

Chapter 7: Laplace Theory

Sample Problem 1. Piecewise Continuous Inputs

Consider a passenger SUV on a one-day trip from Salt Lake City to Pine Bluffs, Wyoming, on the Nebraska border. The route is I-80 E, 471 miles through Utah and Wyoming. Google maps estimates 6 hours and 52 minutes hours driving time. The table below shows the distances, time, road segment and average speed with total trip time 7 hours and 42 minutes. Cities enroute reduce the freeway speed by 10 mph, the trip time effect not shown in the table.

Miles	Minutes	Speed mph	Road Segment	Posted limit mph
18.1	20	54.3	Parley's Walmart to Kimball	65
11.3	12	56.3	Kimball to Wanship	65 – 55
9.1	13	42	Wanship to Coalville	70
5.7	7	48.9	Coalville to Echo Dam	70
16.5	18	55	Echo Dam to 75 mph sign	70
39	36	65	75 mph sign to Evanston	75
308	269	68.7	Evanston to Laramie	75
50.6	50	61.2	Laramie to Cheyenne	75
43	37	69.7	Cheyenne to Pine bluffs	75

The **velocity function** for the SUV is approximated by

$$V_{pc}(t) = \left\{ \begin{array}{lll} \text{Speed mph} & \text{Time interval minutes} & \text{Road segment} \\ \hline 54.3 & 0 < t < 20 & \text{Parley's Walmart to Kimball} \\ 56.3 & 20 < t < 32 & \text{Kimball to Wanship} \\ 42.0 & 32 < t < 45 & \text{Wanship to Coalville} \\ 48.9 & 45 < t < 52 & \text{Coalville to Echo Dam} \\ 55.0 & 52 < t < 70 & \text{Echo Dam to 75 mph sign} \\ 65.0 & 70 < t < 106 & \text{75 mph sign to Evanston} \\ 68.7 & 106 < t < 375 & \text{Evanston to Laramie} \\ 61.2 & 375 < t < 425 & \text{Laramie to Cheyenne} \\ 69.7 & 425 < t < 462 & \text{Cheyenne to Pine bluffs} \\ 0.00 & 462 < t < \infty & \text{SUV stopped} \end{array} \right.$$

The velocity function $V_{pc}(t)$ is piecewise continuous, because it has the general form

$$f(t) = \left\{ \begin{array}{ll} f_1(t) & t_1 < t < t_2 \\ f_2(t) & t_2 < t < t_3 \\ \vdots & \vdots \\ f_n(t) & t_n < t < t_{n+1} \end{array} \right.$$

where functions f_1, f_2, \dots, f_n are **continuous on the whole real line** $-\infty < t < \infty$. We don't define $f(t)$ at division points, because of many possible ways to make the definition. As long as these values are not used, then it will make no difference. Both right and left hand limits exist at a division point. For Laplace theory, we like the definition $f(t_k) = \lim_{h \rightarrow 0^+} f(t_k + h)$, which makes the function right-continuous.

The Problem. The SUV travels from $t = 0$ to $t = \frac{462}{60} = 7.7$ hours. The odometer trip meter reading $x(t)$ is in miles (assume $x(0) = 0$). The function $V_{pc}(t)$ is an approximation to the speedometer reading. Laplace's method can solve the approximation model

$$\frac{dx}{dt} = V_{pc}(60t), \quad x(0) = 0, \quad x \text{ in miles, } t \text{ in hours,}$$

obtaining $x(t) = \int_0^t V_{\text{pc}}(60w)dw$, the same result as the method of quadrature. Show the details. Then display the piecewise linear continuous trip meter reading $x(t)$.

Solution.

Method of Quadrature. The meaning of the differential equation is that $x'(t)$ is piecewise continuous. We want $x(t)$ to be continuous, because it is the odometer trip meter reading. But $x'(t)$ cannot be continuous, if we require $\frac{dx}{dt} = V_{\text{pc}}(60t)$, because the right side is piecewise defined and discontinuous at division points.

Theorem (Fundamental Theorem of Calculus)

If $f'(x)$ is piecewise continuous and $f(x)$ is continuous on $a \leq x \leq b$, then $\int_a^b f'(x)dx = f(b) - f(a)$.

The theorem implies that the method of quadrature works for the equation $x'(t) = V_{\text{pc}}(60t)$. The quadrature method gives the correct answer

$$x(t) = \int_0^t V_{\text{pc}}(60w)dw.$$

Another plan is to split $x'(t) = V_{\text{pc}}(60t)$ into 10 simple equations, $x' = 54.3$, $x(0) = 0$ on $0 \leq t < 20$ being the first equation. The next equation is $x' = 56.3$, $x(20) = x_0$, on $20 < t < 32$. To make $x(t)$ continuous, we must choose $x_0 = 1086$, which is the value at the division point $t = 20$ assumed by the first problem ($x' = 54.3$, $x(0) = 0$ on $0 \leq t < 20$). This tedious process has to be continued for all 10 segments. The result is that $x(t)$ is **piecewise linear** between division points.

Laplace's Method. The piecewise continuous input $V_{\text{pc}}(60t)$ is of exponential order, because it is zero after $t = 462/60$. Laplace theory says it has a Laplace transform $\mathcal{L}(V_{\text{pc}}(60t))$. Assuming a continuous solution $x(t)$, with $x'(t)$ piecewise continuous, then the equation to be satisfied is

$$s\mathcal{L}(x(t)) - x(0) = \mathcal{L}(x'(t)) = \mathcal{L}(V_{\text{pc}}(60t)).$$

The **Laplace integral theorem** implies

$$\mathcal{L}(x(t)) = \frac{1}{s}\mathcal{L}(V_{\text{pc}}(60t)) = \mathcal{L}\left(\int_0^t V_{\text{pc}}(60w)dw\right).$$

Lerch's theorem then implies that the symbol \mathcal{L} cancel from each side, giving the odometer trip meter reading in terms of the integral of the piecewise continuous input $V_{\text{pc}}(60t)$:

$$x(t) = \int_0^t V_{\text{pc}}(60w)dw.$$

We'll use technology to program and evaluate the integral, even though it can be done by hand. The plan is to plot the trip meter reading, then comment on the slow and fast segments of the route, by using a clever plot involving the average speed. The last display is the piecewise linear trip meter reading $x(t)$.

