be applied when conditions suggest rapid and reliable computation details. Error propagation possibilities dictate that systems of size 4 or larger be subjected to an answer check.

Polynomial × Exponential Method.
The method applies to special equations $ay'' + by' + cy = p(x)e^{kx}$ where $p(x)$ is a polynomial. The idea, due to Kümmer, uses the transformation $y = e^{kx}Y$ to obtain the auxiliary equation

$$[a(D+k)^2 + b(D+k) + c]Y = p(x), \quad D = \frac{d}{dx}.$$

The polynomial method applies to find Y. Multiplication by e^{kx} gives y. Computational details are in Example 10, page 180. Justification appears in Theorem 10. In words, to find the differential equation for Y:

> In the differential equation, replace D by $D + k$ and cancel e^{kx} on the RHS.

Polynomial × Exponential × Cosine Method.
The method applies to equations $ay'' + by' + cy = p(x)e^{kx}\cos(mx)$ where $p(x)$ is a polynomial. Kümmer's transformation $y = e^{kx}\,\mathcal{R}\mathrm{e}(e^{imx}Y)$ gives the auxiliary problem

$$[a(D+z)^2 + b(D+z) + c]Y = p(x), \quad z = k + im, \quad D = \frac{d}{dx}.$$

The polynomial method applies to find Y. Symbol $\mathcal{R}\mathrm{e}$ extracts the real part of a complex number. Details are in Example 11, page 181. The formula is justified in Theorem 11. In words, to find the equation for Y:

> In the differential equation, replace D by $D + k + im$ and cancel $e^{kx}\cos mx$ on the RHS.

Polynomial × Exponential × Sine Method.
The method applies to equations $ay'' + by' + cy = p(x)e^{kx}\sin(mx)$ where $p(x)$ is a polynomial. Kümmer's transformation $y = e^{kx}\,\mathcal{I}\mathrm{m}(e^{imx}Y)$ gives the auxiliary problem

$$[a(D+z)^2 + b(D+z) + c]Y = p(x), \quad z = k + im, \quad D = \frac{d}{dx}.$$

The polynomial method applies to find Y. Symbol $\mathcal{I}\mathrm{m}$ extracts the imaginary part of a complex number. Details are in Example 12, page 182. The formula is justified in Theorem 11. In words, to find the equation for Y:

> In the differential equation, replace D by $D + k + im$ and cancel $e^{kx}\sin mx$ on the RHS.

Kümmer's Method. The methods known above as the polynomial × exponential method, the polynomial × exponential × cosine method, and the polynomial × exponential × sine method, are collectively called **Kümmer's method**, because of their origin.

Trial Solution Shortcut

The library of special methods leads to a related method for finding a particular solution, called the **trial solution method**. The idea of the method is to write down a trial solution having undetermined coefficients, then substitute this trial solution into the full differential equation in order to determine the values of the coefficients. The method is perhaps the most popular one, possibly because of advertisement in leading differential equation textbooks published over the past 50 years.

How Kümmer's Method Predicts Trial Solutions. Given $ay'' + by' + cy = f(x)$ where $f(x) =$(polynomial)$e^{kx}\cos mx$, then the method of Kümmer predicts $y = e^{kx}\,\mathcal{R}\mathrm{e}\,(Y(x)(\cos mx + i\sin mx))$, where $Y(x)$ is a polynomial solution of a different, associated differential equation. In the simplest case, $Y(x) = \sum_{j=0}^{n} A_j x^j + i\sum_{j=0}^{n} B_j x^j$, a polynomial of degree n with complex coefficients, matching the degree of the polynomial in $f(x)$. Expansion of the Kümmer formula for y plus definitions $a_j = A_j - B_j$, $b_j = B_j + A_j$ gives a **trial solution**

$$y = \left(\cos(mx)\sum_{j=0}^{n} a_j x^j + \sin(mx)\sum_{j=0}^{n} b_j x^j\right)e^{kx}. \tag{9}$$

The undetermined coefficients are $a_0, \ldots, a_n, b_0, \ldots, b_n$. Exactly the same trial solution results when $f(x) =$(polynomial)$e^{kx}\sin mx$.

A root $r = 0$ of the characteristic equation for the associated differential equation corresponds exactly to root $r = k + m\sqrt{-1}$ for $ar^2 + br + c = 0$. Therefore, Y must be multiplied by x for each time $k + m\sqrt{-1}$ is a root of $ar^2 + br + c = 0$. The result is that y must be multiplied by x, correspondingly.

Shortcuts using (9) have been known for some time. The shortcuts are called **trial solution table lookup methods**. The results can be summarized in words as follows.

> If the right side of $ay'' + by' + cy = f(x)$ is a polynomial of degree n times $e^{kx}\cos(mx)$ or $e^{kx}\sin(mx)$, then an initial trial solution y is given by relation (9), with undetermined coefficients $a_0, \ldots, a_n, b_0, \ldots, b_n$. Correct the trial solution y by multiplication by x, once for each time $r = k + m\sqrt{-1}$ is a root of the characteristic equation $ar^2 + br + c = 0$.

Once the **corrected trial solution** y is determined, then substitute y into the differential equation. Find the undetermined coefficients by matching terms of the form $x^j e^{kx}\cos(mx)$ and $x^j e^{kx}\sin(mx)$, which appear on the left and right side of the equation after substitution.

There is a penalty, in general, for using the trial solution shortcut method: the differentiation of the trial solution y can be a lot of work, with many opportunities for errors. Further, the equations that result by matching terms can be so complicated that a full course in linear algebra is required to solve them.

