

Math 2250 Maple Project 2, March 2004. Tacoma Narrows.

NAME \_\_\_\_\_ CLASSTIME \_\_\_\_ VERSION A-E, F-K, L-R, S-Z

Circle the version - see problem 2.1. There are six (6) problems in this project. Please answer the questions A, B, C , ... associated with each problem. The original worksheet "project2-fall-2003.mws" is a template for the solution; you must fill in the code and all comments. Sample code can be copied with the mouse. Use pencil freely to annotate the worksheet and to clarify the code and figures.

The problem headers for the Spring 2004 revision of David Eyre's project (original was year 2000).

- 2.1. OVER-DAMPED FREE OSCILLATIONS.
- 2.2. CRITICALLY DAMPED FREE OSCILLATIONS.
- 2.3. UNDER-DAMPED FREE OSCILLATIONS.
- 2.4. UNDAMPED FORCED OSCILLATIONS.
- 2.5. PRACTICAL RESONANCE.
- 2.6. MCKENNA NON-HOOKES LAW CABLE MODEL.

### 2.1. PROBLEM (OVER-DAMPED FREE OSCILLATIONS)

FREE OSCILLATIONS. Consider the general problem of free linear oscillations

$$m x'' + c x' + k x = 0,$$
$$x(0) = x_0, \quad x'(0) = v_0.$$

Assume  $m$ ,  $c$  and  $k$  are non-negative constants. The symbols  $x_0$  and  $v_0$  are the initial position and initial velocity, respectively. Treated here is the over-damped case  $c^2 > 4km$ , page 317 of E&P. Depending on the first letter of your last name, assume:

Version A-E:  $m=3, k=20$                       Version F-K:  $m=4, k=24$   
Version L-R:  $m=3, k=24$                       Version S-Z:  $m=5, k=20$

- A. Suggest a value for parameter  $c > 0$  so that the free oscillations are over-damped. This value will be used in item B below. Check your answer by solving the characteristic equation using Maple's "solve" command.
  
- B. Use  $x(0)=1$  and  $x'(0) = -4$  for the initial conditions and Maple's "dsolve" to find the explicit real solution  $x(t)$ . Plot the solution  $x(t)$  for  $t=0$  to  $t=5$  using Maple's "plot" command. Check the plot against Figure 5.4.6 page 317.

EXAMPLE(Wrong parameters! Change it!)

```
# Use semicolons to see what you have done.
de:=3*diff(x(t),t,t)+1.5*diff(x(t),t)
+4*x(t)=0:                                     # Define the differential equation
solve(3*r^2+1.5*r+4=0,r);                       # Solve characteristic equation.
ic:=x(0)=0,D(x)(0)= -1:                         # Define the initial conditions
p:=dsolve({de,ic},x(t),method=laplace):# Symbolically solve for x(t)
X:=unapply(rhs(p),t):                           # Capture the dsolve symbolic
# answer as a function X(t)
```

```

plot(X(t),t=0..5);           # Plot the solution
>
> #2.1-A
> # over-damped means  $mr^2+cr+k=0$  has two real roots.
> #2.1-B
>
2.2 PROBLEM (CRITICALLY DAMPED FREE OSCILLATIONS)

```

FREE OSCILLATIONS. Consider the free linear oscillation problem

$$m x'' + c x' + k x = 0,$$

$$x(0) = 0, \quad x'(0) = 1.$$

Here,  $m$ ,  $c$  and  $k$  are non-negative constants. The critically damped case is studied here,  $c^2 = 4km$ , as on page 318 in E&P. Depending on the first letter of your last name, assume:

Version A-E: $m=1, c=5$	Version F-K: $m=2, c=6$
Version L-R: $m=3, c=7$	Version S-Z: $m=4, c=8$

