Math 2250 Maple Project 2, October 2003. Tacoma Narrows.

NAME ______________________ CLASSTIME ____ VERSION A-E, F-K, L-R, S-Z

Circle the version - see problem 2.1. There are six (6) problems in this project. Please answer the questions A, B, C , ... associated with each problem. The original worksheet "project2-fall-2003.mws" is a template for the solution; you must fill in the code and all comments. Sample code can be copied with the mouse. Use pencil freely to annotate the worksheet and to clarify the code and figures.

The problem headers for the Fall 2003 revision of David Eyre's project (original was year 2000).

__________ 2.1. OVERDAMPED FREE OSCILLATIONS.
__________2.2. CRITCALLY DAMPED FREE OSCILLATIONS.
__________2.3. UNDERDAMPED FREE OSCILLATIONS.
__________2.4. UNDAMPED FORCED OSCILLATIONS.
__________2.5. PRACTICAL RESONANCE.
__________2.6. MCKENNA NON-HOOKES LAW CABLE MODEL.

2.1. PROBLEM (OVERDAMPED FREE OSCILLATIONS)

FREE OSCILLATIONS. Consider the general problem of free linear oscillations

```
m x'' + c x' + k x=0,
x(0)=x0,  x'(0)=v0.
```

Assume m, c and k are non-negative constants. The symbols x0 and v0 are the initial position and initial velocity, respectively. Treated here is the overdamped case c^2 > 4km, page 317 of E&P. Depending on the first letter of your last name, assume:

```
Version A-E:  m=1, k=16       Version F-K: m=2, k=20
Version L-R: m=3, k=18        Version S-Z: m=4, k=24
```

A. Suggest a value for parameter c > 0 so that the free oscillations are overdamped. This value will be used in item B below. Check your answer by solving the characteristic equation using Maple's "solve" command.

B. Use x(0)=1 and x'(0)= -2 for the initial conditions and Maple's "dsolve" to find the explicit real solution x(t). Plot the solution x(t) for t=0 to t=5 using Maple's "plot" command. Check the plot against Figure 5.4.6 page 317.

EXAMPLE(Wrong parameters! Change it!)

```
# Use semicolons to see what you have done.
de:=3*diff(x(t),t,t)+1.5*diff(x(t),t)
    +4*x(t)=0:                         # Define the differential equation
solve(3*r^2+1.5*r+4=0,r);              # Solve characteristic equation.
ic:=x(0)=0,D(x)(0)= -1:                # Define the initial conditions
p:=dsolve({de,ic},x(t),method=laplace):# Symbolically solve for x(t)
X:=unapply(rhs(p),t):                  # Capture the dsolve symbolic
                                       # answer as a function X(t)
```

```
plot(X(t),t=0..5);                     # Plot the solution
>
> #2.1-A
> # overdamped means mr^2+cr+k=0 has two real roots.
> #2.1-B
>
```

2.2 PROBLEM (CRITICALLY DAMPED FREE OSCILLATIONS)

FREE OSCILLATIONS. Consider the free linear oscillation problem

```
m x'' + c x' + k x=0,
x(0)=0,  x'(0)=1.
```

Here, m, c and k are non-negative constants. The critically damped case is studied here, $c^2 = 4km$, as on page 318 in E&P. Depending on the first letter of your last name, assume:

Version A-E: m=1, c=4   Version F-K: m=2, c=5
Version L-R: m=3, c=7   Version S-Z: m=4, c=6

A. Display the Hooke's constant k > 0 so that the equation is critically damped and the solution x(t) changes sign at most once. Display the exact symbolic solution x(t), using maple methods from the 2.1 example.

B. Plot the exact symbolic solution x(t) on a suitable t-interval. Check the graphic against Figure 5.4.7 page 317 of E&P.

```
>
> #2.2-A Define  k, then solve.
> # critically damped means mr^2+cr+k=0 has two equal roots.
> #2.2-B Plot.
>
```

2.3. PROBLEM (UNDERDAMPED FREE OSCILLATIONS)

FREE OSCILLATIONS. Consider the problem of free linear oscillations

```
m x'' + c x' + k x=0,
x(0)=0,  x'(0)=1.
```

Here, m, c and k are non-negative constants. The underdamped case is studied here, $c^2 < 4km$, as on page 318 in E&P. Depending on the first letter of your last name, assume:

Version A-E: m=1, c=4   Version F-K: m=2, c=5
Version L-R: m=3, c=7   Version S-Z: m=4, c=6

A. Display the Hooke's constant k > 0 so that the solution x(t) is underdamped and x(t) changes sign infinitely often on t>0. Display the exact solution x(t) obtained by maple methods from the 2.1 example.