A Table Lookup Method. The special cases of trial solution (9) that are of interest in applications are (1) $m = k = 0$, (2) $k \neq 0$, $m = 0$, (3) $k = 0$, $m > 0$. In addition, there is wide use of the case when the polynomial is a constant. The table below summarizes the form of a trial solution in these cases, according to the form of $f(x)$.

Table 2. A Table Method for Trial Solutions.
The table predicts the **initial trial solution** y in the method of undetermined coefficients. Then the **fixup rule** below is applied to find the **corrected trial solution**. Symbol n is the degree of the polynomial in column 1.

Form of $f(x)$	Values	Initial Trial Solution
constant	$k = m = 0$	$y = a_0$ =constant
polynomial	$k = m = 0$	$y = \sum_{j=0}^{n} a_j x^j$
combination of $\cos mx$ and $\sin mx$	$k = 0$, $m > 0$	$y = a_0 \cos mx + b_0 \sin mx$
(polynomial)e^{kx}	$m = 0$	$y = \left(\sum_{j=0}^{n} a_j x^j\right) e^{kx}$
(polynomial)$e^{kx}\cos mx$ or (polynomial)$e^{kx}\sin mx$	$m > 0$	$y = \left(\sum_{j=0}^{n} a_j x^j\right) e^{kx}\cos mx + \left(\sum_{j=0}^{n} b_j x^j\right) e^{kx}\sin mx$

The Fixup Rule. Table 2 was obtained by choosing values for k and m in the trial solution formula (9). Accordingly, the **corrected trial solution** is found by this rule:

> Given an initial trial solution y for $au'' + by' + cy = f(x)$, from Table 2, correct y by multiplication by x, once for each time that $r = k + m\sqrt{-1}$ is a root of the characteristic equation $ar^2 + br + c = 0$.

After k, m and the corrected trial solution y are found, then find the undetermined coefficients $a_0, \ldots, a_n, b_0, \ldots, b_n$ by substituting y into the differential equation.

Details for lines 2-3 of Table 2 appear in Examples 6, 8 on page 179.

Key theorems

The following results, whose proofs are delayed to page 184, form the theoretical basis for the method of undetermined coefficients. University courses might have to assign the proofs as reading to save class time for examples.

Theorem 9 (Polynomial Solutions)
Assume a, b, c are constants, $a \neq 0$. Let $p(x)$ be a polynomial of degree d. Then $ay'' + by' + cy = p(x)$ has a polynomial solution y of degree d, $d + 1$ or $d + 2$. Precisely, these three cases hold:

Case 1. $ay'' + by' + cy = p(x)$ Then $y = y_0 + \cdots + y_d \frac{x^d}{d!}$.
$c \neq 0$.

Case 2. $ay'' + by' = p(x)$ Then $y = \left(y_0 + \cdots + y_d \frac{x^d}{d!}\right) x$.
$b \neq 0$.

Case 3. $ay'' = p(x)$ Then $y = \left(y_0 + \cdots + y_d \frac{x^d}{d!}\right) x^2$.
$a \neq 0$.

Theorem 10 (Polynomial $\times$ Exponential)
Assume a, b, c, k are constants, $a \neq 0$, and $p(x)$ is a polynomial. If Y is a solution of $[a(D+k)^2 + b(D+k) + c]Y = p(x)$, then $y = e^{kx}Y$ is a solution of $ay'' + by' + cy = p(x)e^{kx}$.

Theorem 11 (Polynomial $\times$ Exponential $\times$ Cosine or Sine)
Assume a, b, c, k, m are real, $a \neq 0$, $m > 0$. Let $p(x)$ be a real polynomial and $z = k + im$. If Y is a solution of $[a(D+z)^2 + b(D+z) + c]Y = p(x)$, then $y = e^{kx}\,\mathcal{R}\mathrm{e}(e^{imx}Y)$ is a solution of $ay'' + by' + cy = p(x)e^{kx}\cos(mx)$ and $y = e^{kx}\,\mathcal{I}\mathrm{m}(e^{imx}Y)$ is a solution of $ay'' + by' + cy = p(x)e^{kx}\sin(mx)$.

5 Example (Simple Quadrature)
Solve for y_p in $y'' = 2 - x + x^3$ using the fundamental theorem of calculus, verifying $y_p = x^2 - x^3/6 + x^5/20$.

Solution: Two integrations using the fundamental theorem of calculus give $y = y_0 + y_1 x + x^2 - x^3/6 + x^5/20$. The terms $y_0 + y_1 x$ represent the homogeneous solution y_h. Therefore, $y_p = x^2 - x^3/6 + x^5/20$ is reported. The method works in general for $ay'' + by' + cy = p(x)$, provided $b = c = 0$, that is, in **case 3** of Theorem 9. Some explicit details:

$\int y''(x)dx = \int(2 - x + x^3)dx$	Integrate across on x.
$y' = y_1 + 2x - x^2/2 + x^4/4$	Fundamental theorem.
$\int y'(x)dx = \int(y_1 + 2x - x^2/2 + x^4/4)dx$	Integrate across again on x.

$y = y_0 + y_1 x + x^2 - x^3/6 + x^5/20$ Fundamental theorem.

6 Example (Classical Undetermined Coefficients)
Solve for y_p in the equation $y'' - y' + y = 2 - x + x^3$ by the classical method of undetermined coefficients, verifying $y_p = -5 - x + 3x^2 + x^3$.