- A. Display the Hooke's constant  $k > 0$  so that the equation is critically damped and the solution  $x(t)$  changes sign at most once. Display the exact symbolic solution  $x(t)$ , using maple methods from the 2.1 example.
- B. Plot the exact symbolic solution  $x(t)$  on a suitable  $t$ -interval. Check the graphic against Figure 5.4.7 page 317 of E&P.

```

>
> #2.2-A Define k, then solve.
> # critically damped means  $mr^2+cr+k=0$  has two equal roots.
> #2.2-B Plot.
>
2.3. PROBLEM (UNDER-DAMPED FREE OSCILLATIONS)

```

FREE OSCILLATIONS. Consider the problem of free linear oscillations

$$m x'' + c x' + k x = 0,$$

$$x(0) = 0, \quad x'(0) = 1.$$

Here,  $m$ ,  $c$  and  $k$  are non-negative constants. The under-damped case is studied here,  $c^2 < 4km$ , as on page 318 in E&P. Depending on the first letter of your last name, assume:

Version A-E: $m=1, c=3$	Version F-K: $m=2, c=4$
Version L-R: $m=3, c=5$	Version S-Z: $m=4, c=6$

- A. Display a Hooke's constant  $k > 0$  so that the solution  $x(t)$  is under-damped. Check that  $x(t) = 0$  infinitely many times on  $t > 0$ . Display the exact solution  $x(t)$  obtained by maple methods as in the 2.1 example.
- B. Plot the exact symbolic solution  $x(t)$  on a suitable  $t$ -interval. Check the graphic against Figure 5.4.8 page 318 of E&P.

- C. Estimate from the graph the decimal value of the pseudo-period. Display the graphical estimate and also the exact pseudo-period  $2\pi/w$ , where  $w$  is the natural frequency of the trigonometric term in the solution  $x(t)$  found in item 2.3.A.

Maple tip: Click with the mouse on the graphic to print the cursor location (left upper corner of the maple window). The coordinates printed are of the form  $(x,y)$ . From this coordinate information, a simple subtraction estimates the period

```
>
> #2.3-A Define k, then solve.
> # under-damped means  $mr^2+cr+k=0$  has two conjugate complex roots.
> #2.3-B Plot.
> #2.3-C Pseudo-period calculations.
>
```

#### 2.4. PROBLEM (UNDAMPED FORCED OSCILLATIONS )

FORCED LINEAR OSCILLATIONS. Consider the undamped ( $c=0$ ) forced problem

$$mx'' + kx = 5 \cos(\omega t),$$

$$x(0)=0, \quad x'(0)=0,$$

where  $m$ ,  $k$  and  $w$  are non-negative constants. Depending on the first letter of your last name, assume:

Version A-E: $m=1, k=2.5$	Version F-K: $m=2, k=3.5$
Version L-R: $m=3, k=4.5$	Version S-Z: $m=4, k=5.5$

- A. Choose the forcing angular frequency  $w$  to be 3 times larger than the natural angular frequency  $w_0$ ,  $w_0^2=k/m$ . Solve for  $x(t)$  using `dsolve()`. Plot the solution  $x(t)$  on a suitable interval in order to show the global behavior of the solution  $x(t)$ . See Figure 5.6.2, page 341.
- B. The solution  $x(t)$  is the sum of two functions, one of period  $2\pi/w$  and the other of period  $2\pi/w_0$ . Display the exact period, as calculated from the solution formula for  $x(t)$  -- see page 341 for details.
- C. Suggest a value for the forcing frequency  $w$  so that the oscillations exhibit resonance. Show resonant behavior on a graph. Check against Figure 5.6.4, page 343.

```
> #2.4-A
> #2.4-B
> #2.4-C
>
```

#### 2.5. PROBLEM (PRACTICAL RESONANCE)

Consider the damped forced problem

$$mx'' + c x' + kx = 5 \cos(\omega t),$$

$$x(0)=0, \quad x'(0)=0.$$

Depending on the first letter of your last name, assume:

Version A-E: m=1, k=25                      Version F-K: m=2, k=36  
Version L-R: m=3, k=50                      Version S-Z: m=4, k=60

