Chapter 3. Linear Differential Equations and Applications

Chapter 3. Sample Problem 1. Harmonic Vibration

A mass of $m = 250$ grams attached to a spring of Hooke's constant k undergoes free undamped vibration. At equilibrium, the spring is stretched 25 cm by a force of 8 Newtons. At time $t = 0$, the spring is stretched 0.5 m and the mass is set in motion with initial velocity 5 m/s directed away from equilibrium. Find:

(a) The numerical value of Hooke's constant k.

(b) The initial value problem for vibration $x(t)$.

Solution

(a): Hooke's law Force=k(elongation) is applied with force 8 Newtons and elongation $25/100 = 1/4$ meter. Equation $8 = k(1/4)$ implies $k = 32$ N/m.

(b): Given $m = 250/1000$ kg and $k = 32$ N/m from part (a), then the free vibration model $mx'' + kx = 0$ becomes $\frac{1}{4}x'' + 32x = 0$. Initial conditions are $x(0) = 0.5$ m and $x'(0) = 5$ m/s. The initial value problem is

$$\begin{cases} \dfrac{d^2x}{dt^2} + 128x = 0, \\ x(0) = 0.5, \\ x'(0) = 5. \end{cases}$$

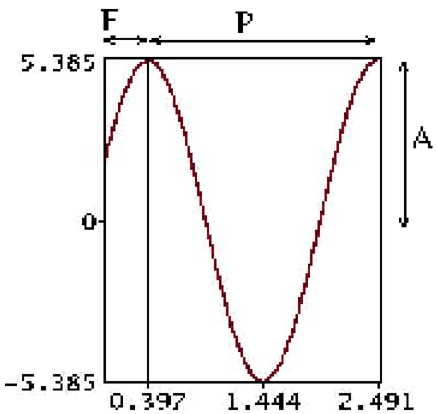
Chapter 3. Sample Problem 2. Phase-Amplitude Conversion
Write the vibration equation

$$x(t) = 2\cos(3t) + 5\sin(3t)$$

in phase-amplitude form $x = A\cos(\omega t - \alpha)$. Create a graphic of $x(t)$ with labels for period, amplitude and phase shift.

Solution
The answer and the graphic appear below.

$$x(t) = \sqrt{29}\cos(3t - 1.190289950) = \sqrt{29}\cos(3(t - 0.3967633167)).$$



Harmonic Oscillation

The graph of $2\cos(3t) + 5\sin(3t)$. It has amplitude $A = \sqrt{29} = 5.385$, period $P = 2\pi/3$ and phase shift $F = 0.3967633167$. The graph is on $0 \le t \le P + F$.

Algebra Details. The plan is to re-write $x(t)$ in the form $x(t) = A\cos(\omega t - \alpha)$, called the phase-amplitude form of the harmonic oscillation.

Start with $x(t) = 2\cos(3t) + 5\sin(3t)$. Trig identity $x(t) = A\cos(\omega t - \alpha) = A\cos(\alpha)\cos(\omega t) + A\sin(\alpha)\sin(\omega t)$ causes the definitions

$$\omega = 3, \quad A\cos(\alpha) = 2, \quad A\sin(\alpha) = 5.$$

The Pythagorean identity $\cos^2\alpha + \sin^2\alpha = 1$ implies $A^2 = 2^2 + 5^2 = 29$ and then the amplitude is $A = \sqrt{29}$. Because $\cos\alpha = 2/A$, $\sin\alpha = 5/A$, then both the sine and cosine are positive, placing angle α in quadrant I. Divide equations $\cos\alpha = 2/A$, $\sin\alpha = 5/A$ to obtain $\tan(\alpha) = 5/2$, which by calculator implies $\alpha = \arctan(5/2) = 1.190289950$ radians or 68.19859051 degrees. Then $x(t) = A\cos(\omega t - \alpha) = \sqrt{29}\cos(3t - 1.190289950)$.

Computer Details. Either equation for $x(t)$ can be used to produce a computer graphic. A hand-drawn graphic would use only the phase-amplitude form. The period is $P = 2\pi/\omega = 2\pi/3$. The amplitude is $A = \sqrt{29} = 5.385164807$ and the phase shift is $F = \alpha/\omega = 0.3967633167$. The graph is on $0 \le t \le P + F$.

```
# Maple
F:=evalf(arctan(5/2)/3); P:=2*Pi/3;A:=sqrt(29);
X:=t->2*cos(3*t)+5*sin(3*t);
opts:=xtickmarks=[0,F,P/2+F,P+F],ytickmarks=[-A,0,A],
      axes=boxed,thickness=3,labels=["",""];
plot(X(t),t=0..P+F,opts);
```

Chapter 3. Sample Problem 3. Undamped Spring-Mass System
A mass of 6 Kg is attached to a spring that elongates 20 centimeters due to a force of 12 Newtons. The motion starts at equilibrium with velocity -5 m/s. Find an equation for $x(t)$ using the free undamped vibration model $mx'' + kx = 0$.

Solution

The answer is $x(t) = -\sqrt{\frac{5}{2}}\sin(\sqrt{10}t)$.

The mass is $m = 6$ kg. Hooke's law $F = kx$ is applied with $F = 12$ N and $x = 20/100$ m. Then Hooke's constant is $k = 60$ N/m. Initial conditions are $x(0) = 0$ m (equilibrium) and $x'(0) = -5$ m/s.

The Model.

$$\begin{cases} 6\dfrac{d^2x}{dt^2} + 60x = 0, \\ x(0) = 0, \\ x'(0) = -5. \end{cases}$$

Solving the Model.