Solution: Let's begin by *calculating the trial solution* $y = c_0 + c_1 x + c_2 x^2/2 + x^3$. This is done by differentiation of $y'' - y' + y = 2 - x + x^3$ until the right side is constant:

$$y^v - y^{iv} + y''' = 6.$$

The equilibrium method solves the truncated equation $0 + 0 + y''' = 6$ by quadrature to give $y = c_0 + c_1 x + c_2 x^2/2 + x^3$.

The **undetermined coefficients** c_0, c_1, c_2 will be found by a classical technique in which the trial solution y is back-substituted into the differential equation. We begin by computing the derivatives of y:

$y = c_0 + c_1 x + c_2 x^2/2 + x^3$	Calculated above; see Theorem 9.
$y' = c_1 + c_2 x + 3x^2$	Differentiate.
$y'' = c_2 + 6x$	Differentiate.

The relations above are back-substituted into the differential equation $y'' - y' + y = 2 - x + x^3$ as follows:

$$\begin{aligned} 2 - x + x^3 &= y'' - y' + y && \text{Write the DE backwards.} \\ &= \ [c_2 + 6x] \\ &\quad - [c_1 + c_2 x + 3x^2] && \text{Substitute for } y,\ y',\ y''. \\ &\quad + [c_0 + c_1 x + c_2 x^2/2 + x^3] \\ &= \ [c_2 - c_1 + c_0] \\ &\quad + [6 - c_2 + c_1]x \\ &\quad + [-3 + c_2/2]x^2 && \text{Collect on powers of } x. \\ &\quad + [1]x^3 \end{aligned}$$

The coefficients c_0, c_1, c_2, c_3 are found by **matching powers** on the LHS and RHS of the expanded equation:

$$(10) \qquad \begin{aligned} 2 &= [c_2 - c_1 + c_0] && \text{match constant term,} \\ -1 &= [6 - c_2 + c_1] && \text{match } x\text{-term,} \\ 0 &= [-3 + c_2/2] && \text{match } x^2\text{-term.} \end{aligned}$$

These equations are solved by back-substitution, starting with the last equation and proceeding to the first equation. The answers are successively $c_2 = 6$, $c_1 = -1$, $c_0 = -5$. For more detail on back-substitution, see the next example. Substitution into $y = c_0 + c_1 x + c_2 x^2/2 + x^3$ gives the particular solution $y_p = -5 - x + 3x^2 + x^3$.

7 Example (Undetermined Coefficients by Taylor's Method)
Solve for y_p in the equation $y'' - y' + y = 2 - x + x^3$ by Taylor's method, verifying $y_p = -5 - x + 3x^2 + x^3$.

Solution: Theorem 9 implies that there is a polynomial solution $y = c_0 + c_1x + c_2x^2/2 + c_3x^3/6$. The **undetermined coefficients** c_0, c_1, c_2, c_3 will be found by a technique related to **Taylor's method** in calculus. The Taylor technique requires differential equations obtained by successive differentiation of $y'' - y' + y = 2 - x + x^3$, as follows.

$y'' - y' + y = 2 - x + x^3$	The original.
$y''' - y'' + y' = -1 + 3x^2$	Differentiate the original once.
$y^{iv} - y''' + y'' = 6x$	Differentiate the original twice.
$y^{v} - y^{iv} + y''' = 6$	Differentiate the original three times. The process stops when the right side is constant.

Set $x = 0$ in the above differential equations. Then substitute the Taylor polynomial derivative relations

$$y(0) = c_0,\ y'(0) = c_1,\ y''(0) = c_2,\ y'''(0) = c_3.$$

It is also true that $y^{iv}(0) = y^{v}(0) = 0$, since y is a cubic. This produces the following equations for *undetermined coefficients* c_0, c_1, c_2, c_3:

$$\begin{aligned} c_2 - c_1 + c_0 &= 2 \\ c_3 - c_2 + c_1 &= -1 \\ -c_3 + c_2 &= 0 \\ c_3 &= 6 \end{aligned}$$

These equations are solved by **back-substitution**, working in reverse order. No experience with linear algebra is required, because this is strictly a low-level college algebra method. Successive back-substitutions, working from the last equation in reverse order, give the answers

$c_3 = 6$,	Use the fourth equation first.
$c_2 = c_3$	Solve for c_2 in the third equation.
$\quad = 6$,	Back-substitute c_3.
$c_1 = -1 + c_2 - c_3$	Solve for c_1 in the second equation.
$\quad = -1$,	Back-substitute c_2 and c_3.
$c_0 = 2 + c_1 - c_2$	Solve for c_0 in the first equation.
$\quad = -5$.	Back-substitute c_1 and c_2.

The result is $c_0 = -5$, $c_1 = -1$, $c_2 = 6$, $c_3 = 6$. Substitution into $y = c_0 + c_1x + c_2x^2/2 + c_3x^3/6$ gives the particular solution $y_p = -5 - x + 3x^2 + x^3$.

8 Example (Sine–Cosine Trial solution) Verify for $y'' + 4y = \sin x - \cos x$ that $y_p(x) = 5\cos x + 3\sin x$, by using the trial solution $y = A\cos x + B\sin x$.

Solution: Substitute $y = A\cos x + B\sin x$ into the differential equation and use $u'' = -u$ for $u = \sin x$ or $u = \cos x$ to obtain the relation

$$\begin{aligned} \sin x - \cos x &= y'' + 4y \\ &= (-A + 4)\cos x + (-B + 4)\sin x. \end{aligned}$$

Comparing sides, matching sine and cosine terms, gives

$$\begin{aligned} -A+4 &= -1, \\ -B+4 &= 1. \end{aligned}$$

Solving, $A = 5$ and $B = 3$. The trial solution $y = A\cos x + B\sin x$ becomes $y_p(x) = 5\cos x + 3\sin x$. Generally, this method produces linear algebraic equations that must be solved by linear algebra techniques (back-substitution is no longer an option).