- A. Consider the damping constants  $c=2$ ,  $c=1$  and  $c=1/2$ . Compute the amplitude function  $C(w)$  [page 346] for these three equations, then plot for  $w=0$  to  $w=20$  the three amplitude graphs on a single set of axes. Compare against Figure 5.6.9 page 348 of E&P (it has one curve, yours has 3 curves).
- B. For each case  $c=2$ ,  $c=1$ ,  $c=1/2$ , print the values  $w^*$ ,  $C^*$  where  $C^*=C(w^*)=\max \{C(w) : 0 \leq w \leq 20\}$ . The three data pairs should show that  $C^*$  becomes larger as  $c$  tends to zero. SAVE YOUR MAPLE FILE FREQUENTLY

Maple Hint: Use Maple's mouse interface on the graphic of Part C. Specifically, click on a possible maximum (horizontal tangent) in the graph to display the values  $w^*$ ,  $C^*$  on the screen. Copy the values on paper.

EXAMPLE(Beware! Wrong values!)

```
F:=15: m:=1: k:=25: c:='c': w:='w':  
C:=(w,c)->F/sqrt((k-m*w*w)^2+(c*w)^2):  
plot({C(w,4),C(w,3),C(w,2)},w=0..15,color=black);
```

```
> #2.5-A Plot C(w), three graphics on one set of axes  
> #2.5-B Table of six data values for w*, C*
```

## 2.6. PROBLEM (NONLINEAR MCKENNA MODELS)

There are six (6) parts 2.6A to 2.6F to complete. Mostly, this is mouse copying. Retyping the maple code by hand is not recommended.

NONLINEAR TORSIONAL MODEL WITH GEOMETRY INCLUDED.

Consider the nonlinear, forced, damped oscillator equation for torsional motion, with bridge geometry included,

$$x'' + 0.05 x' + 2.4 \sin(x)\cos(x) = 0.06 \cos(12 t/10) ,$$
$$x(0) = x_0, \quad x'(0) = v_0$$

and its corresponding linearized equation

$$x'' + 0.05 x' + 2.4 x = 0.06 \cos(12 t/10) ,$$
$$x(0) = x_0, \quad x'(0) = v_0.$$

The spring-mass system parameters are  $m=1$ ,  $c = 0.05$ ,  $k = 2.4$ ,  $w = 1.2$ ,  $F = 0.06$ . Maple code used to solve and plot the solutions appears below.

```
# WARNING: set the parameters on the second line!  
# Use "copy as maple text" for maple 6+.  
m:=1: F := 0.06: w := 1.2: m:=1: c:= 0.05: k:= 2.4:  
x0:=0: v0:=0: a:=0: b:=50:  
deNonLinear:= m*diff(x(t),t,t) + c*diff(x(t),t) +  
              k*sin(x(t))*cos(x(t)) = F*cos(w*t):
```

```

deLinear:= m*diff(x(t),t,t) + c*diff(x(t),t) + k*x(t) = F*cos(w*t):
with(DEtools):  opts:=stepsize=0.1:IC=[[x(0)=x0,D(x)(0)=v0]]:
DEplot(deNonLinear,x(t),t=a..b,IC,opts,title='NonLinear');
DEplot(deLinear,x(t),t=a..b,IC,opts,title='Linear');

```

- A. Let  $x_0=0$ ,  $v_0=0$ . Plot the solutions of the linear and nonlinear equations from  $t=200$  to  $t=300$ . These plots represent the steady state solutions of the two equations.
- B. Let  $x_0=1.2$ ,  $v_0=0$ . Plot the solutions of the linear and nonlinear equations from  $t=220$  to  $t=320$ . These plots represent the steady state solutions of the two equation, with new starting value  $x_0=1.2$ .

