Homework 7 Solutions

Sec. 6.3 (11)

$$\frac{dx}{dt} = 5x - x^2 - xy \quad = f(x,y)$$
$$\frac{dy}{dt} = -2y + xy \quad = g(x,y)$$

$$\vec{DF} = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix} = \begin{pmatrix} 5-2x-y & -x \\ y & -2+x \end{pmatrix} \quad \text{in general}$$

at $(0,0)$ $\quad D\vec{F}(0,0) = \begin{pmatrix} 5 & 0 \\ 0 & -2 \end{pmatrix} \Rightarrow$ Eigenvalues $\lambda_1 = 5$, $\lambda_2 = -2$

$\Rightarrow (0,0)$ a saddle point.

(12) at $(5,0)$ $\quad D\vec{F}(5,0) = \begin{pmatrix} -5 & -5 \\ 0 & +3 \end{pmatrix}$ $\quad$ that is

$$\begin{pmatrix} u \\ v \end{pmatrix}' = \begin{pmatrix} -5 & -5 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} \quad \text{or} \quad \begin{aligned} u' &= -5u - 5v \\ v' &= 3v \end{aligned}$$

$\Rightarrow$ Eigenvalues are $-5, 3$ and thus $(5,0)$ is a saddle point.

(13) At $(2,3)$ $\quad D\vec{F}(2,3) = \begin{pmatrix} -2 & -2 \\ 3 & 0 \end{pmatrix}$ $\quad$ so linearization is

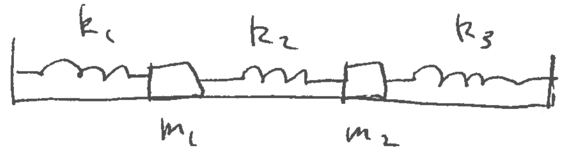
$$\Rightarrow \begin{pmatrix} u \\ v \end{pmatrix}' = \begin{pmatrix} -2 & -2 \\ 3 & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} \quad \text{or} \quad \begin{aligned} u' &= -2u - 2v \\ v' &= 3u \end{aligned}$$

Eigenvalues: $\begin{vmatrix} -2-\lambda & -2 \\ 3 & -\lambda \end{vmatrix} = \lambda(2+\lambda) + 6 = 0$, $\quad \lambda^2 + 2\lambda + 6 = 0$

$$\lambda = \frac{-2 \pm \sqrt{4-24}}{2} = -1 \pm \sqrt{5}\, i$$

$\mathrm{Re}\,\lambda < 0$ both eigenvalues $\Rightarrow$ stable spiral.
$\mathrm{Im}\,\lambda \neq 0$

Sec. 5.4 (2)



$m_1 = m_2 = 1$

$k_1 = 1,\ k_2 = 4,\ k_3 = 1$

$$M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad K = \begin{bmatrix} -(k_1+k_2) & k_2 \\ k_2 & -(k_2+k_3) \end{bmatrix} = \begin{bmatrix} -5 & 4 \\ 4 & -5 \end{bmatrix}$$

$$M\vec{x}'' = K\vec{x} \quad \Rightarrow \quad \vec{x}'' = K\vec{x}$$

Eigenvalues of K: $\begin{vmatrix} -5-\lambda & 4 \\ 4 & -5-\lambda \end{vmatrix} = (5+\lambda)^2 - 16 = 0,\ (5+\lambda)^2 = 16$

$\Rightarrow\ 5+\lambda = \pm 4 \ \Rightarrow\ \lambda = \pm 4 - 5,\quad \lambda_1 = -1,\ \lambda_2 = -9$

$\Rightarrow$ $\omega_1 = 1,\ \omega_2 = 3$ natural frequencies

Eigenvectors: $\lambda_1 = -1$: $\left(\begin{array}{cc|c} -4 & 4 & 0 \\ 4 & -4 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow \begin{array}{l} x_1 = x_2 \\ x_2 = x_2 \end{array}$

Setting $x_2 = 1 \Rightarrow \vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\lambda_2 = -9$: $\left(\begin{array}{cc|c} 4 & 4 & 0 \\ 4 & 4 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \to \begin{array}{l} x_1 = -x_2 \\ x_2 = x_2 \end{array}$

Setting $x_2 = 1 \Rightarrow \vec{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

Thus the natural modes of oscillation are:

$$(a_1 \cos t + b_1 \sin t)\begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \underline{\text{and}} \quad (a_2 \cos 3t + b_2 \sin 3t)\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

(8) $\vec{x}'' = \begin{pmatrix}-5 & 4\\ 4 & -5\end{pmatrix}\vec{x} + \cos 5t\begin{pmatrix}96\\ 0\end{pmatrix}$