9 Example (Recursive Polynomial Method)

In the equation $y'' - y' = 2 - x + x^3$, verify $y_p = -7x - 5x^2/2 - x^3 - x^4/4$ by the recursive polynomial method.

Solution: A **recursive method** will be applied, based upon the fundamental theorem of calculus, as in Example 5.

Step 1. Differentiate $y'' - y' = 2 - x + x^3$ until the right side is constant, to obtain

Equation 1: $y'' - y' = 2 - x + x^3$	The original.
Equation 2: $y''' - y'' = -1 + 3x^2$	Differentiate the original once.
Equation 3: $y^{iv} - y''' = 6x$	Differentiate the original twice.
Equation 4: $y^{v} - y^{iv} = 6$	Differentiate the original three times. The process stops when the right side is constant.

Step 2. There are 4 equations. Theorem 9 implies that there is a polynomial solution y of degree 4. Then $y^v = 0$.

The last equation $y^v - y^{iv} = 6$ then gives $y^{iv} = -6$, which can be solved for y''' by the fundamental theorem of calculus. Then $y''' = -6x + c$. Evaluate c by requiring that y satisfy equation 3: $y^{iv} - y''' = 6x$. Substitution of $y''' = -6x + c$, followed by setting $x = 0$ gives $-6 - c = 0$. Hence $c = -6$. The conclusion: $y''' = -6x - 6$.

Step 3. Solve $y''' = -6x - 6$, giving $y'' = -3x^2 - 6x + c$. Evaluate c as in *Step 2* using equation 2: $y''' - y'' = -1 + 3x^2$. Then $-6 - c = -1$ gives $c = -5$. The conclusion: $y'' = -3x^2 - 6x - 5$.

Step 4. Solve $y'' = -3x^2 - 6x - 5$, giving $y' = -x^3 - 3x^2 - 5x + c$. Evaluate c as in *Step 2* using equation 1: $y'' - y' = 2 - x + x^3$. Then $-5 - c = 2$ gives $c = -7$. The conclusion: $y' = -x^3 - 3x^2 - 5x - 7$.

Step 5. Solve $y' = -x^3 - 3x^2 - 5x - 7$, giving $y = -x^4/4 - x^3 - 5x^2/2 - 7x + c$. Just one solution is sought, so take $c = 0$. Then $y = -7x - 5x^2/2 - x^3 - x^4/4$. Theorem 9 also drops the constant term, because it is included in the homogeneous solution y_h. While this method duplicates all the steps in Example 7, it remains attractive due to its simplistic implementation. The method is best appreciated when it terminates at step 2 or 3.

10 Example (Polynomial × Exponential)

Solve for y_p in $y'' - y' + y = (2 - x + x^3)e^{2x}$, verifying that $y_p = e^{2x}(x^3/3 - x^2 + x + 1/3)$.

Solution: Let $y = e^{2x}Y$ and $[(D+2)^2 - (D+2) + 1]Y = 2 - x + x^3$, as per the *polynomial* $\times$ *exponential method*, page 174. The equation $Y'' + 3Y' + 3Y = 2 - x + x^3$ will be solved by the polynomial method of Example 7.

Differentiate $Y'' + 3Y' + 3Y = 2 - x + x^3$ until the right side is constant.

$$\begin{aligned}
&Y'' + 3Y' + 3Y = 2 - x + x^3 \\
&Y''' + 3Y'' + 3Y' = -1 + 3x^2 \\
&Y^{iv} + 3Y''' + 3Y'' = 6x \\
&Y^{v} + 3Y^{iv} + 3Y''' = 6
\end{aligned}$$

The last equation, by the equilibrium method, implies Y is a polynomial of degree 4, $Y = c_0 + c_1x + c_2x^2/2 + c_3x^3/6$. Set $x = 0$ and $c_i = Y^{(i)}(0)$ in the preceding equations to get the system

$$\begin{aligned}
c_2 + 3c_1 + 3c_0 &= 2 \\
c_3 + 3c_2 + 3c_1 &= -1 \\
c_4 + 3c_3 + 3c_2 &= 0 \\
c_5 + 3c_4 + 3c_3 &= 6
\end{aligned}$$

in which $c_4 = c_5 = 0$. Solving by back-substitution gives the answers $c_3 = 2$, $c_2 = -2$, $c_1 = 1$, $c_0 = 1/3$. Then $Y = x^3/3 - x^2 + x + 1/3$.

Finally, Kümmer's transformation $y = e^{2x}Y$ implies $y = e^{2x}(x^3/3 - x^2 + x + 1/3)$.

11 Example (Polynomial $\times$ Exponential $\times$ Cosine)

Solve in $y'' - y' + y = (3 - x)e^{2x}\cos(3x)$ for y_p, verifying that $y_p = \frac{1}{507}((26x - 107)e^{2x}\cos(3x) + (115 - 39x)e^{2x}\sin(3x))$.

Solution: Let $z = 2 + 3i$. If Y satisfies $[(D+z)^2 - (D+z) + 1]Y = 3 - x$, then $y = e^{2x}\,\mathcal{R}\text{e}(e^{3ix}Y)$, by the method on page 174. The differential equation simplifies into $Y'' + (3+6i)Y' + (9i-6)Y = 3 - x$. It will be solved by the recursion method of Example 9.

Step 1. Differentiate $Y'' + (3+6i)Y' + (9i-6)Y = 3 - x$ until the right side is constant, to obtain $Y''' + (3+6i)Y'' + (9i-6)Y' = -1$. The conclusion: $Y' = 1/(6-9i)$.