The characteristic equation $6r^2 + 60 = 0$ is solved for $r = \pm i\sqrt{10}$, then the Euler solution atoms are $\cos(\sqrt{10}t)$, $\sin(\sqrt{10}t)$ and we write the general solution as

$$x(t) = c_1\cos(\sqrt{10}t) + c_2\sin(\sqrt{10}t).$$

The task remaining is determination of constants c_1, c_2 subject to initial conditions $x(0) = 0$, $x'(0) = -5$. The linear algebra problem uses the derivative formula

$$x'(t) = -\sqrt{10}c_1\sin(\sqrt{10}t) + \sqrt{10}c_2\cos(\sqrt{10}t).$$

The 2×2 system of linear algebraic equations for c_1, c_2 is obtained from the two equations $x(0) = 0$, $x'(0) = -5$ as follows.

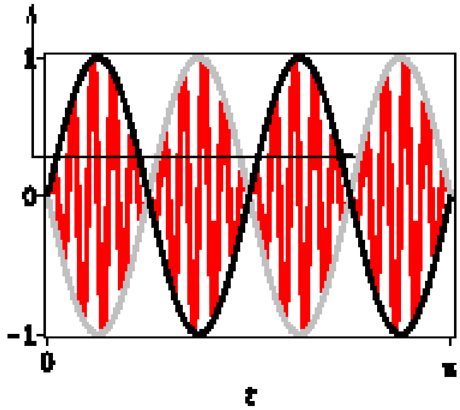
$$\left\{\begin{array}{rcrcrl} \cos(0)c_1 & + & \sin(0)c_2 & = & 0, & \text{Equation } x(0) = 0 \\ -\sqrt{10}\sin(0)c_1 & + & \sqrt{10}\cos(0)c_2 & = & -5, & \text{Equation } x'(0) = -5 \end{array}\right.$$

Because $\cos(0) = 1$, $\sin(0) = 0$, then $c_1 = 0$ and $c_2 = -5/\sqrt{10} = -\sqrt{5/2}$. Insert answers c_1, c_2 into the general solution to find the answer to the initial value problem

$$x(t) = -\sqrt{\frac{5}{2}}\sin(\sqrt{10}t).$$

Chapter 3. Sample Problem 4. Beats

The physical phenomenon of **beats** refers to the periodic interference of two sound waves of slightly different frequencies. A destructive interference occurs during a very brief interval, so our impression is that the sound periodically stops, only briefly, and then starts again with a beat, a section of sound that is instantaneously loud again. Human heartbeat uses the same terminology. Our pulse rate is $40 - 100$ **beats** per minute at rest. An illustration of the graphical meaning appears in the figure below.



Beats

Shown in red is a periodic oscillation $x(t) = 2\sin 4t \sin 40t$ with rapidly–varying factor $\sin 40t$ and the two slowly–varying envelope curves $x_1(t) = 2\sin 4t$ (black), $x_2(t) = -2\sin 4t$ (grey).

The undamped, forced spring-mass problem $x'' + 1296x = 640\cos(44t)$, $x(0) = x'(0) = 0$ has by trig identities the solution $x(t) = \cos(36t) - \cos(44t) = 2\sin 4t \sin 40t$.

A key example is piano tuning. A tuning fork is struck, then the piano string is tuned until the beats are not heard. The number of beats per second (unit Hz) is approximately the frequency difference between the two sources, e.g., two tuning forks of frequencies 440 Hz and 437 Hz would produce 3 beats per second.

The average human ear can detect beats only if the two interfering sound waves have a frequency difference of about 7 Hz or less. Ear-tuned pianos are subject to the same human ear limitations. Two piano keys are more than 7 Hz apart, even for a badly tuned piano, which is why simultaneously struck piano keys are heard as just one sound (no beats).

The beat we hear corresponds to maxima in the figure. We see not the two individual sound waves, but their **superposition**. When the tuning fork and the piano string have the same exact frequency ω, then the figure would show a simple harmonic wave, because the two sounds would **superimpose** to a graph that looks like $\cos(\omega t - \alpha)$.

The origin of the phenomenon of **beats** can be seen from the formula

$$x(t) = 2\sin at \sin bt.$$

There is no sound when $x(t) \approx 0$: this is when destructive interference occurs. When a is small compared to b, e.g., $a = 4$ and $b = 40$, then there are long intervals between the zeros of $A(t) = 2\sin at$, at which destructive interference occurs. Otherwise, the amplitude of the sound wave is the average value of $A(t)$, which is 1. The sound stops at a zero of $A(t)$ and then it is rapidly loud again, causing the beat.

The Problem. Solve the initial value problem

$$x'' + 1296x = 640\cos(44t), \quad x(0) = x'(0) = 0$$

by undetermined coefficients and linear algebra, obtaining the solution $x(t) = \cos(36t) - \cos(44t)$. Then show the trig details for $x(t) = 2\sin(4t)\sin(40t)$. Finally, graph $x(t)$ and its slowly varying envelope curves on $0 \le t \le \pi$.

Solution to Problem 4.

The trial solution for undetermined coefficients has the form $x = d_1 \cos(44t) + d_2 \sin(44t)$. This is the Rule I trial solution. But Rule II does not modify the trial solution, because the Euler solution atoms $\cos(44t), \sin(44t)$ are not solutions of the homogeneous equation $x'' + 1296x = 0$.