$\vec{x}_p = \cos 5t\,\vec{c}$ Solve for $\vec{c}$: $\vec{x}_p'' = -25\cos 5t\,\vec{c}$

$$-25\cancel{\cos 5t}\,\vec{c} = \begin{pmatrix}-5 & 4\\ 4 & -5\end{pmatrix}\cos 5t\,\vec{c} + \cancel{\cos 5t}\begin{pmatrix}96\\ 0\end{pmatrix}$$

$$\begin{pmatrix}-5 & 4\\ 4 & -5\end{pmatrix}\vec{c} + 25\vec{c} = \begin{pmatrix}-96\\ 0\end{pmatrix}$$

$$\left(\begin{array}{cc|c}20 & 4 & -96\\ 4 & 20 & 0\end{array}\right) \to \left(\begin{array}{cc|c}1 & 5 & 0\\ 20 & 4 & -96\end{array}\right) \to \left(\begin{array}{cc|c}1 & 5 & 0\\ 0 & 96 & -96\end{array}\right)$$

$$\to \left(\begin{array}{cc|c}1 & 5 & 0\\ 0 & 1 & -1\end{array}\right) \to \left(\begin{array}{cc|c}1 & 0 & 5\\ 0 & 1 & -1\end{array}\right) \qquad \vec{c} = \begin{pmatrix}5\\ -1\end{pmatrix}$$

$\vec{x}(t) = \vec{x}_c(t) + \vec{x}_p(t)$, $\vec{x}(0) = \begin{pmatrix}0\\ 0\end{pmatrix}$, $\vec{x}'(0) = \begin{pmatrix}0\\ 0\end{pmatrix}$

$$\vec{x}(t) = (a_1\cos t + b_1\sin t)\begin{pmatrix}1\\ 1\end{pmatrix} + (a_2\cos 3t + b_2\sin 3t)\begin{pmatrix}-1\\ 1\end{pmatrix} + \cos 5t\begin{pmatrix}5\\ -1\end{pmatrix}$$

$$\vec{x}'(t) = (-a_1\sin t + b_1\cos t)\begin{pmatrix}1\\ 1\end{pmatrix} + (-3a_2\sin 3t + 3b_2\cos 3t)\begin{pmatrix}-1\\ 1\end{pmatrix} - 5\sin 5t\begin{pmatrix}5\\ -1\end{pmatrix}$$

$$\vec{x}(0) = a_1\begin{pmatrix}1\\ 1\end{pmatrix} + a_2\begin{pmatrix}-1\\ 1\end{pmatrix} + \begin{pmatrix}5\\ -1\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}$$

$$\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}\begin{pmatrix}a_1\\ a_2\end{pmatrix} = \begin{pmatrix}-5\\ 1\end{pmatrix}, \quad \left(\begin{array}{cc|c}1 & 1 & 1\\ 1 & -1 & -5\end{array}\right) \to \left(\begin{array}{cc|c}1 & 1 & 1\\ 0 & -2 & -6\end{array}\right)$$

$$\to \left(\begin{array}{cc|c}1 & 1 & 1\\ 0 & 1 & 3\end{array}\right) \to \left(\begin{array}{cc|c}1 & 0 & -2\\ 0 & 1 & 3\end{array}\right) \qquad \begin{array}{l}a_1 = -2\\ a_2 = 3\end{array}$$

$$\vec{x}\,'(0) = b_1\begin{pmatrix}1\\1\end{pmatrix} + 3b_2\begin{pmatrix}-1\\1\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \Rightarrow \begin{pmatrix}1&-1\\1&1\end{pmatrix}\begin{pmatrix}b_1\\b_2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

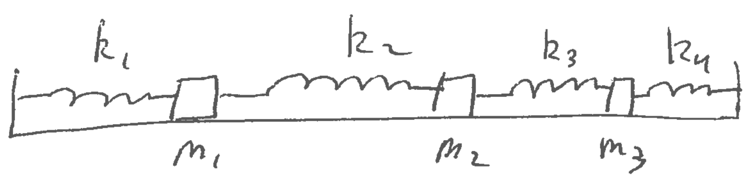
$$\Longrightarrow \quad b_1 = b_2 = 0 \quad \text{so}$$

$$\boxed{\vec{x}(t) = -2\cos t\begin{pmatrix}1\\1\end{pmatrix} + 3\cos 3t\begin{pmatrix}-1\\1\end{pmatrix} + \cos 5t\begin{pmatrix}5\\-1\end{pmatrix}}$$

or

$$\boxed{\begin{aligned} x_1(t) &= -2\cos t - 3\cos 3t + 5\cos 5t \\ x_2(t) &= -2\cos t + 3\cos 3t - 5\cos 5t \end{aligned}}$$

(12)



$m_1 = m_2 = m_3 = 1$

$k_1 = k_2 = k_3 = k_4 = 1$

$$M = I, \quad K = \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix} \qquad M\vec{x}\,'' = K\vec{x} \Rightarrow \vec{x}\,'' = K\vec{x}.$$

Eigenvalues:

$$\begin{vmatrix} -2-\lambda & 1 & 0 \\ 1 & -2-\lambda & 1 \\ 0 & 1 & -2-\lambda \end{vmatrix} = (-2-\lambda)\left[(-2-\lambda)(-2-\lambda) - 1\right] - (-2-\lambda)$$

$$= (-2-\lambda)\left[(\lambda^2+4\lambda+4) - 1 - 1\right]$$

$$= -(\lambda+2)\left[\lambda^2 + 4\lambda + 2\right]$$

$$\lambda = -2, \quad \lambda = \frac{-4 \pm \sqrt{16-8}}{2} = \frac{-4 \pm 2\sqrt{2}}{2} = -2 \pm \sqrt{2}$$