Maple

```
Xpc:=t->piecewise(t<0,0,
t < 20 ,54.3, t < 32, 56.3, t < 45, 42, t < 52, 48.9,
t < 70 ,55, t < 106, 65, t < 375, 68.7, t < 425, 61.2,
t < 462, 69.7, 0.0);
X:=t->int(Xpc(60*w),w=0..t);
plot(X(t),t=0..480/60); # Almost a straight line.
```

Average Speed

Define the average value of a function $f(w)$ on $a \leq w \leq b$ by $\frac{1}{b-a} \int_a^b f(w)dw$. Then the average speed in the example is

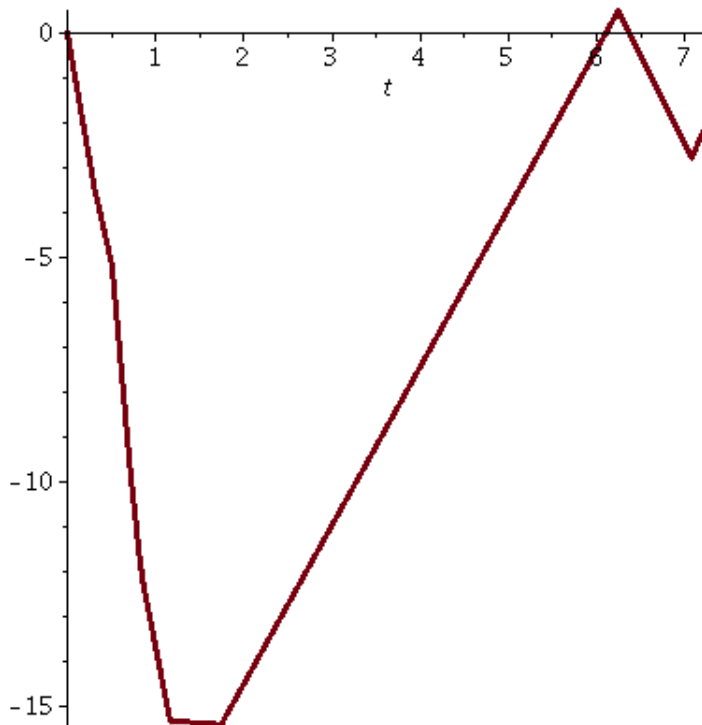
$$\frac{\int_0^{462/60} V_{\text{pc}}(60w)dw}{462/60} = 65.14956710.$$

A Clever Plot

An average driver would try to maintain 65.15 mph. The clever plot will create a graphic of $x(t) - 65.15t$ on interval $0 \leq t \leq T_1$, where T_1 is the 471 mile trip time at 65.15 mph.

Maple code

```
Xpc:=t->piecewise(t<0,0,
t < 20 ,54.3, t < 32, 56.3, t < 45, 42, t < 52, 48.9,
t < 70 ,55, t < 106, 65, t < 375, 68.7, t < 425, 61.2,
t < 462, 69.7, 0.0);
X:=t->int(Xpc(60*w),w=0..t);
AVEspeed:=X(462/60)/(462/60); # AVEspeed = 65.14956710 mph
T1:=solve(AVEspeed*t=471,t); # T1 = 7.229518491 hours
plot(X(t)-AVEspeed*t,t=0..T1);
```



We see from the graphic that segments of the road cause a slowdown of up to 15 mph, but for a brief interval it is possible to exceed the average speed, due to a 75 mph speed limit.

Maple code for piecewise linear display

```
X:=t->int(Xpc(60*w),w=0..t);
convert(X(t),piecewise,t):evalf(%,4);
```

$$\text{Trip meter at time } t = \begin{cases} 0.0 & t \leq 0.0 \\ 54.30t & t \leq 0.3333 \\ 56.30t - 0.6667 & t \leq 0.53 \\ 42.0t + 6.960 & t \leq 0.75 \\ 48.90t + 1.785 & t \leq 0.8667 \\ 55.0t - 3.502 & t \leq 1.167 \\ 65.0t - 15.17 & t \leq 1.767 \\ 68.70t - 21.70 & t \leq 6.25 \\ 61.20t + 25.17 & t \leq 7.083 \\ 69.70t - 35.04 & t \leq 7.7 \\ 501.7 & 7.7 < t \end{cases}$$

Chapter 7: Laplace Theory

Background. Switches and Impulses

Laplace's method solves differential equations. It is the preferred method for solving equations containing switches or impulses.

Unit Step Define $u(t - a) = \begin{cases} 1 & t \geq a, \\ 0 & t < a. \end{cases}$. It is a **switch**, turned on at $t = a$.

Ramp Define **ramp** $(t - a) = (t - a)u(t - a) = \begin{cases} t - a & t \geq a, \\ 0 & t < a. \end{cases}$, whose graph shape is a continuous **ramp** at 45-degree incline starting at $t = a$.

Unit Pulse Define **pulse** $(t, a, b) = \begin{cases} 1 & a \leq t < b, \\ 0 & \text{otherwise} \end{cases} = u(t - a) - u(t - b)$. The switch is **ON** at time $t = a$ and then **OFF** at time $t = b$.

Impulse of a Force

Define the **impulse** of an applied force $F(t)$ on time interval $a \leq t \leq b$ by the equation

$$\text{Impulse of } F = \int_a^b F(t)dt = \left(\frac{\int_a^b F(t)dt}{b - a} \right) (b - a) = \text{Average Force} \times \text{Duration Time}.$$

Dirac Unit Impulse

A Dirac impulse acts like a hammer hit, a brief injection of energy into a system. It is a special idealization of a real hammer hit, in which only the **impulse** of the force is deemed important, and not its magnitude nor duration.

Define the **Dirac Unit Impulse** by the equation $\delta(t - a) = \frac{du}{dt}(t - a)$, where $u(t - a)$ is the unit step. Symbol δ makes sense only under an integral sign, and the integral in question must be a generalized Riemann integral (definition pending), with new evaluation rules. Symbol δ is an abbreviation like **etc** or **e.g.**, because it abbreviates a paragraph of descriptive text.

- Symbol $M\delta(t - a)$ represents an ideal impulse of magnitude M at time $t = a$. Value M is the change in momentum, but $M\delta(t - a)$ contains no detail about the applied force or the duration. A common force approximation for a hammer hit of very small duration $2h$ and impulse M is Dirac's approximation

$$F_h(t) = \frac{M}{2h} \text{pulse}(t, a - h, a + h).$$

- Symbol δ is not manipulated as an ordinary function. It is a special modeling tool with rules for application and rules for algebraic manipulation.

THEOREM (Second Shifting Theorem). Let $f(t)$ and $g(t)$ be piecewise continuous and of exponential order. Then for $a \geq 0$,

$$\begin{aligned} e^{-as} \mathcal{L}(f(t)) &= \mathcal{L}(f(t)u(t)|_{t:=t-a}), \\ \mathcal{L}(g(t)u(t - a)) &= e^{-as} \mathcal{L}(g(t)|_{t:=t+a}). \end{aligned}$$

Sample Problem 2. Solve the following by Laplace methods.

(a) Forward table. Compute the Laplace integral for the unit step, ramp and pulse, in these special cases:

$$(1) \mathcal{L}(10u(t - \pi)) \quad (2) \mathcal{L}(\mathbf{ramp}(t - 2\pi)), \quad (3) \mathcal{L}(10 \mathbf{pulse}(t, 3, 5)).$$

(b) Backward table. Find $f(t)$ in the following special cases.

$$(1) \mathcal{L}(f) = \frac{5e^{-3s}}{s} \quad (2) \mathcal{L}(f) = \frac{e^{-4s}}{s^2} \quad (3) \mathcal{L}(f) = \frac{5}{s} (e^{-2s} - e^{-3s}).$$

Sample Problem 3. Solve the following Dirac Impulse and Second Shifting theorem problems.