B. Plot the exact symbolic solution x(t) on a suitable t-interval. Check the graphic against Figure 5.4.8 page 318 of E&P.

```
    C. Estimate from the graph the decimal value of the pseudoperiod.
       Display the graphical estimate and also the exact pseudoperiod
       2Pi/w, where w is the natural frequency of the trigonometric term
       in the solution x(t) found in item 2.3.A.

Maple tip: Click with the mouse on the graphic to print the cursor
location (left upper corner of the maple window). The coordinates
printed are of the form (x,y). From this coordinate information, a
simple subtraction estimates the period

>
> #2.3-A Define  k, then solve.
> # underdamped means mr^2+cr+k=0 has two conjugate complex roots.
> #2.3-B Plot.
> #2.3-C Pseudoperiod calculations.
>
2.4. PROBLEM (UNDAMPED FORCED OSCILLATIONS )

FORCED LINEAR OSCILLATIONS.  Consider the undamped (c=0) forced problem

      mx'' +  k x = 10 cos(wt),
      x(0)=0,  x'(0)=0,

where m, k and w are non-negative constants. Depending on the first
letter of your last name, assume:

Version A-E:  m=1, k=1.5          Version F-K: m=2, k=2.5
Version L-R: m=3, k=3.5           Version S-Z: m=4, k=4.5

    A.  Choose the forcing angular frequency w to be 3 times larger than
        the natural angular frequency w0, w0^2=k/m. Solve for x(t) using
        dsolve(). Plot the solution x(t) on a suitable interval in order
        to show the global behavior of the solution x(t). See Figure
        5.6.2, page 341.

    B. The solution x(t) is the sum of two functions, one of period
       2Pi/w and the other of period 2Pi/w0. Display the exact period,
       as calculated from the solution formula for x(t) -- see page 341
       for details.

    C. Suggest a value for the forcing frequency w so that the
       oscillations exhibit resonance.  Show resonant behavior on a
       graph. Check against Figure 5.6.4, page 343.

> #2.4-A
> #2.4-B
> #2.4-C
>
2.5. PROBLEM (PRACTICAL RESONANCE)
   Consider the damped forced problem

      mx'' + c x' + k x = 10 cos(w t),
      x(0)=0,  x'(0)=0.
```

```
Depending on the first letter of your last name, assume:

Version A-E:  m=1, k=21         Version F-K: m=2, k=32
Version L-R: m=3, k=48          Version S-Z: m=4, k=64

    A. Consider the damping constants c=2, c=1 and c=1/2.  Compute the
       amplitude function C(w) [page 346] for these three equations,
       then plot for w=0 to w=20 the three amplitude graphs on a single
       set of axes. Compare against Figure 5.6.9 page 348 of E&P (it has
       one curve, yours has 3 curves).

    B. For each case c=2, c=1, c=1/2, print the values w*, C* where
       C*=C(w*)=max {C(w) : 0 <= w <= 20}. The three data pairs should
       show that C* becomes larger as c tends to zero. SAVE YOUR MAPLE
       FILE FREQUENTLY

    Maple Hint: Use Maple's mouse interface on the graphic of Part C.
    Specifically, click on a possible maximum (horizontal tangent) in
    the graph to display the values w*, C* on the screen. Copy the
    values on paper.

EXAMPLE(Beware! Wrong values!)
F:=15: m:=1: k:=25: c:='c': w:='w':
C:=(w,c)->F/sqrt((k-m*w*w)^2+(c*w)^2):
plot({C(w,4),C(w,3),C(w,2)},w=0..15,color=black);

> #2.5-A Plot C(w), three graphics on one set of axes
> #2.5-B Table of six data values for w*, C*

2.6. PROBLEM (NONLINEAR MCKENNA MODELS)

There are six (6) parts 2.6A to 2.6F to complete. Mostly, this is mouse
copying. Retyping the maple code by hand is not recommended.

NONLINEAR TORSIONAL MODEL WITH GEOMETRY INCLUDED.
Consider the nonlinear, forced, damped oscillator equation for torsional
motion, with bridge geometry included,

    x'' + 0.05 x' + 2.4 sin(x)cos(x) = 0.06 cos (12 t/10) ,
    x(0) = x0,  x'(0) = v0

and its corresponding linearized equation

   x'' + 0.05 x' + 2.4 x = 0.06 cos (12 t/10) ,
   x(0) = x0,  x'(0) = v0.

The spring-mass system parameters are m=1, c = 0.05, k = 2.4, w = 1.2 ,
F = 0.06. Maple code used to solve and plot the solutions appears below.

   # WARNING: set the parameters on the second line!
   # Use "copy as maple text" for maple 6+.
   m:=1:  F := 0.06:  w := 1.2: m:=1: c:= 0.05: k:= 2.4:
   x0:=0: v0:=0: a:=0: b:=50:
   deNonLinear:= m*diff(x(t),t,t) + c*diff(x(t),t) +
                 k*sin(x(t))*cos(x(t)) = F*cos(w*t):
```

```
deLinear:= m*diff(x(t),t,t) + c*diff(x(t),t) + k*x(t) = F*cos(w*t):
with(DEtools):  opts:=stepsize=0.1:IC:=[[x(0)=x0,D(x)(0)=v0]]:
DEplot(deNonLinear,x(t),t=a..b,IC,opts,title='NonLinear');
DEplot(deLinear,x(t),t=a..b,IC,opts,title='Linear');
```

```
 A. Let x0=0, v0=0.  Plot the solutions of the linear and nonlinear
    equations from t=160 to t=260. These plots represent the steady
    state solutions of the two equations.

 B. Let x0=1.2, v0=0. Plot the solutions of the linear and nonlinear
    equations from t=220 to t=320. These plots represent the steady
    state solutions of the two equation, with new starting value
    x0=1.2.

    The two linear plots in A and B have to be identical to the plot
    of xss(t). The reason is the superposition formula
    x(t)=xh(t)+xss(t), even though the homogeneous solution xh(t) is
    different for the two plots.  This is because xh(t) has limit
    zero at t=infinity.

 C. Determine the ratio of the apparent amplitudes (a number > 1) for
   the nonlinear plots in A and B. Explain why "large sustained
   oscillations" is an appropriate description of the nonlinear
   steady-state behavior.
```

```
>
> #2.6-A
> #2.6-B
> #2.6-C
>
```