Insert the trial solution into $x'' + 1296x = 640\cos(44t)$ to obtain the equation

$$(1296 - 44^2)d_1 \cos(44t) + (1296 - 44^2)d_2 \sin(44t) = 640\cos(44t).$$

Then matching atoms across the equal sign implies $d_1 = 640/(1296 - 44^2) = -1$, $d_2 = 0$. The particular solution is the trial solution with $d_1 = -1$, $d_2 = 0$. The formula obtained so far is

$$x_p(t) = -\cos(44t).$$

The homogeneous solution $x_h(t)$ is found from the characteristic equation $r^2 + 1296 = 0$, with complex conjugate roots $r = \pm 36i$. Then

$$x_h(t) = c_1 \cos(36t) + c_2 \sin(36t).$$

The initial conditions $x(0) = x'(0) = 0$ are used together with the general solution and its derivative

$$\begin{array}{rcccl} x(t) & = & x_h(t) + x_p(t) & = & c_1\cos(36t) + c_2\sin(36t) - \cos(44t) \\ x'(t) & = & x_h'(t) + x_p'(t) & = & -36c_1\sin(36t) + 36c_2\cos(36t) + 44\sin(44t) \end{array}$$

to obtain the 2×2 linear algebraic system of equations

$$\left\{ \begin{array}{rcrcrl} \cos(0)c_1 & + & \sin(0)c_2 & = & \cos(0), & \text{Equation } x(0) = 0 \\ -36\sin(0)c_1 & + & 36\cos(0)c_2 & = & -44\sin(0). & \text{Equation } x'(0) = 0 \end{array} \right.$$

Then $c_1 = 1, c_2 = 0$ and $x(t) = \cos(36t) - \cos(44t)$.

The trig identities $\cos(a-b) = \cos(a)\cos(b) + \sin(a)\sin(b)$, $\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$ are subtracted to give $\cos(a-b) - \cos(a+b) = 2\sin(a)\sin(b)$. Let $a - b = 36t, a + b = 44t$ and solve for $a = 40t, b = 4t$. Then

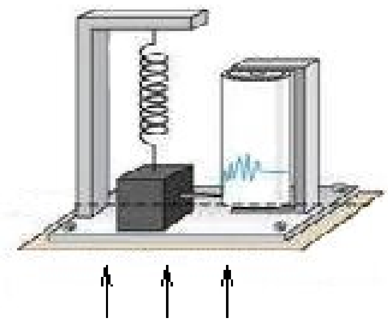
$$\begin{array}{rcl} x(t) & = & \cos(36t) - \cos(44t) \\ & = & \cos(a-b) - \cos(a+b) \\ & = & 2\sin a \sin b \\ & = & 2\sin 4t \sin 40t. \end{array}$$

The graphic of the problem was obtained from `MAPLE`.

```
a:=4:b:=10*a:
opts:=scaling=constrained,axes=boxed,axesfont=[Courier,bold,16],
labelfont=[Courier,bold,16],thickness=4,color=[red,gray,black]:
 plot([sin(a*t)*sin(b*t),-sin(a*t),sin(a*t)],t=0..2*(2*Pi/a),opts);
```

Chapter 3. Sample Problem 5. Vertical Motion Seismoscope

The 1875 **horizontal motion seismoscope** of F. Cecchi (1822-1887) reacted to an earthquake. It started a clock, and then it started motion of a recording surface, which ran at a speed of 1 cm per second for 20 seconds. The clock provided the observer with the earthquake hit time.



A Simplistic Vertical Motion Seismoscope

The apparently stationary heavy mass on a spring writes with the attached stylus onto a rotating drum, as the ground moves up. The generated trace is $x(t)$.

The motion of the heavy mass m in the figure can be modeled initially by a forced spring-mass system with damping. The initial model has the form

$$mx'' + cx' + kx = f(t)$$

where $f(t)$ is the vertical ground force due to the earthquake. In terms of the vertical ground motion $u(t)$, we write via Newton's second law the force equation $f(t) = -mu''(t)$ (compare to falling body $-mg$). The final model for the motion of the mass is then

$$\begin{cases} x''(t) + 2\beta\Omega_0 x'(t) + \Omega_0^2 x(t) = -u''(t), \\ \dfrac{c}{m} = 2\beta\Omega_0, \quad \dfrac{k}{m} = \Omega_0^2, \\ x(t) = \text{center of mass position measured from equilibrium}, \\ u(t) = \text{vertical ground motion due to the earthquake}. \end{cases} \tag{1}$$

Terms **seismoscope, seismograph, seismometer** refer to the device in the figure. Some observations:

Slow ground movement means $x' \approx 0$ and $x'' \approx 0$, then (1) implies $\Omega_0^2 x(t) = -u''(t)$. The seismometer records ground acceleration.

Fast ground movement means $x \approx 0$ and $x' \approx 0$, then (1) implies $x''(t) = -u''(t)$. The seismometer records ground displacement.

A **release test** begins by starting a vibration with u identically zero. Two successive maxima $(t_1, x_1), (t_2, x_2)$ are recorded. This experiment determines constants β, Ω_0.

The objective of (1) is to determine $u(t)$, by knowing $x(t)$ from the seismograph.

The Problem.

(a) Explain how a **release test** can find values for β, Ω_0 in the model $x''+2\beta\Omega_0 x'+\Omega_0^2 x = 0$.

(b) Assume the seismograph trace can be modeled at time $t = 0$ (a time after the earthquake struck) by $x(t) = Ce^{-at}\sin(bt)$ for some positive constants C, a, b. Assume a release test determined $2\beta\Omega_0 = 12$ and $\Omega_0^2 = 100$. Explain how to find a formula for the ground motion $u(t)$, then provide a formula for $u(t)$, using technology.

Solution to the Seismoscope Problem.

(a) A **release test** is an experiment which provides initial data $x(0) > 0$, $x'(0) = 0$ to the seismoscope mass. The model is $x'' + 2\beta\Omega_0 x' + \Omega_0^2 x = 0$ (ground motion zero). The recorder graphs $x(t)$ during the experiment, until two successive maxima $(t_1, x_1), (t_2, x_2)$ appear in the graph. This is enough information to find values for β, Ω_0.