(c) Forward table problems.

$$(1) \mathcal{L}(10\delta(t - \pi)), \quad (2) \mathcal{L}(5\delta(t - 1) + 10\delta(t - 2) + 15\delta(t - 3)), \quad (2) \mathcal{L}((t - \pi)\delta(t - \pi)).$$

The sum of Dirac impulses in (2) is called an **impulse train**.

Solutions to sample problems 2 and 3

Solution (a). The forward second shifting theorem applies.

$$(1) \quad \mathcal{L}(10u(t - \pi)) = \mathcal{L}(g(t)u(t - a)) \text{ where } g(t) = 10 \text{ and } a = \pi. \text{ Then } \mathcal{L}(10u(t - \pi)) = \mathcal{L}(g(t)u(t - a)) = e^{-as} \mathcal{L}(g(t)|_{t=t+a}) = e^{-\pi s} \mathcal{L}(10|_{t=t+\pi}) = \frac{10}{s} e^{-\pi s}.$$

$$(2) \quad \mathcal{L}(\mathbf{ramp}(t - 2\pi)) = \mathcal{L}((t - 2\pi)u(t - 2\pi)) = \mathcal{L}(tu(t)|_{t=t-2\pi}) = e^{-2\pi s} \mathcal{L}(t) = \frac{1}{s^2} e^{-2\pi s}.$$

$$(3) \quad \mathcal{L}(10 \mathbf{pulse}(t, 3, 5)) = 10 \mathcal{L}(u(t - 3) - u(t - 5)) = \frac{10}{s}(e^{3s} - e^{5s}).$$

Solution (b). Presence of an exponential e^{-as} signals step $u(t - a)$ in the answer, the main tool being the backward second shifting theorem.

$$(1) \quad \mathcal{L}(f) = \frac{5e^{-3s}}{s} = e^{-3s} \frac{5}{s} = e^{-3s} \mathcal{L}(5) = \mathcal{L}(5u(t)|_{t=t+3}) = \mathcal{L}(5u(t - 3)). \text{ Lerch implies } f = 5u(t - 3).$$

$$(2) \quad \mathcal{L}(f) = \frac{e^{-4s}}{s^2} = \frac{e^{-as}}{s^2}(t) \text{ where } a = 4. \text{ Then } \mathcal{L}(f) = \frac{e^{-as}}{s^2}(t) = \mathcal{L}(tu(t)|_{t=t-a}) = \mathcal{L}((t - 4)u(t - 4)) = \mathcal{L}(\mathbf{ramp}(t - 4)). \text{ Lerch implies } f = \mathbf{ramp}(t - 4).$$

$$(3) \quad \mathcal{L}(f) = e^{-2s} \frac{5}{s} - e^{-3s} \frac{5}{s} = \mathcal{L}(5u(t - 2)) - \mathcal{L}(5u(t - 3)) = \mathcal{L}(5 \mathbf{pulse}(t, 2, 3)). \text{ Lerch implies } f = 5 \mathbf{pulse}(t, 2, 3).$$

Solution (c). The main result for Dirac unit impulse δ is the equation

$$\int_0^\infty g(t)\delta(t - a)dt = g(a),$$

valid for $g(t)$ continuous on $0 \leq t < \infty$. When $g(t) = e^{-st}$, then the equation implies the Laplace formula $\mathcal{L}(\delta(t - a)) = e^{-as}$.

$$(1) \quad \mathcal{L}(10\delta(t - \pi)) = 10e^{-\pi s}, \text{ by the displayed equation with } g(t) = 10e^{-st}, \text{ or by using linearity and the formula } \mathcal{L}(\delta(t - a)) = e^{-as}.$$

$$(2) \quad \mathcal{L}(5\delta(t - 1) + 10\delta(t - 2) + 15\delta(t - 3)) = 5 \mathcal{L}(\delta(t - 1)) + 10 \mathcal{L}(\delta(t - 2)) + 15 \mathcal{L}(\delta(t - 3)) = 5e^{-s} + 10e^{-2s} + 15e^{-3s}.$$

$$(3) \quad \mathcal{L}((t - \pi)\delta(t - 2\pi)) = \int_0^\infty (t - \pi)e^{st}\delta(t - 2\pi)dt = (t - \pi)e^{-st}|_{t=2\pi} = \pi e^{-2\pi s}, \text{ using } g(t) = (t - \pi)e^{-st} \text{ and } a = 2\pi \text{ in the equation.}$$

Sample Problem 4. Experiment to Find the Transfer Function $h(t)$

Consider a second order problem

$$ax''(t) + bx'(t) + cx(t) = f(t)$$

which by Laplace theory has a particular solution defined as the convolution of the transfer function $h(t)$ with the input $f(t)$,

$$x_p(t) = \int_0^t f(w)h(t-w)dw.$$

Examined in this problem is another way to find $h(t)$, which is the system response to a Dirac unit impulse with zero data. Then $h(t)$ is the solution of

$$ah''(t) + bh'(t) + ch(t) = \delta(t), \quad h(0) = h'(0) = 0.$$

The Problem. Assume a, b, c are constants and define $g(t) = \int_0^t h(w)dw$.

(a) Show that $g(0) = g'(0) = 0$, which means g has zero data.

(b) Let $u(t)$ be the unit step. Argue that g is the solution of

$$ag''(t) + bg'(t) + cg(t) = u(t), \quad g(0) = g'(0) = 0.$$

The fundamental theorem of calculus says that $h(t) = g'(t)$. Therefore, to compute the transfer function $h(t)$, find the response $g(t)$ to the unit step with zero data, followed by computing the derivative $g'(t)$, which equals $h(t)$.

The experimental impact is important. Turning on a switch creates a unit step, generally easier than designing a hammer hit.

(c) Illustrate the method for finding the transfer function $h(t)$ in the special case

$$x''(t) + 2x'(t) + 5x(t) = f(t).$$

Solutions to sample problem 4

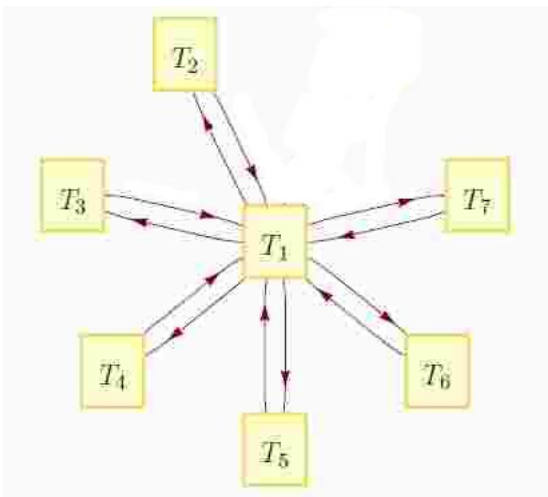
(a) $g(0) = \int_0^0 h(w)dw = 0$, $g'(0) = h(0) = 0$.

(b) Let $u(t)$ be the unit step. Initial data was decided in part (a). The Laplace applied to $ag''(t) + bg'(t) + cg(t) = u(t)$ gives $(as^2 + bs + c)\mathcal{L}(g) = \mathcal{L}(u(t))$. Then $\mathcal{L}(g) = \mathcal{L}(h(t))\mathcal{L}(u(t)) = \mathcal{L}(h(t))\frac{1}{s}\mathcal{L}\left(\int_0^t h(r)du\right)$ by the **integral theorem**. Lerch's theorem then says $g(t) = \int_0^t h(r)dr$.