$\boxed{\lambda_1 = -2}$ $\quad \boxed{\lambda_2 = -2+\sqrt{2}}$ $\quad \boxed{\lambda_3 = -2-\sqrt{2}}$

$$\boxed{\text{Natural Frequencies: } \omega_1 = \sqrt{2}, \quad \omega_2 = \sqrt{2-\sqrt{2}}, \quad \omega_3 = \sqrt{2+\sqrt{2}}}$$

Eigenvalues: $\lambda_1 = -2$:
$$\left(\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$x_1 = -x_3$
$x_2 = 0$
$x_3 = x_3$

Can take $\vec{v}_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

$\lambda_3 = -2-\sqrt{2}$:
$$\left(\begin{array}{ccc|c} \sqrt{2} & 1 & 0 & 0 \\ 1 & \sqrt{2} & 1 & 0 \\ 0 & 1 & \sqrt{2} & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & \sqrt{2} & 1 & 0 \\ 0 & 1 & \sqrt{2} & 0 \\ \sqrt{2} & 1 & 0 & 0 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 1 & \sqrt{2} & 1 & 0 \\ 0 & 1 & \sqrt{2} & 0 \\ 0 & -1 & -\sqrt{2} & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & \sqrt{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$x_1 = x_3$
$x_2 = -\sqrt{2}\, x_3$
$x_3 = x_3$

set $x_3 = \sqrt{2} \Rightarrow \begin{pmatrix} \sqrt{2} \\ -2 \\ \sqrt{2} \end{pmatrix} = \vec{v}_3$ or $\begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix} \vec{v}_3$

$x_2 = -2$
$x_1 = \sqrt{2}$

$\lambda_3 = -2+\sqrt{2}$
$$\begin{pmatrix} -\sqrt{2} & 1 & 0 \\ 1 & -\sqrt{2} & 1 \\ 0 & 1 & -\sqrt{2} \end{pmatrix} \to \left(\begin{array}{ccc|c} 1 & -\sqrt{2} & 1 & 0 \\ 0 & 1 & -\sqrt{2} & 0 \\ 0 & -1 & \sqrt{2} & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & -\sqrt{2} & 1 & 0 \\ & 1 & -\sqrt{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \to \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -\sqrt{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$x_1 = x_3$

$x_2 = \sqrt{2}\, x_3$

$x_3 = x_3$

$$\boxed{\vec{v}_2 = \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}}$$

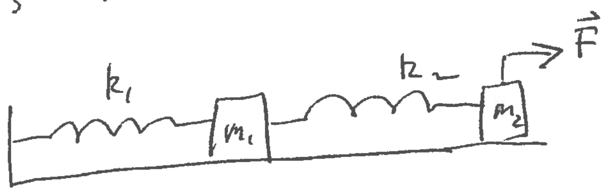
Ratio of amplitude for $\omega_1 = \sqrt{2}$ $(\lambda_1 = -2)$

$-1 : 0 : 1$

For $\lambda_2 = -2 + \sqrt{2}$: $1 : \sqrt{2} : 1$
$(\omega_2 = \sqrt{2 - \sqrt{2}})$

For $\omega_3 = \sqrt{2 + \sqrt{2}}$: $1 : -\sqrt{2} : 1$

(15)



$m_1 = 2, \; m_2 = \frac{1}{2}$

$k_1 = 75, \; k_2 = 25$

$\vec{F} = \cos 10t \begin{pmatrix} 0 \\ 100 \end{pmatrix}$

$\vec{x}(0) = \vec{x}'(0) = 0$

$$M = \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \qquad M^{-1} \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 2 \end{pmatrix}$$

$$M^{-1}\vec{F} = \cos 10t \begin{pmatrix} 0 \\ 200 \end{pmatrix}$$

$$K = \begin{pmatrix} -(k_1+k_2) & k_2 \\ k_2 & -k_2 \end{pmatrix} = \begin{pmatrix} -100 & 25 \\ 25 & -25 \end{pmatrix}$$

$$\vec{x}'' = A\vec{x} + M^{-1}\vec{F} \qquad A = M^{-1}K = \begin{pmatrix} -50 & 12.5 \\ 50 & -50 \end{pmatrix}$$

$$\vec{x}_p = \cos 10t\, \vec{c}, \quad \vec{x}_p'' = -100 \cos 10t\, \vec{c}$$

$$-100 \cancel{\cos 10t}\, \vec{c} = \cancel{\cos 10t} \begin{pmatrix} -50 & 12.5 \\ 50 & -50 \end{pmatrix} \vec{c} + \cancel{\cos 10t} \begin{pmatrix} 0 \\ 200 \end{pmatrix}$$

$$\begin{pmatrix} -50 & 12.5 \\ 50 & -50 \end{pmatrix} \vec{C} + 100 I \vec{C} = \begin{pmatrix} 0 \\ -200 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} 50 & 12.5 & 0 \\ 50 & 50 & -200 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 1 & -4 \\ 50 & 12.5 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 1 & -4 \\ 0 & -37.5 & 200 \end{array}\right)$$

$$\to \left(\begin{array}{cc|c} 1 & 1 & -4 \\ 0 & 1 & -5.33\ldots \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 0 & 1.3\bar{3} \\ 0 & 1 & -5.3\bar{3} \end{array}\right)$$

$$\vec{x}_p = \cos 10t \begin{pmatrix} 4/3 \\ 16/3 \end{pmatrix}$$

Eigenvalues of A: $\begin{vmatrix} -50-\lambda & 12.5 \\ 50 & -50-\lambda \end{vmatrix} = (\lambda+50)^2 - 625 = 0$