In an under-damped oscillation, the characteristic equation is $(r+p)^2 + \omega^2 = 0$ corresponding to complex conjugate roots $-p \pm \omega i$. The phase-amplitude form is $x(t) = Ce^{-pt}\cos(\omega t - \alpha)$, with period $2\pi/\omega$.

The equation $x'' + 2\beta\Omega_0 x' + \Omega_0^2 x = 0$ has characteristic equation $(r+\beta)^2 + \Omega_0^2 = 0$. Therefore $x(t) = Ce^{-\beta t}\cos(\Omega_0 t - \alpha)$.

The period is $t_2 - t_1 = 2\pi/\Omega_0$. Therefore, Ω_0 is known. The maxima occur when the cosine factor is 1, therefore

$$\frac{x_2}{x_1} = \frac{Ce^{-\beta t_2} \cdot 1}{Ce^{-\beta t_1} \cdot 1} = e^{-\beta(t_2 - t_1)}.$$

This equation determines β.

(b) The equation $-u''(t) = x''(t) + 2\beta\Omega_0 x'(t) + \Omega_0^2 x(t)$ (the model written backwards) determines $u(t)$ in terms of $x(t)$. If $x(t)$ is known, then this is a quadrature equation for the ground motion $u(t)$.

For the example $x(t) = Ce^{-at}\sin(bt)$, $2\beta\Omega_0 = 12, \Omega_0^2 = 100$, then the quadrature equation is

$$-u''(t) = x''(t) + 12x'(t) + 100x(t).$$

After substitution of $x(t)$, the equation becomes

$$-u''(t) = C\mathrm{e}^{-at}\left(\sin(bt)\,a^2 - \sin(bt)\,b^2 - 2\cos(bt)\,ab - 12\sin(bt)\,a + 12\cos(bt)\,b + 100\sin(bt)\right)$$

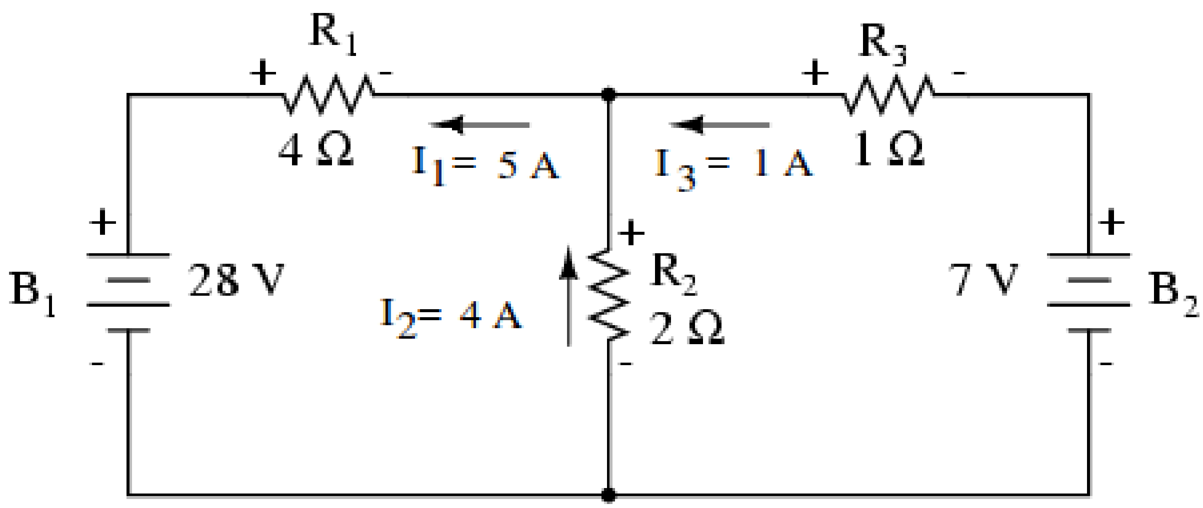
which can be integrated twice using Maple, for simplicity. All integration constants will be assumed zero. The answer:

$$\begin{aligned} u(t) &= \frac{C\mathrm{e}^{-at}\left(12\,a^2 b + 12\,b^3 - 200\,ab\right)\cos(bt)}{\left(a^2+b^2\right)^2} \\ &\quad - \frac{C\mathrm{e}^{-at}\left(a^4 + 2\,a^2 b^2 + b^4 - 12\,a^3 - 12\,ab^2 + 100\,a^2 - 100\,b^2\right)\sin(bt)}{\left(a^2+b^2\right)^2} \end{aligned}$$

The Maple session has this brief input:

```
de:=-diff(u(t),t,t) = diff(x(t),t,t) + 12*diff(x(t),t) + 100* x(t);
x:=t->C*exp(-a*t)*sin(b*t);
dsolve(de,u(t));subs(_C1=0,_C2=0,%);
```

Chapter 3. Sample Problem 6. **Resistive Network** with 2 Loops and DC Sources.



The **Branch Current Method** can be used to find a 3×3 linear system for the **branch currents** I_1, I_2, I_3.

$$\begin{array}{rcrcrcrl} I_1 & - & I_2 & - & I_3 & = & 0 & \text{KCL, upper node} \\ 4I_1 & + & 2I_2 & & & = & 28 & \text{KVL, left loop} \\ & & 2I_2 & - & I_3 & = & 7 & \text{KVL, right loop} \end{array}$$

Symbol **KCL** means *Kirchhoff's Current Law*, which says the algebraic sum of the currents at a node is zero. Symbol **KVL** means *Kirchhoff's Voltage Law*, which says the algebraic sum of the voltage drops around a closed loop is zero.