(c) For equation $x''(t) + 2x'(t) + 5x(t) = f(t)$ we replace $x(t)$ by $g(t)$ and $f(t)$ by the unit step $u(t)$, then solve $g''(t) + 2g'(t) + 5g(t) = u(t)$, obtaining $\mathcal{L}(g) = \frac{1}{s} \frac{1}{s^2 + 2s + 5} = \mathcal{L}\left(\frac{1}{5} - \frac{1}{10}e^{-t}(2\cos(2t) + \sin(2t))\right)$. Then $g(t) = \frac{1}{5} - \frac{1}{10}e^{-t}(2\cos(2t) + \sin(2t))$ and $h(t) = g'(t) = \frac{1}{2}e^{-t}\sin(2t)$.

Chapters 4, 5: Linear Systems of Differential Equations

Sample Problem 5. Brine Tanks



Flow through each pipe is f gallons per unit time.

Each tank has constant volume V .

Symbols $x_1(t)$ to $x_7(t)$ are the salt amounts in tanks T_1 to T_7 , respectively.

The differential equations are obtained by the classical balance law, which says that the rate of change in salt amount is the rate in minus the rate out. Individual rates in/out are of the form (flow rate)(salt concentration), where flow rate f has units volume per unit time and $x_i(t)/V$ is the concentration = amount/volume.

$$x_1'(t) = \frac{f}{V} (x_2(t) + x_3(t) + x_4(t) + x_5(t) + x_6(t) + x_7(t) - 6x_1(t))$$

$$x_2'(t) = \frac{f}{V} (x_1(t) - x_2(t)),$$

$$x_3'(t) = \frac{f}{V} (x_1(t) - x_3(t)),$$

$$x_4'(t) = \frac{f}{V} (x_1(t) - x_4(t)),$$

$$x_5'(t) = \frac{f}{V} (x_1(t) - x_5(t)),$$

$$x_6'(t) = \frac{f}{V} (x_1(t) - x_6(t)),$$

$$x_7'(t) = \frac{f}{V} (x_1(t) - x_7(t)).$$

Part (a). Change variables $t = Vr/f$ to obtain the new system

$$\frac{dx_1}{dr} = x_2 + x_3 + x_4 + x_5 + x_6 + x_7 - 6x_1$$

$$\frac{dx_2}{dr} = x_1 - x_2,$$

$$\frac{dx_3}{dr} = x_1 - x_3,$$

$$\frac{dx_4}{dr} = x_1 - x_4,$$

$$\frac{dx_5}{dr} = x_1 - x_5,$$

$$\frac{dx_6}{dr} = x_1 - x_6,$$

$$\frac{dx_7}{dr} = x_1 - x_7.$$

Solution (a):

Because $\frac{dx(t)}{dt} = \frac{dx}{dr} \frac{dr}{dt} = \frac{dx}{dr} \frac{f}{V}$, then the fraction f/V cancels in the equations, resulting in the new system.

Part (b). Formulate the equations in 1(a) in the system form $\frac{d}{dr}\vec{u} = A\vec{u}$.

Answer:

$$A = \begin{pmatrix} -6 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}, \quad \vec{u} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{pmatrix}$$

Part (c). Find the eigenvalues of A .

Answer: $\lambda = 0, -1, -1, -1, -1, -1, -7$

Solution (c).

Let $D = |A - \lambda I|$. Replace $-1 - \lambda$ by symbol u . Then

$$D = \begin{vmatrix} -5 + u & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & u & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & u & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & u & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & u & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & u & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & u \end{vmatrix}$$

Add each of rows 2, 3, 4, 5, 6 to row 1. Then $1 + u$ is a common factor of row 1 and the determinant multiply rule implies

$$D = (1 + u) \begin{vmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & u & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & u & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & u & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & u & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & u & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & u \end{vmatrix}$$

Cofactor expansion along the last row, plus induction, gives the answer $D = (u + 1)(u - 6)u^5 = (-\lambda)(-\lambda - 7)(-\lambda - 1)^5$ with roots $\lambda = 0, -7, -1, -1, -1, -1, -1$.

Part (d). Find the eigenvectors of A .

Solution (d).

The root $\lambda = -1$ causes us to solve $(A + I)\vec{v} = \vec{0}$, which has coefficient matrix

$$B = \begin{pmatrix} -5 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \text{with } \mathbf{rref}(B) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

There are 2 lead variables and 5 free variables, hence 5 basis vectors

$$\begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

The eigenvector for $\lambda = 0$ has all components equal to 1. This fact is found from the equation $(A - (0)I)\vec{v} = \vec{0}$, which has coefficient matrix A .

The eigenvector for $\lambda = -7$ has first component -6 and the remaining equal to 1. The task begins with the equation $(A - (-7)I)\vec{v} = \vec{0}$, which has coefficient matrix

$$\begin{pmatrix} 7-6 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 7-1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 7-1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 7-1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 7-1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 7-1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 7-1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 6 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 6 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 6 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 6 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 6 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 6 \end{pmatrix}$$

The eigenvectors for $\lambda = 0$ and $\lambda = -7$ are respectively

$$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -6 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

Part (e). Solve the differential equation $\frac{d\vec{u}}{dr} = A\vec{u}$ by the eigenanalysis method.

Three Methods for Solving $\frac{d}{dt}\vec{u}(t) = A\vec{u}(t)$

- **Eigenanalysis Method.** The eigenpairs of matrix A are required. The matrix A must be diagonalizable, meaning there are n eigenpairs $(\lambda_1, \vec{v}_1), (\lambda_2, \vec{v}_2), \dots, (\lambda_n, \vec{v}_n)$. The main theorem says that the general solution of $\vec{u}' = A\vec{u}$ is

$$\vec{u}(t) = c_1 e^{\lambda_1 t} \vec{v}_1 + c_2 e^{\lambda_2 t} \vec{v}_2 + \dots + c_n e^{\lambda_n t} \vec{v}_n.$$

- **Laplace's Method.** Solve the scalar equations by the Laplace transform method. The **resolvent method** automates this process: $\vec{u}(t) = \mathcal{L}^{-1}((sI - A)^{-1})\vec{u}(0)$.
- **Cayley-Hamilton-Ziebur Method.** The solution $\vec{u}(t)$ is a vector linear combination of the Euler solution atoms f_1, \dots, f_n found from the roots of the characteristic equation $|A - \lambda I| = 0$. The vectors $\vec{d}_1, \dots, \vec{d}_n$ in the linear combination

$$\vec{u}(t) = f_1(t)\vec{d}_1 + f_2(t)\vec{d}_2 + \dots + f_n(t)\vec{d}_n$$

are determined by the explicit formula

$$\langle \vec{d}_1 | \vec{d}_2 | \dots | \vec{d}_n \rangle = \langle \vec{u}_0 | A\vec{u}_0 | \dots | A^{n-1}\vec{u}_0 \rangle \left(W(0)^T\right)^{-1},$$

where $W(t)$ is the Wronskian matrix of atoms f_1, \dots, f_n and \vec{u}_0 is the initial data.