$\lambda + 50 = \pm 25$, $\lambda = \pm 25 - 50$, $\lambda_1 = -25$, $\lambda_2 = -75$

$\omega_1 = 5$, $\omega_2 = \sqrt{75} = 5\sqrt{3}$

Eigenvectors $\lambda_1 = -25$: $\left(\begin{array}{cc|c} -25 & 12.5 & 0 \\ 50 & -25 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & -1/2 & 0 \\ 0 & 0 & 0 \end{array}\right)$

$x_1 = \frac{1}{2} x_2 \Rightarrow$ take $\vec{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$\lambda_2 = -75$: $\left(\begin{array}{cc|c} 25 & 12.5 & 0 \\ 50 & 25 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 1/2 & 0 \\ 0 & 0 & 0 \end{array}\right)$

$x_1 = -1/2\, x_2 \Rightarrow$ take $\vec{v}_2 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$

$$\vec{x}(t) = (a_1 \cos 5t + b_1 \sin 5t)\begin{pmatrix}1\\2\end{pmatrix} + (a_2 \cos 5\sqrt{3}t + b_2 \sin 5\sqrt{3}t)\begin{pmatrix}-1\\2\end{pmatrix}$$

$$+ \cos 10t \begin{pmatrix}4/3\\16/3\end{pmatrix}$$

$$\vec{x}'(t) = (-5a_1 \sin 5t + 5b_1 \cos 5t)\begin{pmatrix}1\\2\end{pmatrix} + (-5\sqrt{3}a_2 \sin\sqrt{3}5t + 5\sqrt{3}b_2 \cos 5\sqrt{3}t)\begin{pmatrix}-1\\2\end{pmatrix}$$

$$-10 \sin 10t \begin{pmatrix}4/3\\16/3\end{pmatrix}$$

$$\vec{x}(0) = a_1\begin{pmatrix}1\\2\end{pmatrix} + a_2\begin{pmatrix}-1\\2\end{pmatrix} + \begin{pmatrix}4/3\\16/3\end{pmatrix} = 0$$

$$\begin{pmatrix}1 & -1\\2 & 2\end{pmatrix}\begin{pmatrix}a_1\\a_2\end{pmatrix} = \begin{pmatrix}-4/3\\-16/3\end{pmatrix}, \quad \left(\begin{array}{cc|c}1 & -1 & -4/3\\2 & 2 & -16/3\end{array}\right) \to \left(\begin{array}{cc|c}1 & -1 & -4/3\\0 & 4 & -8/3\end{array}\right)$$

$$\left(\begin{array}{cc|c}1 & -1 & -4/3\\0 & 1 & -2/3\end{array}\right) \to \left(\begin{array}{cc|c}1 & 0 & -2\\0 & 1 & -2/3\end{array}\right) \qquad \begin{aligned}a_1 &= -2\\ a_2 &= -2/3\end{aligned}$$

$$\vec{x}'(0) = 5b_1\begin{pmatrix}1\\2\end{pmatrix} + 5\sqrt{3}b_2\begin{pmatrix}1\\-2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

$$\begin{pmatrix}1 & \sqrt{3}\\2 & -2\sqrt{3}\end{pmatrix}\begin{pmatrix}b_1\\b_2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \Rightarrow b_1 = b_2 = 0$$

$$\boxed{\vec{x}(t) = -2\cos 5t\begin{pmatrix}1\\2\end{pmatrix} - \frac{2}{3}\cos 5\sqrt{3}t\begin{pmatrix}-1\\2\end{pmatrix} + \cos 10t\begin{pmatrix}4/3\\16/3\end{pmatrix}}$$

or

$$\boxed{\begin{aligned}x_1(t) &= -2\cos 5t + \tfrac{2}{3}\cos 5\sqrt{3}t + 4/3\cos 10t\\ x_2 &= -4\cos 5t - \tfrac{4}{3}\cos 5\sqrt{3}t + \tfrac{16}{3}\cos 10t\end{aligned}}$$

(21) $x_1'(0) = 2v_0$, $x_2'(0) = 0$, $x_3'(0) = -v_0$ $\quad v_0 = 48$

From the analysis in Example we have (eq. (23))

$$\vec{x}(t) = a_1\begin{pmatrix}1\\1\\1\end{pmatrix} + b_1 t\begin{pmatrix}1\\1\\1\end{pmatrix} + a_2\begin{pmatrix}1\\0\\-1\end{pmatrix}\cos 2t + b_2\begin{pmatrix}1\\0\\-1\end{pmatrix}\sin 2t + a_3\begin{pmatrix}1\\-3\\1\end{pmatrix}\cos 4t$$

$$+ b_3\begin{pmatrix}1\\-3\\1\end{pmatrix}\sin 4t$$

as long as springs are engaged. Springs are engaged when $x_2 - x_1 < 0$ and $x_3 - x_2 < 0$. Let's find coefficient:

$$\vec{x}(0) = a_1\begin{pmatrix}1\\1\\1\end{pmatrix} + a_2\begin{pmatrix}1\\0\\-1\end{pmatrix} + a_3\begin{pmatrix}1\\-3\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