(a) Solve the equations to verify the currents reported in the figure: $I_1 = 5, I_2 = 4, I_3 = 1$ Amperes.

(b) Compute the voltage drops across resistors R_1, R_2, R_3. Answer: $20, 8, 1$ volts.

References. Edwards-Penney 3.7, electric circuits. All About Circuits Volume I – DC, by T. Kuphaldt:
http://www.allaboutcircuits.com/.
Course slides on Electric Circuits:
`http://www.math.utah.edu/~gustafso/s2019/2280/lectureslides/electricalCircuits.pdf`
Solved examples of electrical networks can be found in the lecture notes of Ruye Wang:
http://fourier.eng.hmc.edu/e84/lectures/ch2/node2.html.

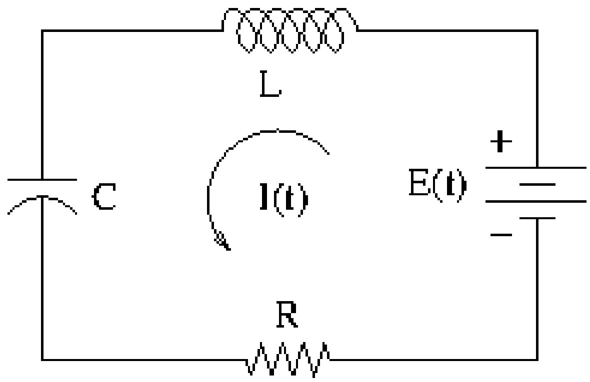
Solutions to Problem 6

Part (a). Write the system as a matrix equation $A\vec{x} = \vec{b}$. Solve for $\vec{x}$ by any method, including technology, to get $\vec{x} = \begin{pmatrix} 5 \\ 4 \\ 1 \end{pmatrix}$, whose components are currents I_1, I_2, I_3.

Part (b). Use Ohm's Law $V = RI$ to compute V, which is the voltage drop across resistor R. Use the answers from part (a).

Handwritten solutions to Problem 6 appear below, after the solution to Problem 7.

Chapter 3. Sample Problem 7. *RLC*-**Circuit**



The Problem. Suppose $E = 100\sin(20t)$, $L = 5$ H, $R = 250\,\Omega$ and $C = 0.002$ F. The model for the charge $Q(t)$ is $LQ'' + RQ' + \frac{1}{C}Q = E(t)$.

(a) Differentiate the charge model and substitute $I = \frac{dQ}{dt}$ to obtain the current model $5I'' + 250I' + 500I = 2000\cos(20t)$.

(b) Find the **reactance** $S = \omega L - \frac{1}{\omega C}$, where $\omega = 20$ is the input frequency, the natural frequency of $E = 100\sin(20t)$ and $E' = 2000\cos(20t)$.

(c) Substitute $I = A\cos(20t) + B\sin(20t)$ into the current model (a) and solve for $A = \frac{-12}{109}$, $B = \frac{40}{109}$. Then the steady-state current is

$$I(t) = A\cos(20t) + B\sin(20t) = \frac{-12\cos(20t) + 40\sin(20t)}{109}.$$

(d) Write the answer in (c) in phase-amplitude form $I = I_0\sin(20t - \delta)$ with $I_0 > 0$ and $\delta \geq 0$. Then compute the **time lag** δ/ω.

Answers: $I_0 = \frac{4}{\sqrt{109}}$, $\delta = \arctan(3/10)$, $\delta/\omega = 0.01457$.

References

Course slides on Electric Circuits:
`http://www.math.utah.edu/~gustafso/s2019/2280/lectureslides/electricalCircuits.pdf`
Edwards-Penney *Differential Equations and Boundary Value Problems*, sections 3.4, 3.5, 3.6, 3.7.

Solutions to Problem 7

Part (a) Start with $5Q'' + 250Q' + 500Q = 100\sin(20t)$. Differentiate across to get $5Q''' + 250Q'' + 500Q' = 2000\cos(20t)$. Change Q' to I.

Part (b) $S = (20)(5) - 1/(20 * 0.002) = 75$

Part (c) It helps to use the differential equation $u'' + 400u = 0$ satisfied by both $u_1 = \cos(20t)$ and $u_2 = \sin(20t)$. Functions u_1, u_2 are Euler solution atoms, hence independent. Along the solution path, we'll use $u_1' = -20\sin(20t) = -20u_2$ and $u_2' = 20\cos(20t) = 20u_1$. The arithmetic is simplified by dividing the equation first by 5. We then substitute $I = Au_1 + Bu_2$.

$$\begin{array}{l}
I'' + 50I' + 100I = 400\sin(20t) \\
A(u_1'' + 50u_1' + 100u_1) + B(u_2'' + 50u_2' + 100u_2) = 400\sin(20t) \\
A(-400u_1 + 50(-20u_2) + 100u_1) + B(-400u_2 + 50(20u_1) + 100u_2) = 400\sin(20t) \\
(-400A + 100A + 1000B)u_1 + (-1000A - 400B + 100B)u_2 = 400u_2
\end{array}$$

By independence of u_1, u_2, coefficients of u_1, u_2 on each side of the equation must match. The linear algebra property is called *unique representation of linear combinations.* This implies the 2×2 system of equations

$$\begin{array}{rcrcl}
-300A & + & 1000B & = & 0, \\
-1000A & - & 300B & = & 400.
\end{array}$$

The solution by Cramer's rule (the easiest method) is $A = -12/109, B = 40/109$. Then the steady-state current is

$$I(t) = A\cos(20t) + B\sin(20t) = \frac{-12\cos(20t) + 40\sin(20t)}{109}.$$

The **steady-state current** is defined to be the sum of those terms in the general solution of the differential equation that remain after all terms that limit to zero at $t = \infty$ have been removed. The logic is that only these terms contribute to a graphic or to a numerical calculation after enough time has passed (as $t \to \infty$).