Step 2. Solve $Y' = 1/(6-9i)$ for $Y = x/(6-9i) + c$. Evaluate c by requiring Y to satisfy the original equation $Y'' + (3+6i)Y' + (9i-6)Y = 3 - x$. Substitution of $Y' = x/(6-9i) + c$, followed by setting $x = 0$ gives $0 + (3+6i)/(6-9i) + (9i-6)c = 3$. Hence $c = (-15+33i)/(6-9i)^2$. The conclusion: $Y = x/(6-9i) + (-15 + 33i)/(6-9i)^2$.

Step 3. Use variable $y = e^{2x}\,\mathcal{R}\text{e}(e^{3ix}Y)$ to complete the solution. This is the point where complex arithmetic must be used. Let $y = e^{2x}\mathcal{Y}$ where $\mathcal{Y} = \mathcal{R}\text{e}(e^{3ix}Y)$. Some details:

$$Y = \frac{x}{6-9i} + \frac{-15+33i}{(6-9i)^2}$$
The plan: write $Y = Y_1 + iY_2$.

$$= x\frac{6+9i}{6^2+9^2} + \frac{(-15+33i)(6+9i)^2}{(6^2+9^2)^2}$$
Use $1/Z = \overline{Z}/|Z|^2$, $Z = a+ib$, $\overline{Z} = a - ib$, $|Z| = a^2 + b^2$.

$$= \frac{2x}{39} + \frac{xi}{13} + \frac{-2889 - 3105i}{117^2}$$
Use $6^2 + 9^2 = 117 = (9)(13)$.

$$= \frac{26x-107}{507} + i\frac{39x-115}{507} \qquad \text{Split off real and imaginary.}$$

$$Y_1 = \frac{26x-107}{507}, \quad Y_2 = \frac{39x-115}{507} \qquad \text{Decomposition found.}$$

$$\begin{aligned}\mathcal{Y} &= \mathcal{R}\text{e}((\cos 3x + i\sin 3x)(Y_1+iY_2)) && \text{Use } e^{3ix} = \cos 3x + i\sin 3x.\\ &= Y_1\cos 3x - Y_2\sin 3x && \text{Take the real part.}\\ &= \frac{26x-107}{507}\cos 3x + \frac{115-39x}{507}\sin 3x && \text{Substitute for } Y_1,\, Y_2.\end{aligned}$$

The solution $y = e^{2x}\mathcal{Y}$ multiplies the above display by e^{2x}. This verifies the formula $y_p = \frac{1}{507}((26x-107)e^{2x}\cos(3x) + (115-39x)e^{2x}\sin(3x))$.

12 Example (Polynomial × Exponential × Sine)

Solve in $y'' - y' + y = (3-x)e^{2x}\sin(3x)$ for y_p, verifying that a particular solution is $y_p = \frac{1}{507}\left((39x-115)e^{2x}\cos(3x) + (26x-107)e^{2x}\sin(3x)\right)$.

Solution: Let $z = 2+3i$. Kümmer's transformation $y = e^{2x}\,\mathcal{I}\text{m}(e^{3ix}Y)$ as on page 174 implies that Y satisfies $[(D+z)^2-(D+z)+1]Y = 3-x$. This equation has been solved in the previous example: $Y = Y_1+iY_2$ with $Y_1 = (26x-107)/507$ and $Y_2 = (39x-115)/507$. Let $\mathcal{Y} = \mathcal{I}\text{m}(e^{3ix}Y)$. Then

$$\begin{aligned}\mathcal{Y} &= \mathcal{I}\text{m}((\cos 3x + i\sin 3x)(Y_1+iY_2)) && \text{Expand complex factors.}\\ &= Y_2\cos 3x + Y_1\sin 3x && \text{Extract the imaginary part.}\\ &= \frac{(39x-115)\cos 3x + (26x-107)\sin 3x}{507} && \text{Substitute for } Y_1 \text{ and } Y_2.\end{aligned}$$

The solution $y = e^{2x}\mathcal{Y}$ multiplies the display by e^{2x}. This verifies the formula $y = \frac{1}{507}\left((39x-115)e^{2x}\cos(3x) + (26x-107)e^{2x}\sin(3x)\right)$.

13 Example (Undetermined Coefficient Algorithm)

Solve $y'' - y' + y = 1 + e^x + \cos(x)$, verifying $y = c_1e^{x/2}\cos(\sqrt{3}x/2) + c_2e^{x/2}\sin(\sqrt{3}x/2) + 1 + e^x - \sin(x)$.

Solution: There are $n = 3$ easily solved equations: $y_1'' - y_1' + y_1 = 1$, $y_2'' - y_2' + y_2 = e^x$ and $y_3'' - y_3' + y_3 = \cos(x)$. The plan is that each such equation is solvable by one of the **library methods**. Then $y_p = y_1 + y_2 + y_3$ is the sought particular solution.

Equation 1: $y_1'' - y_1' + y_1 = 1$. It is solved by *the equilibrium method*, which gives immediately solution $y_1 = 1$.

Equation 2: $y_2'' - y_2' + y_2 = e^x$. Then $y_2 = e^xY$ and $[(D+1)^2 - (D+1) + 1]Y = 1$, by the *polynomial × exponential method*. The equation simplifies to $Y'' + Y' + Y = 1$. Obtain $Y = 1$ by the *equilibrium method*, then $y_2 = e^x$.