Since the eigenvectors are linearly independent $a_1 = a_2 = a_3 = 0$

$$\vec{x}'(t) = b_1\begin{pmatrix}1\\1\\1\end{pmatrix} + 2b_2\begin{pmatrix}1\\0\\-1\end{pmatrix}\cos 2t + 4b_3\begin{pmatrix}1\\-3\\1\end{pmatrix}\cos 4t$$

$$\vec{x}'(0) = b_1\begin{pmatrix}1\\1\\1\end{pmatrix} + 2b_2\begin{pmatrix}1\\0\\-1\end{pmatrix} + 4b_3\begin{pmatrix}1\\-3\\1\end{pmatrix} = \begin{pmatrix}96\\0\\-48\end{pmatrix} = \begin{pmatrix}2v_0\\0\\-v_0\end{pmatrix}$$

OK.

$$\left(\begin{array}{ccc|c}1 & 2 & 4 & 96\\1 & 0 & -12 & 0\\1 & -2 & 4 & 48\end{array}\right) \to \left(\begin{array}{ccc|c}1 & 2 & 4 & 96\\0 & -2 & -16 & -96\\0 & -4 & 0 & -48\end{array}\right) \to \left(\begin{array}{ccc|c}1 & 2 & 4 & 96\\0 & 1 & 0 & 12\\0 & -2 & -16 & -96\end{array}\right)$$

$$\left(\begin{array}{ccc|c}1 & 0 & 4 & 84\\0 & 1 & 0 & 12\\0 & 0 & -16 & 72\end{array}\right) \to \left(\begin{array}{ccc|c}1 & 0 & 4 & 84\\0 & 1 & 0 & 12\\0 & 0 & 1 & -4.5\end{array}\right) \to \left(\begin{array}{ccc|c}1 & 0 & 0 & 102\\0 & 1 & 0 & 12\\0 & 0 & 1 & -4.5\end{array}\right)$$

$b_1 = 102$, $b_2 = 12$, $b_3 = -4.5$

$$\begin{aligned} b_1 + 2b_2 + 4b_3 &= 2v_0 \\ b_1 \qquad\quad - 12 b_3 &= 0 \\ b_1 - 2b_2 + 4b_3 &= -v_0 \end{aligned} \Rightarrow \left(\begin{array}{ccc|c} 1 & 2 & 4 & 2v_0 \\ 1 & 0 & -12 & 0 \\ 1 & -2 & 4 & -v_0 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & -2 & 4 & -v_0 \\ 1 & 0 & -12 & 0 \\ 0 & 4 & 0 & 3v_0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & -2 & 4 & -v_0 \\ 0 & 2 & -16 & v_0 \\ 0 & 4 & 0 & 3v_0 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & -2 & 4 & -v_0 \\ 0 & 1 & 0 & 3/4\, v_0 \\ 0 & 2 & -16 & v_0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 4 & 1/2\, v_0 \\ 0 & 1 & 0 & 3/4\, v_0 \\ 0 & 0 & -16 & -1/2\, v_0 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 4 & 1/2\, v_0 \\ 0 & 1 & 0 & 3/4\, v_0 \\ 0 & 0 & 1 & v_0/32 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 3/8\, v_0 \\ 0 & 1 & 0 & 3/4\, v_0 \\ 0 & 0 & 1 & v_0/32 \end{array}\right)$$

$$\vec{x}(t) = \frac{3v_0}{8} t \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \frac{3}{4} v_0 \sin 2t \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \frac{v_0}{32} \sin 4t \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix}$$

$$x_1 = \frac{3}{8} v_0 t + \frac{3}{4} v_0 \sin 2t + \frac{v_0}{32} \sin 4t$$

$$x_2 = \frac{3}{8} v_0 t \qquad\qquad - \frac{3}{32} v_0 \sin 4t$$

$$x_3 = \frac{3}{8} v_0 t - \frac{3}{4} v_0 \sin 2t + \frac{v_0}{32} \sin 4t$$

Springs 1 and 2 are engaged until the first positive time that $x_2 - x_1 = 0$ and $x_3 - x_2 = 0$ respectively:

$$\cancel{v_0}\, \frac{1}{8} \sin 4t = -\frac{3}{4} \cancel{v_0} \sin 2t$$

and $$v_0 \frac{1}{8} \sin 4t = \frac{3}{4} v_0 \sin 2t$$

a $$\frac{1}{2} \cos 2t \sin 2t = -\frac{3}{4} \sin 2t$$

and $$\frac{1}{2} \cos 2t \sin 2t = \frac{3}{4} \sin 2t$$

Either of these only occurs when $\sin 2t = 0$, (otherwise, we would get $|\cos 2t| = 3/2$, impossible) which first happens when $\boxed{t = \frac{\pi}{2}}$.