Part (d) Let $\cos(\delta) = B/I_0, \sin(\delta) = -A/I_0, I_0 = \sqrt{A^2 + B^2}$. Use the trig identity

$$\sin(a - b) = \sin(a)\cos(b) - \cos(a)\sin(b)$$

to rearrange the current formula as follows:

$$I(t) = A\cos(20t) + B\sin(20t) = I_0(\sin(20t)\cos(\delta) - \sin(\delta)\cos(20t)) = I_0\sin(20t - \delta).$$

Compute $I_0 = \sqrt{A^2 + B^2} = \frac{4}{\sqrt{109}}$. Compute $\tan(\delta) = \frac{\sin\delta}{\cos\delta} = -A/B = 12/40$. Then $\delta = \arctan(12/40)$ and finally $\delta/\omega = \arctan(3/10)/20 = 0.01457$.

Another method, using Edwards-Penney Section 3.7: Compute the **impedance** $Z = \sqrt{R^2 + S^2} = \sqrt{250^2 + 75^2} = \sqrt{68125} = 25\sqrt{109}$ and then $I_0 = E_0/Z = 4/\sqrt{109}$. The phase $\delta = \arctan(S/R) = \arctan(75/250) = \arctan(3/10)$. Then the time lag is $\delta/\omega = \frac{\arctan(0.3)}{20} = 0.01457$.

Resistive Network. Solutions to Problem 6.

Resistive Network with 2 Loops and DC Sources

(a) The system in augmented matrix form is

$$\left(\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 4 & 2 & 0 & 28 \\ 0 & 2 & -1 & 7 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 6 & 4 & 28 \\ 0 & 2 & -1 & 7 \end{array}\right) \quad \text{combo}(1,2,-4)$$

$$\left(\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 0 & 7 & 7 \\ 0 & 2 & -1 & 7 \end{array}\right) \quad \text{combo}(3,2,-3)$$

$$\left(\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 2 & -1 & 7 \\ 0 & 0 & 7 & 7 \end{array}\right) \quad \text{swap}(2,3)$$

$$\left(\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 2 & -1 & 7 \\ 0 & 0 & 1 & 1 \end{array}\right) \quad \text{mult}(3, 1/7)$$

$$\left(\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 2 & 0 & 8 \\ 0 & 0 & 1 & 1 \end{array}\right) \quad \text{combo}(3,2,1)$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & 2 & 0 & 8 \\ 0 & 0 & 1 & 1 \end{array}\right) \quad \text{combo}(3,1,1)$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 1 \end{array}\right) \quad \text{mult}(2, 1/2)$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 1 \end{array}\right) \quad \text{combo}(2,1,1)$$

Last frame test passed ✓

Solution: $\begin{cases} I_1 = 5 \\ I_2 = 4 \\ I_3 = 1 \end{cases}$ unique solution case

(b) The voltage drop across a resistor is given by Ohm's Law: $V_R = RI$

Drop across $R_1 = 4\ \Omega$: $V_{R_1} = R_1 I_1 = 20$

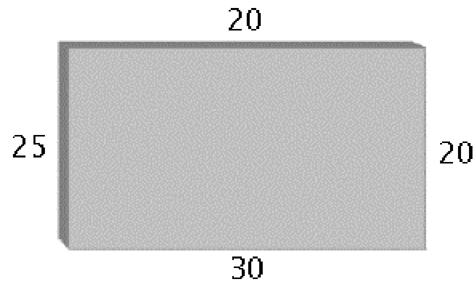
Drop across $R_2 = 2\ \Omega$: $V_{R_2} = R_2 I_2 = 8$

Drop across $R_3 = 1\ \Omega$: $V_{R_3} = R_3 I_3 = 1$

Enrichment. Mean Value Property.

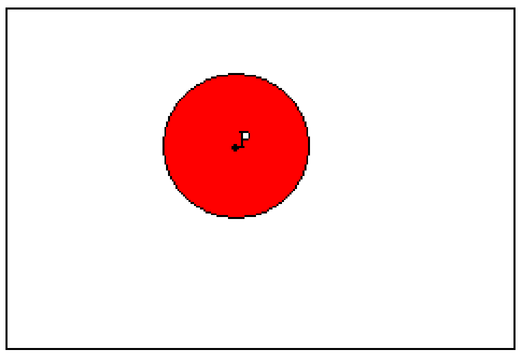
Sample Problem 8. **Heat Transfer** and the Mean Value Property.

Consider the cross section of a long rectangular dam on a river, represented in the figure.



The boundaries of the dam are subject to three factors: the temperature in degrees Celsius of the air (20), the water (25), and the ground at its base (30).

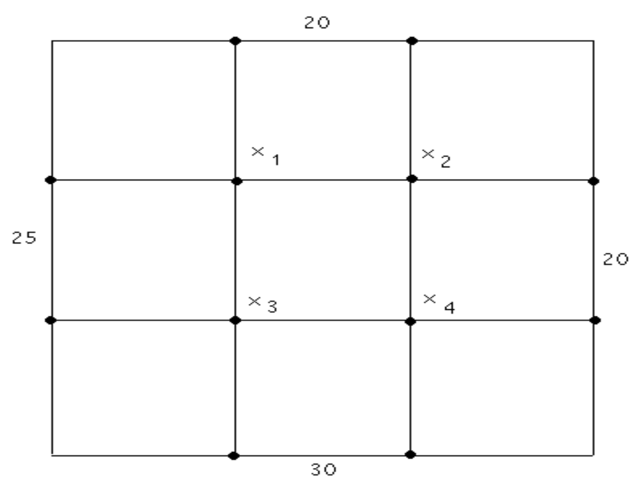
An analysis of the heat transfer from the three sources will be done from the equilibrium temperature, which is found by the Mean Value Property below.