The two linear plots in A and B have to be identical to the plot of  $x_{ss}(t)$ . The reason is the superposition formula (see E&P)  $x(t)=x_h(t)+x_{ss}(t)$ , even though the homogeneous solution  $x_h(t)$  is different for the two plots. This is because  $x_h(t)$  has limit zero at  $t=\infty$ .

- C. Determine the ratio of the apparent amplitudes (a number  $> 1$ ) for the nonlinear plots in A and B. Do "large sustained oscillations" appear in the plot of the nonlinear steady-state?

```

>
> #2.6-A
> #2.6-B
> #2.6-C
>

```

#### MCKENNA'S NON-HOOKE'S LAW CABLE MODEL FOR THE TACOMA NARROWS BRIDGE

The model of McKenna studies the bridge with a nonlinear, forced, damped oscillator equation for torsional motion that accounts for the non-Hooke's law cables coupled to the equations for vertical motion. The equations in this case couple the torsional motion with the vertical motion. The equations are:

$$\begin{aligned}
 x'' + c x' - k G(x,y) &= F \sin wt, & x(0) &= x_0, & x'(0) &= x_1, \\
 y'' + c y' + (k/3) H(x,y) &= g, & y(0) &= y_0, & y'(0) &= y_1,
 \end{aligned}$$

where  $x(t)$  is the torsional motion and  $y(t)$  is the vertical motion. The functions  $G(x,y)$  and  $H(x,y)$  are the models of the force generated by the cable when it is contracted and stretched. Below is sample code for writing the differential equations and for plotting the solutions. It is ready to copy with the mouse.

```

with(DEtools):
w := 1.3:  F := 0.05:  f(t) := F*sin(w*t):
c := 0.01:  k1 := 0.2:  k2 := 0.4:  g := 9.8:  L := 6:
STEP:=x->piecewise(x<0,0,1):
fp(t) := y(t)+(L*sin(x(t))):
fm(t) := y(t)-(L*sin(x(t))):
Sm(t) := STEP(fm(t))*fm(t):
Sp(t) := STEP(fp(t))*fp(t):

```

```

sys := {
    diff(x(t),t,t) + c*diff(x(t),t) - k1*cos(x(t))*(Sm(t)-Sp(t))=f(t),
    diff(y(t),t,t) + c*diff(y(t),t) + k2*(Sm(t)+Sp(t)) = g}:
ic := [[x(0)=0, D(x)(0)=0, y(0)=27.25, D(y)(0)=0]]:
vars:=[x(t),y(t)]:
opts:=stepsize=0.1:
DEplot(sys,vars,t=0..300,ic,opts,scene=[t,x]);

```

The amazing thing that happens in this simulation is that the large vertical oscillations take all the tension out of the springs and they induce large torsional oscillations.

D. TORSIONAL OSCILLATION PLOT. Get the sample code above to produce the plot of  $x(t)$  [that's what `scene=[t,x]` means].

E. Estimate the number of degrees the roadway tilts based on the plot. Recall that  $x$  in the plot is reported in radians. Comment on the agreement of this result with historical data and the video evidence in the film clip.

Tip: Average the five largest amplitudes in the plot to find an average maximum amplitude for  $t=0$  to  $t=300$ . Convert to degrees using  $\text{Pi radians} = 180 \text{ degrees}$ . The film clip shows roadway maximum tilt of 30 to 45 degrees, approximately.

F. VERTICAL OSCILLATION PLOT. Modify the DEplot code to `scene=[t,y]` and plot the oscillation  $y(t)$  on  $t=0$  to  $t=300$ . The plot is supposed to show 30-foot vertical oscillations along the roadway that dampen to 7-foot vertical oscillations after 300 seconds.

The agreement between these oscillation results and the historical data for Tacoma Narrows, especially the visual data present in the film clip of the bridge disaster, should be clear from the plots. This is your only answer check for the plot results.

```

>
> #2.6-D Torsional plot t-versus-x
> #2.6-E Roadway oscillation estimate in degrees + comments.
> #2.6-F Vertical plot t-versus-y.
>

```