Equation 3: $y_3'' - y_3' + y_3 = \cos(x)$. Then $[(D+i)^2 - (D+i) + 1]Y = 1$ and $y_3 = \mathcal{R}\text{e}(e^{ix}Y)$, by the *polynomial × exponential × cosine method*. The equation simplifies to $Y'' + (2i-1)Y' - iY = 1$. Obtain $Y = i$ by the *equilibrium method*. Then $y_3 = \mathcal{R}\text{e}(e^{ix}Y)$ implies $y_3 = -\sin(x)$.

Solution y_p. The particular solution is given by addition, $y_p = y_1 + y_2 + y_3$. Therefore, $y_p = 1 + e^x - \sin(x)$.

Solution y_h. The homogeneous solution y_h is the linear equation *recipe* solution for $y'' - y' + y = 0$, which uses the characteristic equation $r^2 - r + 1 = 0$. The latter has roots $r = (1 \pm i\sqrt{3})/2$ and then $y_h = c_1 e^{x/2}\cos(\sqrt{3}x/2) + c_2 e^{x/2}\sin(\sqrt{3}x/2)$ where c_1 and c_2 are arbitrary constants.

General Solution. Add y_h and y_p to obtain the general solution

$$y = c_1 e^{x/2}\cos(\sqrt{3}x/2) + c_2 e^{x/2}\sin(\sqrt{3}x/2) + 1 + e^x - \sin(x).$$

14 Example (Trial Solution Shortcut I)

Solve $y'' - y' + y = 2 + e^x + \sin(x)$ by the trial solution shortcut method, verifying $y = c_1 e^{x/2}\cos(\sqrt{3}x/2) + c_2 e^{x/2}\sin(\sqrt{3}x/2) + 2 + e^x + \cos(x)$.

Solution: There are $n = 3$ easily solved equations: $y_1'' - y_1' + y_1 = 2$, $y_2'' - y_2' + y_2 = e^x$ and $y_3'' - y_3' + y_3 = \sin(x)$. The plan is that each such equation is solvable by trial solution methods, giving $y_1 = 2$, $y_2 = e^x$ and $y_3 = \cos x$. Then $y_p = y_1 + y_2 + y_3$ is the sought particular solution.

Equation 1: $y_1'' - y_1' + y_1 = 2$. An initial trial solution by Table 2 is $y = a_0$. The values are $k = 0$, $m = 0$ and the root $r = k + m\sqrt{-1}$ is $r = 0$. Because $r = 0$ is not a root of the characteristic equation $r^2 - r + 1 = 0$, then the Fixup Rule implies $y = a_0$ is the corrected trial solution. Substitution gives $a_0 = 2$. Then $y_1 = 2$.

Equation 2: $y_2'' - y_2' + y_2 = e^x$. Table 2 says $y = a_0 e^x$ is the initial trial solution. The values are $k = 1$, $m = 0$ and $r = k + m\sqrt{-1}$ is $r = 1$, which is not a root of the characteristic equation $r^2 - r + 1 = 0$. The Fixup Rule says $y = a_0 e^x$ is the corrected trial solution. Substitution into $y'' - y' + y = e^x$ gives $(a_0 - a_0 + a_0)e^x = e^x$. Hence $a_0 = 1$. Then $y_2 = e^x$.

Equation 3: $y_3'' - y_3' + y_3 = \sin(x)$. Table 2 says $y = a_0\cos x + b_0 \sin x$ is the initial trial solution. The values are $k = 0$, $m = 1$ and $r = k + m\sqrt{-1}$ is $r = i$, which is not a root of the characteristic equation $r^2 - r + 1 = 0$. The Fixup Rule says $y = a_0\cos x + b_0\sin x$ is the corrected trial solution. Substitution into $y'' - y' + y = e^x$ gives $-a_0\cos x - b_0\sin x - (-a_0\sin x + b_0\cos x) + (a_0\cos x + b_0\sin x) = \sin x$. Matching sine and cosine terms left and right gives $-b_0 = 0$, $a_0 = 1$. Then $y_3 = \cos x$.

Solution y_p. The particular solution is given by addition, $y_p = y_1 + y_2 + y_3$. Therefore, $y_p = 2 + e^x + \cos(x)$.

Solution y_h. The homogeneous solution y_h is the linear equation *recipe* solution for $y'' - y' + y = 0$, which uses the characteristic equation $r^2 - r + 1 = 0$. The latter has roots $r = (1 \pm i\sqrt{3})/2$ and then $y_h = c_1 e^{x/2}\cos(\sqrt{3}x/2) + c_2 e^{x/2}\sin(\sqrt{3}x/2)$ where c_1 and c_2 are arbitrary constants.

General Solution. Add y_h and y_p to obtain the general solution

$$y = c_1 e^{x/2}\cos(\sqrt{3}x/2) + c_2 e^{x/2}\sin(\sqrt{3}x/2) + 2 + e^x + \cos(x).$$

15 Example (Trial Solution Shortcut II)

Solve for y_p in $y'' - 2y' + y = (1 + x - x^2)e^x$ by the trial solution shortcut method, verifying that $y_p = (x^2/2 + x^3/6 - x^4/12)e^x$.

Solution: An initial trial solution by Table 2 is $y = (a_0 + a_1x + a_2x^2)e^x$. The values are $k = 1$, $m = 0$ and the root $r = k + m\sqrt{-1}$ is $r = 1$. Because $r = 1$ is a root of the characteristic equation $r^2 - 2r + 1 = 0$, of multiplicity 2, then the Fixup Rule implies y should be multiplied twice by x to obtain the corrected trial solution $y = x^2(a_0 + a_1x + a_2x^2)e^x$.