The Mean Value Property

If a plate is at thermal equilibrium, and C is a circle contained in the plate with center P, then the temperature at P is the average value of the temperature function over C.

A version of the Mean Value Property says that the temperature at center P of circle C is the average of the temperatures at four equally-spaced points on C. We construct a grid as in the figure below, label the unknown temperatures at interior grid points as x_1, x_2, x_3, x_4, then use the property to obtain four equations.



Four-Point Temperature Averages

$$x_1 = \frac{1}{4}(20 + 25 + x_2 + x_3)$$

$$x_2 = \frac{1}{4}(20 + 20 + x_1 + x_4)$$

$$x_3 = \frac{1}{4}(25 + 30 + x_1 + x_4)$$

$$x_4 = \frac{1}{4}(20 + 30 + x_2 + x_3)$$

Solve the equations for the four temperatures $x_1 = 23.125, x_2 = 21.875, x_3 = 25.625, x_4 = 24.375$ by any method.

References. EPH Chapters 12, 13, on heat transfer. Used in Partial Differential Equations 3150. Intro Differential Equations 2280 uses Chapter 9 of a different Edwards-Penney textbook.

Sample Quiz 7 Solutions

Extra Credit

Problem 3. write the equations as

$$\begin{pmatrix} 4 & -1 & -1 & 0 \\ -1 & 4 & 0 & -1 \\ -1 & 0 & 4 & -1 \\ 0 & -1 & -1 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 45 \\ 40 \\ 55 \\ 50 \end{pmatrix}$$

The vector $\vec{x}$ is the vector of equilibrium temperatures, located at the grid points of the diagram.

we do swap, combo, mult on the augmented matrix

$$\left(\begin{array}{cccc|c} 4 & -1 & -1 & 0 & 45 \\ -1 & 4 & 0 & -1 & 40 \\ -1 & 0 & 4 & -1 & 55 \\ 0 & -1 & -1 & 4 & 50 \end{array}\right)$$

or alternatively compute the inverse of the coefficient matrix A:

$$A^{-1} = \begin{pmatrix} 7 & 2 & 2 & 1 \\ 2 & 7 & 1 & 2 \\ 2 & 1 & 7 & 2 \\ 1 & 2 & 2 & 7 \end{pmatrix} \cdot \frac{1}{24}$$

The solution of $A\vec{x} = \vec{b}$ is always $\vec{x} = A^{-1}\vec{b}$, so

$$\vec{x} = \frac{1}{24}\begin{pmatrix} 7 & 2 & 2 & 1 \\ 2 & 7 & 1 & 2 \\ 2 & 1 & 7 & 2 \\ 1 & 2 & 2 & 7 \end{pmatrix}\begin{pmatrix} 45 \\ 40 \\ 55 \\ 50 \end{pmatrix} = \begin{pmatrix} 23.125 \\ 21.875 \\ 25.625 \\ 24.375 \end{pmatrix} = \begin{pmatrix} 185 \\ 175 \\ 205 \\ 195 \end{pmatrix} \cdot \frac{1}{8}$$

most of this can be done with technology. As the grid size goes up, so does the difficulty of writing $A, \vec{b}$.

Chapter 3: Linear Differential Equations and Applications

Sample Problem 9. Solving Higher Order Constant-Coefficient Equations

The Algorithm applies to constant-coefficient homogeneous linear differential equations of order N, for example equations like

$$y'' + 16y = 0, \quad y'''' + 4y'' = 0, \quad \frac{d^5y}{dx^5} + 2y''' + y'' = 0.$$

1. Find the Nth degree characteristic equation by Euler's substitution $y = e^{rx}$. For instance, $y'' + 16y = 0$ has characteristic equation $r^2 + 16 = 0$, a polynomial equation of degree $N = 2$.
2. Find all real roots and all complex conjugate pairs of roots satisfying the characteristic equation. List the N roots according to multiplicity.
3. Construct N distinct Euler solution atoms from the list of roots. Then the general solution of the differential equation is a linear combination of the Euler solution atoms with arbitrary coefficients $c_1, c_2, c_3, \ldots$.
   The solution space S of the differential equation is given by

$$S = \mathbf{span}(\text{the } N \text{ Euler solution atoms}).$$

Examples: Constructing Euler Solution Atoms from roots.

Three roots $0, 0, 0$ produce three atoms $e^{0x}, xe^{0x}, x^2e^{0x}$ or $1, x, x^2$.

Three roots $0, 0, 2$ produce three atoms e^{0x}, xe^{0x}, e^{2x}.

Two complex conjugate roots $2 \pm 3i$ produce two atoms $e^{2x}\cos(3x), e^{2x}\sin(3x)$.[1]

Four complex conjugate roots listed according to multiplicity as $2 \pm 3i, 2 \pm 3i$ produce four atoms $e^{2x}\cos(3x), e^{2x}\sin(3x),\ xe^{2x}\cos(3x), xe^{2x}\sin(3x)$.

Seven roots $1, 1, 3, 3, 3, \pm 3i$ produce seven atoms $e^x, xe^x, e^{3x}, xe^{3x}, x^2e^{3x}, \cos(3x), \sin(3x)$.

Two conjugate complex roots $a \pm bi$ ($b > 0$) arising from roots of $(r-a)^2 + b^2 = 0$ produce two atoms $e^{ax}\cos(bx), e^{ax}\sin(bx)$.