Solution (e).

The eigenvectors corresponding to $\lambda = 0, -7, -1, -1, -1, -1, -1$ are respectively

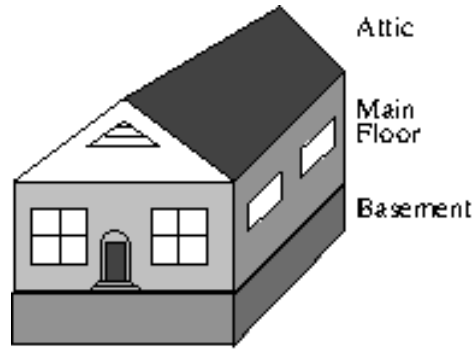
$$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -6 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

The Eigenanalysis method then implies the solution $\vec{x}(r)$ of $\frac{d\vec{x}}{dr} = A\vec{x}$ is given for arbitrary constants c_1, \dots, c_7 by the expression

$$c_1 e^{0r} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + c_2 e^{-7r} \begin{pmatrix} -6 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + c_3 e^{-r} \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + c_4 e^{-r} \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + c_5 e^{-r} \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + c_6 e^{-r} \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + c_7 e^{-r} \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Sample Problem 6. Home Heating

Consider a typical home with attic, basement and insulated main floor.



Heating Assumptions and Variables

- It is usual to surround the main living area with insulation, but the attic area has walls and ceiling without insulation.
- The walls and floor in the basement are insulated by earth.
- The basement ceiling is insulated by air space in the joists, a layer of flooring on the main floor and a layer of drywall in the basement.

The changing temperatures in the three levels is modeled by Newton's cooling law and the variables

$$\begin{aligned}z(t) &= \text{Temperature in the attic,} \\y(t) &= \text{Temperature in the main living area,} \\x(t) &= \text{Temperature in the basement,} \\t &= \text{Time in hours.}\end{aligned}$$

A typical mathematical model is the set of equations

$$\begin{aligned}x' &= \frac{1}{2}(45 - x) + \frac{1}{2}(y - x), \\y' &= \frac{1}{2}(x - y) + \frac{1}{4}(35 - y) + \frac{1}{4}(z - y) + 20, \\z' &= \frac{1}{4}(y - z) + \frac{3}{4}(35 - z).\end{aligned}$$

Part (a). Formulate the system of differential equations as a matrix system $\frac{d}{dt}\vec{u}(t) = A\vec{u}(t) + \vec{b}$. Show details.

Answer. $\vec{u} = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}$, $\vec{b} = \begin{pmatrix} \frac{45}{2} \\ 20 + \frac{35}{4} \\ \frac{105}{4} \end{pmatrix}$, $A = \begin{pmatrix} -1 & \frac{1}{2} & 0 \\ \frac{1}{2} & -1 & \frac{1}{4} \\ 0 & \frac{1}{4} & -1 \end{pmatrix}$

Solution Details.

Expand the right side of the system as follows.

$$\begin{aligned}x' &= \frac{45}{2} - \frac{x}{2} + \frac{y}{2} - \frac{x}{2}, \\y' &= \frac{x}{2} - \frac{y}{2} + \frac{35}{4} - \frac{y}{4} + \frac{z}{4} - \frac{y}{4} + 20 \\z' &= \frac{y}{4} - \frac{z}{4} + \frac{105}{4} - \frac{3z}{4}.\end{aligned}$$

Then collect on the variables:

$$\begin{aligned}x' &= -x + \frac{y}{2} + \frac{45}{2}, \\y' &= \frac{x}{2} - y + \frac{z}{4} + 20 + \frac{35}{4} \\z' &= \frac{y}{4} - z + \frac{105}{4}.\end{aligned}$$

The right side of this system can be written as $A\vec{u} + \vec{b}$. Vector \vec{b} is obtained by formally setting $x = y = z = 0$ on the right. This justifies the answer given.

The matrix A has columns equal to the partial derivatives $\partial_x, \partial_y, \partial_z$ of the right side of the scalar system. This idea is important, because it allows the computation of matrix A **without any of the preceding details**.

Part (b). The heating problem has an **equilibrium solution** $\vec{u}_p(t)$ which is a constant vector of temperatures for the three floors. It is formally found by setting $\frac{d}{dt}\vec{u}(t) = 0$, and then $\vec{u}_p = -A^{-1}\vec{b}$. Justify the algebra and explicitly find $\vec{u}_p(t)$.

Answer (b). $\vec{u}_p(t) = -A^{-1}\vec{b} = \begin{pmatrix} \frac{620}{11} \\ \frac{745}{11} \\ \frac{475}{11} \end{pmatrix} = \begin{pmatrix} 56.36 \\ 67.73 \\ 43.18 \end{pmatrix}.$

Solution Details.

The equation upon setting the derivative equal to zero becomes $\vec{0} = A\vec{u} + \vec{b}$ which implies $A\vec{u} = -\vec{b}$ and finally $\vec{u} = -A^{-1}\vec{b}$.

The calculation is done by technology. The maple code:

```
A:=<-1,1/2,0|1/2,-1,1/4|0,1/4,-1>^+; b:=<45/2,20+35/4,105/4>;
-A^(-1).b; evalf(%);
```

The solution can also be obtained by hand from the augmented matrix of A and $-\vec{b}$, using the linear algebra toolkit of swap, combination and multiply.

Part (c). The homogeneous problem is $\frac{d}{dt}\vec{u}(t) = A\vec{u}(t)$. It can be solved by a variety of methods, three major methods enumerated below. Choose a method and solve for $\vec{x}(t)$.

Answer (c): The homogenous scalar general solution is

$$\begin{aligned}x_1(t) &= -\frac{1}{2}c_1e^{-t} + 2c_2e^{-at} + 2c_3e^{-bt}, \\x_2(t) &= -\sqrt{5}c_2e^{-at} + \sqrt{5}c_3e^{-bt}, \\x_3(t) &= c_1e^{-t} + c_2e^{-at} + c_3e^{-bt}.\end{aligned}$$

Three Methods for Solving $\vec{u}' = A\vec{u}$

- **Eigenanalysis Method.** Three eigenpairs of matrix A are required. The matrix A must be diagonalizable, meaning there are 3 eigenpairs $(\lambda_1, \vec{v}_1), (\lambda_2, \vec{v}_2), (\lambda_3, \vec{v}_3)$. The main theorem says that the general solution of $\vec{u}' = A\vec{u}$ is

$$\vec{u}(t) = c_1e^{\lambda_1 t}\vec{v}_1 + c_2e^{\lambda_2 t}\vec{v}_2 + c_3e^{\lambda_3 t}\vec{v}_3.$$

- **Laplace's Method.** Solve the scalar equations by the Laplace transform method. The **resolvent method** automates this process: $\vec{u}(t) = \mathcal{L}^{-1}((sI - A)^{-1})\vec{u}(0)$.
- **Cayley-Hamilton-Ziebur Method.** The solution $\vec{u}(t)$ is a vector linear combination of the Euler solution atoms found from the roots of the characteristic equation $|A - \lambda I| = 0$. The vectors in the linear combination are determined by an explicit formula.