$$x_1' = \frac{3}{8} v_0 + \frac{3}{2} v_0 \cos 2t + \frac{v_0}{8} \cos 4t$$

$$x_2' = \frac{3}{8} v_0 - \frac{12}{32} \cos 4t \, v_0$$

$$x_3' = \frac{3}{8} v_0 - \frac{3}{2} v_0 \cos 2t + \frac{v_0}{8} \cos 4t$$

So at $t = \pi/2$

$$x_1'(\pi/2) = \frac{3}{8} v_0 - \frac{3}{2} v_0 + \frac{v_0}{8} = -v_0$$

$$x_2'(\pi/2) = \frac{3}{8} v_0 - \frac{3}{8} v_0 = 0$$

$$x_3'(\pi/2) = \frac{3}{8} v_0 + \frac{3}{2} v_0 + \frac{v_0}{8} = 2 v_0$$

Sec. 5.7 ②

$$x' = 2x + 3y + 5$$
$$y' = 2x + y - 2t$$

$$\vec{x}' = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix} \vec{x} + \begin{pmatrix} 5 \\ 0 \end{pmatrix} + t \begin{pmatrix} 0 \\ -2 \end{pmatrix}$$

try $\vec{x}_p = \vec{a} + t\vec{b}$:

$$\vec{x}_p' = \vec{b}$$

$$\vec{b} = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix} (\vec{a} + t\vec{b}) + \begin{pmatrix} 5 \\ 0 \end{pmatrix} + t \begin{pmatrix} 0 \\ -2 \end{pmatrix}$$

$$\vec{b} = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix} \vec{a} + t \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix} \vec{b} + \begin{pmatrix} 5 \\ 0 \end{pmatrix} + t \begin{pmatrix} 0 \\ -2 \end{pmatrix}$$

$$\Leftrightarrow \quad \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix} \vec{a} + \begin{pmatrix} 5 \\ 0 \end{pmatrix} = \vec{b} \quad \text{and} \quad \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix} \vec{b} + \begin{pmatrix} 0 \\ -2 \end{pmatrix} = 0.$$

Solve 2nd eq. for $\vec{b}$: $\left(\begin{array}{cc|c} 2 & 3 & 0 \\ 2 & 1 & 2 \end{array}\right) \to \left(\begin{array}{cc|c} 2 & 3 & 0 \\ 0 & -2 & 2 \end{array}\right)$

$$\to \left(\begin{array}{cc|c} 2 & 3 & 0 \\ 0 & 1 & -1 \end{array}\right) \to \left(\begin{array}{cc|c} 2 & 0 & 3 \\ 0 & 1 & -1 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 0 & 3/2 \\ 0 & 1 & -1 \end{array}\right)$$

$\vec{b} = \begin{pmatrix} 3/2 \\ -1 \end{pmatrix}$. Now 1st equation becomes:

$$\begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix} \vec{a} = \vec{b} - \begin{pmatrix} 5 \\ 0 \end{pmatrix} = \begin{pmatrix} 3/2 \\ -1 \end{pmatrix} - \begin{pmatrix} 5 \\ 0 \end{pmatrix} = \begin{pmatrix} -7/2 \\ -1 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} 2 & 3 & -7/2 \\ 2 & 1 & -1 \end{array}\right) \to \left(\begin{array}{cc|c} 2 & 3 & -7/2 \\ 0 & -2 & 5/2 \end{array}\right) \to \left(\begin{array}{cc|c} 2 & 3 & -7/2 \\ 0 & 1 & -5/4 \end{array}\right)$$

$$\to \left(\begin{array}{cc|c} 2 & 0 & 1/4 \\ 0 & 1 & -5/4 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 0 & 1/8 \\ 0 & 1 & -5/4 \end{array}\right)$$

$\vec{a} = \begin{pmatrix} 1/8 \\ -5/4 \end{pmatrix}$ so $\boxed{\vec{x}_p = \begin{pmatrix} 1/8 \\ -5/4 \end{pmatrix} + t \begin{pmatrix} 3/2 \\ -1 \end{pmatrix}}$

⑧ $x' = x - 5y + \cancel{\cos t}\ 2\sin t$

$y' = x - y - 3\cos t$

$$\vec{x}' = \begin{pmatrix} 1 & -5 \\ 1 & -1 \end{pmatrix}\vec{x} + \sin t\begin{pmatrix} 2 \\ 0 \end{pmatrix} + \cos t\begin{pmatrix} 0 \\ -3 \end{pmatrix}$$

Note: eigenvalues of $\begin{pmatrix} 1 & -5 \\ 1 & -1 \end{pmatrix}$ are $\begin{vmatrix} 1-\lambda & -5 \\ 1 & -1-\lambda \end{vmatrix} = (1-\lambda)(-1-\lambda) + 5 = 0$

$\lambda^2 + 4 = 0$, $\lambda = \pm 2i$ thus there is no duplication.

$$\vec{x}_p = \vec{a}\cos t + \vec{b}\sin t$$

$$\vec{x}_p' = -\vec{a}\,\overset{\sin t}{\cancel{\cos t}} + \vec{b}\cos t$$

$$-\vec{a}\sin t + \vec{b}\cos t = \begin{pmatrix} 1 & -5 \\ 1 & -1 \end{pmatrix}(\vec{a}\cos t + \vec{b}\sin t) + \sin t\begin{pmatrix} 2 \\ 0 \end{pmatrix} + \cos t\begin{pmatrix} 0 \\ -3 \end{pmatrix}$$

$$-\vec{a}\sin t + \vec{b}\cos t = \cos t\begin{pmatrix} 1 & -5 \\ 1 & -1 \end{pmatrix}\vec{a} + \sin t\begin{pmatrix} 1 & -5 \\ 1 & -1 \end{pmatrix}\vec{b} + \sin t\begin{pmatrix} 2 \\ 0 \end{pmatrix} + \cos t\begin{pmatrix} 0 \\ -3 \end{pmatrix}$$