Substitute the corrected trial solution into the full differential equation in order to find the undetermined coefficients a_0, a_1, a_2. To present the details, let $q(x) = x^2(a_0 + a_1x + a_2x^2)$, then

$$\begin{aligned}
\text{LHS} &= y'' - 2y' + y \\
&= [q(x)e^x]'' - 2[q(x)e^x]' + q(x)e^x \\
&= q(x)e^x + 2q'(x)e^x + q''(x)e^x - 2q'(x)e^x - 2q(x)e^x + q(x)e^x \\
&= q''(x)e^{2x} \\
&= [2a_0 + 6a_1x + 12a_2x^2]e^{2x}.
\end{aligned}$$

Because LHS = RHS = $(1+x-x^2)e^x$, then e^x cancels and $2a_0+6a_1x+12a_2x^2 = 1 + x - x^2$. Matching powers of x gives $2a_0 = 1$, $6a_1 = 1$, $12a_2 = -1$. Then $y = x^2(1/2 + x/6 - x^2/12)e^x$.

It is visible that Kümmer's substitution $y = q(x)e^x$ was made in the first four (4) lines of the details. Knowing Kümmer's result is useful to reduce labor on this type of problem. For instance, lines 2, 3 of the details could be skipped, knowing that line 4 must be $[(D + 1)^2 - 2(D + 1) + 1]q(x)$ times e^x.

Historical Notes. The classical method of undetermined coefficients can be presented using only the idea of a **trial solution**; see **Trial Solution Shortcut** above, Page 175. Textbooks that present this method appear in the references, especially [EP2] and [Krey].

If $f(x)$ is a polynomial, then the trial solution is a polynomial $y = a_0 + \cdots + a_dx^d$ with unknown coefficients. It is substituted into the differential equation $Ly = f(x)$ to determine the coefficients $a_0, \ldots, a_d$, as in Example 6. The Taylor method in Example 7 implements the same ideas. In the classical presentation, the three theorems of this section are replaced by Table 2 and the **Fixup Rule** on page 176. Attempts have been made to integrate the fixup rule into the table itself; see [EP] and [EP2].

The *method of annihilators* has been used as an alternative approach; see [KKOP]. The approach gives a deeper insight into higher order differential equations. It requires substantial knowledge of linear algebra.

The idea to employ a recursive method seems to appear first in a paper by Love [LV]. A generalization and expansion of details appears in [Gupta]. While the method is certainly worth learning, doing so does not excuse the reader from also learning the polynomial method. The recursive method is a worthwhile shortcut for special circumstances.

Proof of Theorem 9: The three cases correspond to zero, one or two roots $r = 0$ for the characteristic equation $ar^2 + br + c = 0$. The missing constant and x-terms in **case 2** and **case 3** are justified by including them in the homogeneous solution y_h, instead of in the particular solution y_p.

Assume $p(x)$ has degree d and succinctly write down the successive derivatives of the differential equation as

$$ay^{(2+k)} + by^{(1+k)} + cy^{(k)} = p^{(k)}(x), \quad k = 0, \ldots, d. \tag{11}$$

Assume, to consider simultaneously all three cases, that

$$y = y_0 + y_1 + \cdots + y_{m+d}\frac{x^{m+d}}{(m+d)!}$$

where $m = 0, 1, 2$ corresponding to cases 1,2,3, respectively. It has to be shown that there are coefficients $y_0, \ldots, y_{m+d}$ such that y is a solution of $ay'' + by' + cy = p(x)$.

Let $x = 0$ in equations (11) and use the definition of polynomial y to obtain the equations

$$ay_{2+k} + by_{1+k} + cy_k = p^{(k)}(0), \quad k = 0, \ldots, d. \tag{12}$$

In **case 1** ($c \neq 0$), $m = 0$ and the last equation in (12) gives $y_{m+d} = p^{(d)}(0)/c$. Back-substitution succeeds in finding the other coefficients, in reverse order, because $y^{(d+1)}(0) = y^{(d+2)}(0) = 0$, in this case. *Define* the constants y_0 to y_d to be the solutions of (12). Define $y_{d+1} = y_{d+2} = 0$.

In **case 2** ($c = 0$, $b \neq 0$), $m = 1$ and the last equation in (12) gives $y_{m+d} = p^{(d)}(0)/b$. Back-substitution succeeds in finding the other coefficients, in reverse order, because $y^{(d+2)}(0) = 0$, in this case. However, y_0 is undetermined. Take it to be zero, then *define* y_1 to y_{d+1} to be the solutions of (12). Define $y_{d+2} = 0$.

In **case 3** ($c = b = 0$), $m = 2$ and the last equation in (12) gives $y_{m+d} = p^{(d)}(0)/a$. Back-substitution succeeds in finding the other coefficients, in reverse order. However, y_0 and y_1 are undetermined. Take them to be zero, then *define* y_2 to y_{d+2} to be the solutions of (12).

It remains to prove that the polynomial y so defined is a solution of the differential equation $ay'' + by' + cy = p(x)$. Begin by applying quadrature to the last differentiated equation $ay^{(2+d)} + by^{(1+d)} + cy^{(d)} = p^{(d)}(x)$. The result is $ay^{(1+d)} + by^{(d)} + cy^{(d-1)} = p^{(d-1)}(x) + C$ with C undetermined. Set $x = 0$ in this equation. Then relations (12) say that $C = 0$. This process can be continued until $ay'' + by' + cy = q(x)$ is obtained, hence y is a solution.