Solution Details for Part (c)

The roots of the characteristic polynomial are used in all three methods. This is the polynomial equation $|A - \lambda I| = 0$, having n roots real and complex, for an $n \times n$ matrix A .

Subtract λ from the diagonal of A and form the determinant. Then cofactor expansion on row 3 gives

$$|A - \lambda I| = \begin{vmatrix} -1 - \lambda & \frac{1}{2} & 0 \\ \frac{1}{2} & -1 - \lambda & \frac{1}{4} \\ 0 & \frac{1}{4} & -1 - \lambda \end{vmatrix} = (-1 - \lambda) \left(-\frac{1}{16} + (-1 - \lambda)^2 - \frac{1}{4} \right).$$

The roots are $-1, -a, -b$ where $a = 1 + \sqrt{5}/4 = 1.56, b = 1 - \sqrt{5}/4 = 0.44$. The three roots are distinct, real and negative.

Eigenanalysis Method

The eigenpairs must be found, in order to assemble the solution vector $\vec{u}(t)$. Technology can be used to find the answers, which are

$$\left(-1, \begin{pmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{pmatrix} \right), \quad \left(-a, \begin{pmatrix} 2 \\ -\sqrt{5} \\ 1 \end{pmatrix} \right), \quad \left(-b, \begin{pmatrix} 2 \\ \sqrt{5} \\ 1 \end{pmatrix} \right).$$

Without technology, there are three homogeneous problems to solve of the form $B\vec{v} = \vec{0}$, for eigenvector \vec{v} . Enumerated, they are:

Case $\lambda = -1$. Then $B = A + I = \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{4} \\ 0 & \frac{1}{4} & 0 \end{pmatrix}$

Case $\lambda = -a$. Then $B = A + aI = \begin{pmatrix} \frac{\sqrt{5}}{4} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{\sqrt{5}}{4} & \frac{1}{4} \\ 0 & \frac{1}{4} & \frac{\sqrt{5}}{4} \end{pmatrix}$

Case $\lambda = -b$. Then $B = A + bI = \begin{pmatrix} -\frac{\sqrt{5}}{4} & \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{\sqrt{5}}{4} & \frac{1}{4} \\ 0 & \frac{1}{4} & -\frac{\sqrt{5}}{4} \end{pmatrix}$

In each case, the system $B\vec{v} = \vec{0}$ is solved using the last frame algorithm (there is in each case one free variable). The eigenvector reported is the partial derivative of the general solution on the invented symbol t_1 , which was assigned to the free variable.

Application of the Theorem

System $\vec{u}' = A\vec{u}$ is solved in the diagonalizable case in terms of the eigenpairs of A , denoted as $(\lambda_1, \vec{v}_1), (\lambda_2, \vec{v}_2), (\lambda_3, \vec{v}_3)$. The solution of $\vec{u}' = A\vec{u}$ is given by the formula

$$\vec{u}(t) = c_1 e^{\lambda_1 t} \vec{v}_1 + c_2 e^{\lambda_2 t} \vec{v}_2 + c_3 e^{\lambda_3 t} \vec{v}_3.$$

In the present case, the solution is

$$\vec{u}(t) = c_1 e^{-t} \begin{pmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{pmatrix} + c_2 e^{-at} \begin{pmatrix} 2 \\ -\sqrt{5} \\ 1 \end{pmatrix} + c_3 e^{-bt} \begin{pmatrix} 2 \\ \sqrt{5} \\ 1 \end{pmatrix}.$$

Symbols c_1, c_2, c_3 in the solution are arbitrary constants, uniquely determined by initial conditions. In scalar form, the solution is

$$\begin{aligned}x_1(t) &= -\frac{1}{2}c_1e^{-t} + 2c_2e^{-at} + 2c_3e^{-bt}, \\x_2(t) &= -\sqrt{5}c_2e^{-at} + \sqrt{5}c_3e^{-bt}, \\x_3(t) &= c_1e^{-t} + c_2e^{-at} + c_3e^{-bt}.\end{aligned}$$

Laplace Transform Method

The **Laplace Method** for solving $\vec{u}'(t) = A\vec{u}(t)$ is based upon transforming all differential equations into the frequency domain. Then time variable t no longer appears, being replaced by the frequency variable s .

The homogeneous system of differential equations is

$$\begin{aligned}x' &= -x + \frac{y}{2}, \\y' &= \frac{x}{2} - y + \frac{z}{4} + 20 \\z' &= \frac{y}{4} - z.\end{aligned}$$

Transforming to the s -domain uses the parts rule $\mathcal{L}(f'(t)) = s\mathcal{L}(f(t)) - f(0)$. Then

$$\begin{aligned}s\mathcal{L}(x) - x(0) &= -\mathcal{L}(x) + \frac{1}{2}\mathcal{L}(y), \\s\mathcal{L}(y) - y(0) &= \frac{1}{2}\mathcal{L}(x) - \mathcal{L}(y) + \frac{1}{4}\mathcal{L}(z) + 20 \\s\mathcal{L}(z) - z(0) &= \frac{1}{4}\mathcal{L}(y) - \mathcal{L}(z).\end{aligned}$$

View these equations as linear algebraic equations for the symbols $\mathcal{L}(x), \mathcal{L}(y), \mathcal{L}(z)$. Move terms left and right to re-write the scalar equations as a matrix system

$$\begin{pmatrix} s+1 & -\frac{1}{2} & 0 \\ \frac{1}{2} & s+1 & -\frac{1}{4} \\ 0 & -\frac{1}{4} & s+1 \end{pmatrix} \begin{pmatrix} \mathcal{L}(x) \\ \mathcal{L}(y) \\ \mathcal{L}(z) \end{pmatrix} = \begin{pmatrix} x(0) \\ y(0) \\ z(0) \end{pmatrix}.$$

The system is solved by inverting the coefficient matrix C on the left, using the adjugate formula $C^{-1} = \mathbf{adj}(C)/|C|$. Write the answer as

$$C^{-1}(s) = \begin{pmatrix} s+1 & -\frac{1}{2} & 0 \\ \frac{1}{2} & s+1 & -\frac{1}{4} \\ 0 & -\frac{1}{4} & s+1 \end{pmatrix}^{-1} = \frac{1}{\Delta(s)} \begin{pmatrix} s^2 + 2s + \frac{15}{16} & \frac{s+1}{2} & \frac{1}{8} \\ \frac{s+1}{2} & (s+1)^2 & \frac{s+1}{4} \\ \frac{1}{8} & \frac{s+1}{4} & s^2 + 2s + \frac{3}{4} \end{pmatrix}.$$

Symbol $\Delta(s) = (s+1)(s+a)(s+b)$ is the determinant of $C(s)$. Then

$$\begin{pmatrix} \mathcal{L}(x) \\ \mathcal{L}(y) \\ \mathcal{L}(z) \end{pmatrix} = C^{-1}(s) \begin{pmatrix} x(0) \\ y(0) \\ z(0) \end{pmatrix}.$$