Equating sin + cos terms separately:

$$-\vec{a} = \begin{pmatrix} 1 & -5 \\ 1 & -1 \end{pmatrix}\vec{b} + \begin{pmatrix} 2 \\ 0 \end{pmatrix} \quad \text{and}$$

$$\vec{b} = \begin{pmatrix} 1 & -5 \\ 1 & -1 \end{pmatrix}\vec{a} + \begin{pmatrix} 0 \\ -3 \end{pmatrix}$$

n $\begin{pmatrix} 1 & -5 \\ 1 & -1 \end{pmatrix}\vec{a} - I\vec{b} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$

$$\begin{pmatrix} 1 & -5 \\ 1 & -1 \end{pmatrix}\vec{b} + I\vec{a} = \begin{pmatrix} -2 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} A & -I \\ I & A \end{pmatrix}\begin{pmatrix} \vec{a} \\ \vec{b} \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ -2 \\ 0 \end{pmatrix} \quad \text{where } A = \begin{pmatrix} 1 & -5 \\ 1 & -1 \end{pmatrix}$$

$$\left(\begin{array}{cccc|c} 1 & -5 & -1 & 0 & 0 \\ 1 & -1 & 0 & -1 & 3 \\ 1 & 0 & 1 & -5 & -2 \\ 0 & 1 & 1 & -1 & 0 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 0 & 1 & -5 & -2 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & -5 & -2 & 5 & +2 \\ 0 & -1 & -1 & 4 & 5 \end{array}\right)$$

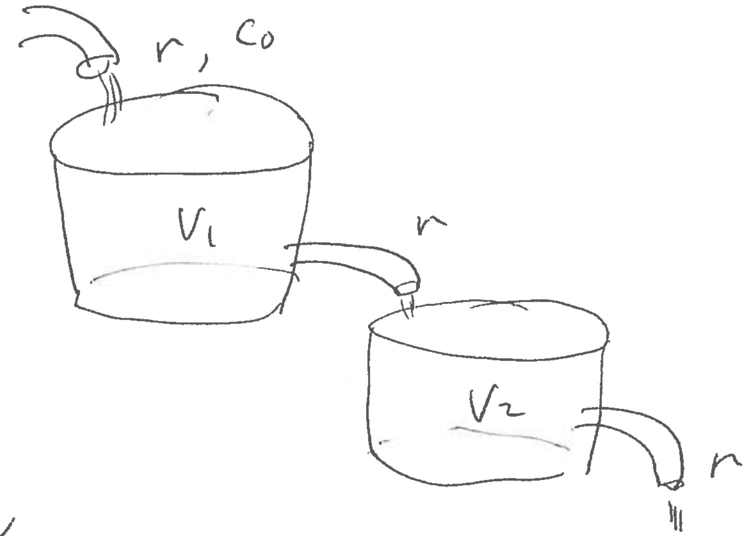
$$\to \left(\begin{array}{cccc|c} 1 & 0 & 1 & -5 & -2 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 3 & 0 & +2 \\ 0 & 0 & 0 & 3 & 1 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 0 & 1 & -5 & -2 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & +2/3 \\ 0 & 0 & 0 & 1 & 1/3 \end{array}\right)$$

$$\to \left(\begin{array}{cccc|c} 1 & 0 & 0 & -5 & -8/3 \\ 0 & 1 & 0 & -1 & -2/3 \\ 0 & 0 & 1 & 0 & +2/3 \\ 0 & 0 & 0 & 1 & 1/3 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1/3 \\ 0 & 0 & 1 & 0 & -2/3 \\ 0 & 0 & 0 & 1 & 1/3 \end{array}\right)$$

$$\vec{a} = \begin{pmatrix} -1 \\ -1/3 \end{pmatrix} \quad \vec{b} = \begin{pmatrix} -2/3 \\ 1/3 \end{pmatrix}$$

$$\vec{x}_p = \cos t \begin{pmatrix} -1 \\ -1/3 \end{pmatrix} + \sin t \begin{pmatrix} -2/3 \\ 1/3 \end{pmatrix}$$

(15)



$V_1 = 100, \quad V_2 = 200$

$r = 10, \quad c_0 = 2$

$$x_1' = rc_0 - r\frac{x}{V_1} \qquad\qquad x_1' = -\frac{10x}{100} + 20$$

$$\Rightarrow$$

$$x_2' = \frac{rx}{V_1} - \frac{ry}{V_2} \qquad\qquad x_2' = \frac{10x}{100} - \frac{10y}{200}$$

or

$$x_1' = -\frac{1}{10}x + 20 \qquad \text{or} \quad \vec{x}\,' = \begin{pmatrix} -\frac{1}{10} & 0 \\ \frac{1}{10} & -\frac{1}{20} \end{pmatrix}\vec{x} + \begin{pmatrix} 20 \\ 0 \end{pmatrix}$$

$$x_2' = \frac{1}{10}x - \frac{1}{20}y$$

$$\vec{x}(0) = \vec{0}$$

Eigenvalues of $\begin{pmatrix} -1/10 & 0 \\ 1/10 & -1/20 \end{pmatrix}$ are $\lambda_1 = -1/10, \ \lambda_2 = -1/20$