MCKENNA'S NON-HOOKE'S LAW CABLE MODEL FOR THE TACOMA NARROWS BRIDGE

The model of McKenna studies the bridge with a nonlinear, forced, damped oscillator equation for torsional motion that accounts for the non-Hooke's law cables coupled to the equations for vertical motion. The equations in this case couple the torsional motion with the vertical motion. The equations are:

```
x'' + c x' - k G(x,y) = F sin wt,   x(0) = x0,  x'(0) = x1,
y'' + c y' + (k/3) H(x,y) = g ,     y(0) = y0,  y'(0) = y1,
```

where x(t) is the torsional motion and y(t) is the vertical motion. The functions G(x,y) and H(x,y) are the models of the force generated by the cable when it is contracted and stretched. Below is sample code for writing the differential equations and for plotting the solutions. It is ready to copy with the mouse.

```
with(DEtools):
w := 1.3:  F := 0.05:  f(t) := F*sin(w*t):
c := 0.01:  k1 := 0.2:  k2 := 0.4:  g := 9.8:  L := 6:
STEP:=x->piecewise(x<0,0,1):
fp(t) := y(t)+(L*sin(x(t))):
fm(t) := y(t)-(L*sin(x(t))):
Sm(t) := STEP(fm(t))*fm(t):
```

```
Sp(t) := STEP(fp(t))*fp(t):
sys := {
     diff(x(t),t,t) + c*diff(x(t),t) - k1*cos(x(t))*(Sm(t)-Sp(t))=f(t),
     diff(y(t),t,t) + c*diff(y(t),t) + k2*(Sm(t)+Sp(t)) = g}:
ic := [[x(0)=0, D(x)(0)=0, y(0)=27.25, D(y)(0)=0]]:
vars:=[x(t),y(t)]:
opts:=stepsize=0.1:
DEplot(sys,vars,t=0..300,ic,opts,scene=[t,x]);
```

The amazing thing that happens in this simulation is that the large vertical oscillations take all the tension out of the springs and they induce large torsional oscillations.

D. TORSIONAL OSCILLATION PLOT. Get the sample code above to produce the plot of x(t) [that's what scene=[t,x] means].

E. Estimate the number of degrees the roadway tilts based on the plot. Recall that x in the plot is reported in radians. Comment on the agreement of this result with historical data and the video evidence in the film clip.

   Tip: Average the five largest amplitudes in the plot to find an average maximum amplitude for t=0 to t=300. Convert to degrees using Pi radians = 180 degrees. The film clip shows roadway maximum tilt of 30 to 45 degrees, approximately.

F. VERTICAL OSCILLATION PLOT. Modify the DEplot code to scene=[t,y] and plot the oscillation y(t) on t=0 to t=300. The plot is supposed to show 30-foot vertical oscillations along the roadway that dampen to 7-foot vertical oscillations after 300 seconds.

   The agreement between these oscillation results and the historical data for Tacoma Narrows, especially the visual data present in the film clip of the bridge disaster, should be clear from the plots. This is your only answer check for the plot results.

```
>
> #2.6-D Torsional plot t-versus-x
> #2.6-E  Roadway oscillation estimate in degrees + comments.
> #2.6-F Vertical plot t-versus-y.
>
```