Backward table steps require solving nine equations of the form $\mathcal{L}(f(t)) = \frac{p(s)}{\Delta(s)}$. A computer algebra system reduces the effort, able to write $C^{-1}(s) = \mathcal{L}(\Phi(t))$, using symbols $f_1 = e^{-t}, f_2 = e^{-at}, f_3 = e^{-bt}$, where

$$\Phi(t) = \begin{pmatrix} \frac{1}{5}f_1 + \frac{2}{5}f_2 + \frac{2}{5}f_3 & \frac{1}{\sqrt{5}}(f_3 - f_2) & \frac{1}{5}f_2 + \frac{1}{5}f_3 - \frac{2}{5}f_1 \\ \frac{1}{\sqrt{5}}(f_3 - f_2) & \frac{1}{2}f_2 + \frac{1}{2}f_3 & \frac{1}{2\sqrt{5}}(f_3 - f_2) \\ \frac{1}{5}f_2 + \frac{1}{5}f_3 - \frac{2}{5}f_1 & \frac{1}{2\sqrt{5}}(f_3 - f_2) & \frac{4}{5}f_1 + \frac{1}{10}f_2 + \frac{1}{10}f_3 \end{pmatrix}$$

Then $\mathcal{L}(\vec{u}(t)) = \mathcal{L}(\Phi(t)\vec{u}(0))$ implies by Lerch's cancelation law the formula

$$\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} \frac{1}{5}f_1 + \frac{2}{5}f_2 + \frac{2}{5}f_3 & \frac{1}{\sqrt{5}}(f_3 - f_2) & \frac{1}{5}f_2 + \frac{1}{5}f_3 - \frac{2}{5}f_1 \\ \frac{1}{\sqrt{5}}(f_3 - f_2) & \frac{1}{2}f_2 + \frac{1}{2}f_3 & \frac{1}{2\sqrt{5}}(f_3 - f_2) \\ \frac{1}{5}f_2 + \frac{1}{5}f_3 - \frac{2}{5}f_1 & \frac{1}{2\sqrt{5}}(f_3 - f_2) & \frac{4}{5}f_1 + \frac{1}{10}f_2 + \frac{1}{10}f_3 \end{pmatrix} \begin{pmatrix} x(0) \\ y(0) \\ z(0) \end{pmatrix}.$$

The Resolvent Method

The scalar system solved above is exactly

$$(sI - A)\mathcal{L}(\vec{u}(t)) = \vec{u}(0), \quad I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad A = \begin{pmatrix} -1 & \frac{1}{2} & 0 \\ \frac{1}{2} & -1 & \frac{1}{4} \\ 0 & \frac{1}{4} & -1 \end{pmatrix}, \quad \vec{u}(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}.$$

The system $(sI - A)\mathcal{L}(\vec{u}(t)) = \vec{u}(0)$ is called the **resolvent equation**. The inverse of the coefficient matrix, $(sI - A)^{-1}$, is called the **resolvent matrix**, because $\mathcal{L}(\vec{u}(t)) = (sI - A)^{-1}\vec{u}(0)$. If these statements make sense to you, then please use them to solve problems. Otherwise, ignore the information, and solve problems in the same manner as outlined earlier.

Engineering and Laplace Transforms

Both mechanical engineering and electrical engineering have rich support for Laplace theory. Using Laplace theory has the advantage that many persons can help you through difficult times. Independent persons prefer to choose the method from their own private toolbox.

Cayley-Hamilton-Ziebur Method

The Ziebur Lemma implies that the solution of the system $\vec{u}'(t) = A\vec{u}(t)$ is given by the formula

$$\vec{u}(t) = \vec{d}_1 e^{-t} + \vec{d}_2 e^{-at} + \vec{d}_3 e^{-bt}.$$

THEOREM. Vectors $\vec{d}_1, \vec{d}_2, \vec{d}_3$ are uniquely determined by initial condition $\vec{u}(0)$, which is a column vector of prescribed constants, by the matrix equation.

$$\langle \vec{d}_1 | \vec{d}_2 | \vec{d}_3 \rangle = \langle \vec{u}(0) | A\vec{u}(0) | A^2\vec{u}(0) \rangle (W(0)^T)^{-1}$$

Symbol $W(t)$ is the Wronskian matrix of the three Euler solution atoms. Notation $\langle \vec{A} | \vec{B} | \vec{C} \rangle$ is the augmented matrix of the three columns vectors $\vec{A}, \vec{B}, \vec{C}$.

Illustration. For the heating example, with $a = 1 + \sqrt{5}/4 = 1.56, b = 1 - \sqrt{5}/4 = 0.44$, the Euler solution atoms are e^{-t}, e^{-at}, e^{-bt} . The Wronskian matrix is

$$W(t) = \begin{pmatrix} e^{-t} & e^{-at} & e^{-bt} \\ -e^{-t} & -ae^{-at} & -be^{-bt} \\ e^{-t} & a^2 e^{-at} & b^2 e^{-bt} \end{pmatrix}, \quad W(0) = \begin{pmatrix} 1 & 1 & 1 \\ -1 & -a & -b \\ 1 & a^2 & b^2 \end{pmatrix}.$$

$$\text{Then } (W(0)^T)^{-1} = \begin{pmatrix} -\frac{11}{5} & \frac{8}{5} - \frac{2}{5}\sqrt{5} & \frac{8}{5} + \frac{2}{5}\sqrt{5} \\ -\frac{32}{5} & \frac{16}{5} - \frac{2}{5}\sqrt{5} & \frac{16}{5} + \frac{2}{5}\sqrt{5} \\ -\frac{16}{5} & \frac{8}{5} & \frac{8}{5} \end{pmatrix}.$$

For initial state $\vec{u}(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\langle \vec{u}(0) | A\vec{u}(0) | A^2\vec{u}(0) \rangle = \begin{pmatrix} 1 & -1 & 5/4 \\ 0 & 1/2 & -1 \\ 0 & 0 & 1/8 \end{pmatrix}$. Then

$$\vec{u}(t) = \langle \vec{d}_1 | \vec{d}_2 | \vec{d}_3 \rangle \begin{pmatrix} e^{-t} \\ e^{-at} \\ e^{-bt} \end{pmatrix} = \begin{pmatrix} 1 & -1 & 5/4 \\ 0 & 1/2 & -1 \\ 0 & 0 & 1/8 \end{pmatrix} \begin{pmatrix} -\frac{11}{5} & \frac{8}{5} - \frac{2}{5}\sqrt{5} & \frac{8}{5} + \frac{2}{5}\sqrt{5} \\ -\frac{32}{5} & \frac{16}{5} - \frac{2}{5}\sqrt{5} & \frac{16}{5} + \frac{2}{5}\sqrt{5} \\ -\frac{16}{5} & \frac{8}{5} & \frac{8}{5} \end{pmatrix} \begin{pmatrix} e^{-t} \\ e^{-at} \\ e^{-bt} \end{pmatrix}$$

For general initial state $\vec{u}(0) = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$,

$$\langle \vec{u}(0) | A \vec{u}(0) | A^2 \vec{u}(0) \rangle = \begin{pmatrix} c_1 & -c_1 + \frac{1}{2}c_2 & \frac{5}{4}c_1 - c_2 + \frac{1}{8}c_3 \\ c_2 & \frac{1}{2}c_1 - c_2 + \frac{1}{4}c_3 & -c_1 + \frac{21}{16}c_2 - \frac{1}{2}c_3 \\ c_3 & \frac{1}{4}c_2 - c_3 & \frac{1}{8}c_1 - \frac{1}{2}c_2 + \frac{17}{16}c_3 \end{pmatrix}.$$

Then $\vec{u}(t)$ is this matrix times $(W(0)^T)^{-1}$ times the column vector of atoms e^{-t}, e^{-at}, e^{-bt} .