Eigenvectors: $\lambda_1 = -\frac{1}{10}$: $\left(\begin{array}{cc|c} 0 & 0 & 0 \\ \frac{1}{10} & \frac{1}{20} & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{array}\right)$ $\quad x_1 = -\frac{1}{2}x_2$, $\quad x_2 = x_2$

can take $\vec{v}_1 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$

$\lambda = -\frac{1}{20}$: $\left(\begin{array}{cc|c} -1/20 & 0 & 0 \\ 1/10 & 0 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right)$ $\quad x_1 = 0$, $\quad x_2 = x_2$

can take $\vec{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

So $\vec{x}_c = c_1 e^{-\frac{1}{10}t}\begin{pmatrix}-1\\2\end{pmatrix} + c_2 e^{-\frac{1}{20}t}\begin{pmatrix}0\\1\end{pmatrix}$ (gen. soln. of homog. prob.)

$x_p = \vec{a} \qquad \vec{x}_p{}' = \vec{0}$

$$\vec{0} = \begin{pmatrix}-1/10 & 0\\ 1/10 & -1/20\end{pmatrix}\vec{a} + \begin{pmatrix}20\\0\end{pmatrix} \Rightarrow \left(\begin{array}{cc|c}-1/10 & 0 & -20\\ 1/10 & -1/20 & 0\end{array}\right)$$

$$\Rightarrow \left(\begin{array}{cc|c}1 & 0 & 200\\ 0 & -\frac{1}{20} & -20\end{array}\right) \to \left(\begin{array}{cc|c}1 & 0 & 200\\ 0 & 1 & 400\end{array}\right)$$

$$\vec{a} = \begin{pmatrix}200\\400\end{pmatrix}$$

$$\vec{x} = \vec{x}_c + \vec{x}_p = c_1 e^{-\frac{1}{10}t}\begin{pmatrix}-1\\2\end{pmatrix} + c_2 e^{-\frac{1}{20}t}\begin{pmatrix}0\\1\end{pmatrix} + \begin{pmatrix}200\\400\end{pmatrix}$$

(b) $\lim\limits_{t\to\infty} \vec{x}(t) = \begin{pmatrix}200\\400\end{pmatrix}$ 200 lb's in tank 1, 400 lb's in tank 2

to find c_1, c_2:

$$\vec{x}(0) = c_1\begin{pmatrix}-1\\2\end{pmatrix} + c_2\begin{pmatrix}0\\1\end{pmatrix} + \begin{pmatrix}200\\400\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

$$\begin{pmatrix}-1 & 0\\ 2 & 1\end{pmatrix}\vec{c} = \begin{pmatrix}-200\\-400\end{pmatrix} \to \left(\begin{array}{cc|c}-1 & 0 & -200\\ 2 & 1 & -400\end{array}\right)$$

$$\left(\begin{array}{cc|c}1 & 0 & 200\\ 0 & 1 & -800\end{array}\right) \qquad c_1 = 200, \quad c_2 = -800$$

$$\boxed{\vec{x}(t) = 200\, e^{-\frac{1}{10}t}\begin{pmatrix}-1\\ 2\end{pmatrix} - 800\, e^{-\frac{1}{20}t}\begin{pmatrix}0\\ 1\end{pmatrix} + \begin{pmatrix}200\\ 400\end{pmatrix}}$$

c) When tank 1 reaches concentration of lb/gal, when

$x_1 = 100$:

$$x_1 = -200\, e^{-\frac{1}{10}t} + 0 + 200 = 100$$

$$\Rightarrow \quad 200\, e^{-1/10\, t} = 100$$

$$\Rightarrow \quad e^{-\frac{1}{10}t} = \frac{1}{2} \quad \Rightarrow \quad -\frac{1}{10}t = \ln\frac{1}{2} \qquad \boxed{t = 10 \ln 2}$$

When $x_2 = 200$:

$$x_2 = 400\, e^{-\frac{1}{10}t} - 800\, e^{-\frac{1}{20}t} + 400 = 200$$

$$800\, e^{-\frac{1}{20}t} - 400\, e^{-\frac{1}{10}t} = 200$$

$u = e^{-\frac{1}{20}t}$ then $e^{-\frac{1}{10}t} = u^2$

$$-400u^2 + 800u = 200$$

$$u^2 - 2u - \frac{1}{2} = 0 \qquad 2u^2 - 4u - 1 = 0$$

$$u = \frac{4 \pm \sqrt{16+8}}{4} = \frac{4 \pm 2\sqrt{6}}{4} = \frac{4 + 2\sqrt{6}}{4} = 1 - \frac{\sqrt{6}}{2} \quad (u < 1)$$

$$e^{-\frac{1}{10}t} = 1 - \frac{\sqrt{6}}{2}, \quad -\frac{1}{10}t = \ln\left(1 - \frac{\sqrt{6}}{2}\right) \qquad \boxed{t = -10 \ln\left(1 - \frac{\sqrt{6}}{2}\right)}$$