Proof of Theorem 10: Kümmer's transformation $y = e^{kx}Y$ is differentiated twice to give the formulas

$$\begin{aligned} y &= e^{kx}Y, \\ y' &= ke^{kx}Y + e^{kx}Y' \\ &= e^{kx}(D+k)Y, \\ y'' &= k^2e^{kx}Y + 2ke^{kx}Y' + e^{kx}Y'' \\ &= e^{kx}(D+k)^2Y. \end{aligned}$$

Insert them into the differential equation $a(D+k)^2Y + b(D+k)Y + cY = p(x)$. Then multiply through by e^{kx} to remove the common factor e^{-kx} on the left, giving $ay'' + by' + cy = p(x)e^{kx}$. This completes the proof.

Proof of Theorem 11: Abbreviate $ay''+by'+cy$ by Ly. Consider the complex equation $Lu = p(x)e^{zx}$, to be solved for $u = u_1 + iu_2$. According to Theorem

10, u can be computed as $u = e^{zx}Y$ where $[a(D+z)^2 + b(D+z) + c]Y = p(x)$. Take the real and imaginary parts of $u = e^{zx}Y$ and $Lu = p(x)e^{zx}$. Then $u_1 = \mathcal{R}\text{e}(e^{zx}Y)$ and $u_2 = \mathcal{I}\text{m}(e^{zx}Y)$ satisfy $Lu_1 = \mathcal{R}\text{e}(p(x)e^{zx}) = p(x)\cos(mx)e^{kx}$ and $Lu_2 = \mathcal{I}\text{m}(p(x)e^{zx}) = p(x)\sin(mx)e^{kx}$. This completes the proof.

Exercises 4.3

Polynomial Solutions. Determine a polynomial solution y_p for the given differential equation. Apply Theorem 9, page 177, and model the solution after Examples 5, 6, 7 and 9.

1. $y'' = x$

2. $y'' = x - 1$

3. $y'' = x^2 - x$

4. $y'' = x^2 + x - 1$

5. $y'' - y' = 1$

6. $y'' - 5y' = 10$

7. $y'' - y' = x$

8. $y'' - y' = x - 1$

9. $y'' - y' + y = 1$

10. $y'' - y' + y = -2$

11. $y'' + y = 1 - x$

12. $y'' + y = 2 + x$

13. $y'' - y = x^2$

14. $y'' - y = x^3$

Polynomial-Exponential Solutions. Determine a solution y_p for the given differential equation. Apply Theorem 10, page 177, and model the solution after Example 10.

15. $y'' + y = e^x$

16. $y'' + y = e^{-x}$

17. $y'' = e^{2x}$

18. $y'' = e^{-2x}$

19. $y'' - y = (x+1)e^{2x}$

20. $y'' - y = (x-1)e^{-2x}$

21. $y'' - y' = (x+3)e^{2x}$

22. $y'' - y' = (x-2)e^{-2x}$

23. $y'' - 3y' + 2y = (x^2+3)e^{3x}$

24. $y'' - 3y' + 2y = (x^2-2)e^{-3x}$

Sine and Cosine Solutions. Determine a solution y_p for the given differential equation. Apply Theorem 11, page 177, and model the solution after Examples 11 and 12.

25. $y'' = \sin(x)$

26. $y'' = \cos(x)$

27. $y'' + y = \sin(x)$

28. $y'' + y = \cos(x)$

29. $y'' = (x+1)\sin(x)$

30. $y'' = (x+1)\cos(x)$

31. $y'' - y = (x+1)e^x \sin(2x)$

32. $y'' - y = (x+1)e^x \cos(2x)$

33. $y'' - y' - y = (x^2+x)e^x \sin(2x)$

34. $y'' - y' - y = (x^2+x)e^x \cos(2x)$

Undetermined Coefficients Algorithm. Determine a solution y_p for the given differential equation. Apply the polynomial algorithm, page 172, and model the solution after Example 13.

35. $y'' = x + \sin(x)$

36. $y'' = 1 + x + \cos(x)$

37. $y'' + y = x + \sin(x)$

38. $y'' + y = 1 + x + \cos(x)$

39. $y'' + y = \sin(x) + \cos(x)$

40. $y'' + y = \sin(x) - \cos(x)$

41. $y'' = x + xe^x + \sin(x)$

42. $y'' = x - xe^x + \cos(x)$

43. $y'' - y = \sinh(x) + \cos^2(x)$

44. $y'' - y = \cosh(x) + \sin^2(x)$

45. $y'' + y' - y = x^2 e^x + xe^x \cos(2x)$

46. $y'' + y' - y = x^2 e^{-x} + xe^x \sin(2x)$

Additional Proofs. The exercises below fill in details in the text.

47. **(Superposition)** Let Ly denote $ay'' + by' + cy$. Show that solutions of $Lu = f(x)$ and $Lv = g(x)$ add to give $y = u+v$ as a solution of $Ly = f(x) + g(x)$.

48. **(Easily Solved Equations)** Let Ly denote $ay'' + by' + cy$. Let $Ly_k = f_k(x)$ for $k = 1, \ldots, n$ and define $y = y_1 + \cdots + y_n$, $f = f_1 + \cdots + f_n$. Show that $Ly = f(x)$.

49. **(Theorem 9)** Supply the details in the proof of Theorem 9 for case 1. In particular, give the details for back-substitution.

50. **(Theorem 9)** Supply the details in the proof of Theorem 9 for case 2. In particular, give the details for back-substitution and explain fully why it is possible to select $y_0 = 0$.

51. **(Theorem 9)** Supply the details in the proof of Theorem 9 for case 3. In particular, explain why back-substitution leaves y_0 and y_1 undetermined, and why it is possible to select $y_0 = y_1 = 0$.