Details for the Theorem

The idea for finding the three vectors is differentiation of Ziebur's equation, two times, to get three equations

$$\begin{aligned} \vec{u}(t) &= \vec{d}_1 e^{-t} + \vec{d}_2 e^{-at} + \vec{d}_3 e^{-bt}, \\ \vec{u}'(t) &= -\vec{d}_1 e^{-t} - a\vec{d}_2 e^{-at} - b\vec{d}_3 e^{-bt}, \\ \vec{u}''(t) &= \vec{d}_1 e^{-t} + a^2\vec{d}_2 e^{-at} + b^2\vec{d}_3 e^{-bt}. \end{aligned}$$

Identities $\vec{u}'(t) = A\vec{u}(t)$ and $\vec{u}''(t) = A\vec{u}'(t) = AA\vec{u}(t)$ imply that the left sides are simplified to

$$\begin{aligned} \vec{u}(t) &= \vec{d}_1 e^{-t} + \vec{d}_2 e^{-at} + \vec{d}_3 e^{-bt}, \\ A\vec{u}(t) &= -\vec{d}_1 e^{-t} - a\vec{d}_2 e^{-at} - b\vec{d}_3 e^{-bt}, \\ A^2\vec{u}(t) &= \vec{d}_1 e^{-t} + a^2\vec{d}_2 e^{-at} + b^2\vec{d}_3 e^{-bt}. \end{aligned}$$

The critical idea is to substitute $t = 0$, which because of $e^0 = 1$ gives the following three equations for unknowns $\vec{d}_1, \vec{d}_2, \vec{d}_3$:

$$\begin{aligned} \vec{u}(0) &= \vec{d}_1 + \vec{d}_2 + \vec{d}_3, \\ A\vec{u}(0) &= -\vec{d}_1 - a\vec{d}_2 - b\vec{d}_3, \\ A^2\vec{u}(0) &= \vec{d}_1 + a^2\vec{d}_2 + b^2\vec{d}_3. \end{aligned}$$

How to solve these vector equations for unknowns $\vec{d}_1, \vec{d}_2, \vec{d}_3$? To begin, solve the scalar system

$$\begin{pmatrix} 1 & 1 & 1 \\ -1 & -a & -b \\ 1 & a^2 & b^2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

where variables x, y, z are the first components of $\vec{d}_1, \vec{d}_2, \vec{d}_3$, and similarly, b_1, b_2, b_3 are the first components of vector $\vec{u}(0), A\vec{u}(0), A^2\vec{u}(0)$:

$$\begin{aligned} x &= \vec{d}_1 \cdot \vec{v}, & b_1 &= \vec{u}(0) \cdot \vec{v}, \\ y &= \vec{d}_2 \cdot \vec{v}, & b_2 &= A\vec{u}(0) \cdot \vec{v}, \\ z &= \vec{d}_3 \cdot \vec{v}, & b_3 &= A^2\vec{u}(0) \cdot \vec{v}, \end{aligned} \quad \vec{v} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

The equations also apply to find the second components, using $\vec{v} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, then the third com-

ponents using $\vec{v} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. The three systems of equations can be written as one huge matrix equation:

$$\begin{pmatrix} 1 & 1 & 1 \\ -1 & -a & -b \\ 1 & a^2 & b^2 \end{pmatrix} \langle \vec{d}_1 | \vec{d}_2 | \vec{d}_3 \rangle^T = \langle \vec{u}(0) | A\vec{u}(0) | A^2\vec{u}(0) \rangle^T$$

Taking the transpose across the equation gives

$$\langle \vec{d}_1 | \vec{d}_2 | \vec{d}_3 \rangle \begin{pmatrix} 1 & 1 & 1 \\ -1 & -a & -b \\ 1 & a^2 & b^2 \end{pmatrix}^T = \langle \vec{u}(0) | A\vec{u}(0) | A^2\vec{u}(0) \rangle$$

Finally, invert the matrix $W(0)^T$ and multiply across the equation on the right to obtain

$$\langle \vec{d}_1 | \vec{d}_2 | \vec{d}_3 \rangle = \langle \vec{u}(0) | A\vec{u}(0) | A^2\vec{u}(0) \rangle \left(\begin{pmatrix} 1 & 1 & 1 \\ -1 & -a & -b \\ 1 & a^2 & b^2 \end{pmatrix}^T \right)^{-1}$$

This is exactly the equation reported in the theorem,

$$\langle \vec{d}_1 | \vec{d}_2 | \vec{d}_3 \rangle = \langle \vec{u}(0) | A\vec{u}(0) | A^2\vec{u}(0) \rangle \left(W(0)^T \right)^{-1}$$

It has been observed that if $f_1 = e^{-t}$, $f_2 = e^{-at}$, $f_3 = e^{-bt}$ are replaced by a new basis of solutions such that $W(0) = I$, then $\vec{d}_1 = \vec{u}(0)$, $\vec{d}_2 = A\vec{u}(0)$, $\vec{d}_3 = A^2\vec{u}(0)$. The resulting solution in this case is

$$\vec{u}(t) = f_1(t)\vec{u}(0) + f_2(t)A\vec{u}(0) + f_3(t)A^2\vec{u}(0).$$

Which Method is the Best?

The eigenanalysis method seems to be the best, because it is a method for simplifying coordinates, hence a shorter answer. Except in the case of complex roots. Or in the case that the matrix A fails to be diagonalizable. In practice, the method used to solve the equation $\vec{u}'(t) = A\vec{u}(t)$ has to be tuned to the expected application. Dynamical systems are an important example. For dynamical systems, the actual solution is less important than its formula, which is a linear combination of Euler solution atoms, according to Cayley-Hamilton-Ziebur.

Laplace theory provides a simple formula for the solution of $\vec{u}'(t) = A\vec{u}(t)$. It has the form

$$\vec{u}(t) = \Phi(t)\vec{u}(0).$$

The matrix $\Phi(t)$ in the Laplace formula is computed from $\mathcal{L}(\Phi(t)) = (sI - A)^{-1}$. Although this computation is nontrivial by hand, computer algebra system automation is possible.

Matrix $\Phi(t)$ is called the **Exponential Matrix**, denoted by e^{At} . Computer algebra system **Maple** computes $\Phi(t)$ by the command `LinearAlgebra[MatrixExponential](A, t)`. Both **Matlab** and **Mathematica** support symbolic computation of the matrix exponential $\Phi(t)$.