The Problem

Solve for the general solution or the particular solution satisfying initial conditions.

(a) $y'' + 16y' = 0$

(b) $y'' + 16y = 0$

(c) $y'''' + 16y'' = 0$

(d) $y'' + 16y = 0$, $y(0) = 1$, $y'(0) = -1$

(e) $y'''' + 9y'' = 0$, $y(0) = y'(0) = 0$, $y''(0) = y'''(0) = 1$

(f) The characteristic equation is $(r-2)^2(r^2-4) = 0$.

(g) The characteristic equation is $(r-1)^2(r^2-1)((r+2)^2+4) = 0$.

(h) The characteristic equation roots, listed according to multiplicity, are $0, 0, 0, -1, 2, 2, 3+4i, 3-4i$.

---

[1] The Reason: $\cos(3x) = \frac{1}{2}e^{3xi} + \frac{1}{2}e^{-3xi}$ by Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$. Then $e^{2x}\cos(3x) = \frac{1}{2}e^{2x+3xi} + \frac{1}{2}e^{2x-3xi}$ is a linear combination of exponentials e^{rx} where r is a root of the characteristic equation. Euler's substitution implies e^{rx} is a solution, so by superposition, so also is $e^{2x}\cos(3x)$. Similar for $e^{2x}\sin(3x)$.

Solutions to Problem 9

(a) $y'' + 16y' = 0$ upon substitution of $y = e^{rx}$ becomes $(r^2 + 16r)e^{rx} = 0$. Cancel e^{rx} to find the **characteristic equation** $r^2 + 16r = 0$. It factors into $r(r+16) = 0$, then the two roots r make the list $r = 0, -16$. The Euler solution atoms for these roots are e^{0x}, e^{-16x}. Report the general solution $y = c_1 e^{0x} + c_2 e^{-16x} = c_1 + c_2 e^{-16x}$, where symbols c_1, c_2 stand for arbitrary constants.

(b) $y'' + 16y = 0$ has characteristic equation $r^2 + 16 = 0$. Because a quadratic equation $(r-a)^2 + b^2 = 0$ has roots $r = a \pm bi$, then the root list for $r^2 + 16 = 0$ is $0 + 4i, 0 - 4i$, or briefly $\pm 4i$. The Euler solution atoms are $e^{0x}\cos(4x), e^{0x}\sin(4x)$. The general solution is $y = c_1\cos(4x) + c_2\sin(4x)$, because $e^{0x} = 1$.

(c) $y'''' + 16y'' = 0$ has characteristic equation $r^4 + 4r^2 = 0$ which factors into $r^2(r^2+16) = 0$ having root list $0, 0, 0 \pm 4i$. The Euler solution atoms are $e^{0x}, xe^{0x}, e^{0x}\cos(4x), e^{0x}\sin(4x)$. Then the general solution is $y = c_1 + c_2 x + c_3\cos(4x) + c_4\sin(4x)$.

(d) $y'' + 16y = 0$, $y(0) = 1$, $y'(0) = -1$ defines a particular solution y. The usual arbitrary constants c_1, c_2 are determined by the initial conditions. From part (b), $y = c_1\cos(4x) + c_2\sin(4x)$. Then $y' = -4c_1\sin(4x) + 4c_2\cos(4x)$. Initial conditions $y(0) = 1, y'(0) = -1$ imply the equations $c_1\cos(0) + c_2\sin(0) = 1, -4c_1\sin(0) + 4c_2\cos(0) = -1$. Using $\cos(0) = 1$ and $\sin(0) = 0$ simplifies the equations to $c_1 = 1$ and $4c_2 = -1$. Then the particular solution is $y = c_1\cos(4x) + c_2\sin(4x) = \cos(4x) - \frac{1}{4}\sin(4x)$.

(e) $y'''' + 9y'' = 0$, $y(0) = y'(0) = 0$, $y''(0) = y'''(0) = 1$ is solved like part (d). First, the characteristic equation $r^4 + 9r^2 = 0$ is factored into $r^2(r^2+9) = 0$ to find the root list $0, 0, 0 \pm 3i$. The Euler solution atoms are $e^{0x}, xe^{0x}, e^{0x}\cos(3x), e^{0x}\sin(3x)$, which implies the general solution $y = c_1 + c_2 x + c_3\cos(3x) + c_4\sin(3x)$. We have to find the derivatives of y: $y' = c_2 - 3c_3\sin(3x) + 3c_4\cos(3x)$, $y'' = -9c_3\cos(3x) - 9c_4\sin(3x)$, $y''' = 27c_3\sin(3x) - 27c_4\cos(3x)$. The initial conditions give four equations in four unknowns c_1, c_2, c_3, c_4:

$$\begin{array}{rcrcrcrcl} c_1 & + & c_2(0) & + & c_3\cos(0) & + & c_4\sin(0) & = & 0, \\ & & c_2 & - & 3c_3\sin(0) & + & 3c_4\cos(0) & = & 0, \\ & & & - & 9c_3\cos(0) & - & 9c_4\sin(0) & = & 1, \\ & & & & 27c_3\sin(0) & - & 27c_4\cos(0) & = & 1, \end{array}$$

which has invertible coefficient matrix $\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & -9 & 0 \\ 0 & 0 & 0 & -27 \end{pmatrix}$ and right side vector $\begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$. The solution is $c_1 = c_2 = 1/9$, $c_3 = -1/9, c_4 = -1/27$. Then the particular solution is $y = c_1 + c_2 x + c_3\cos(3x) + c_4\sin(3x) = \frac{1}{9} + \frac{1}{9}x - \frac{1}{9}\cos(3x) - \frac{1}{27}\sin(3x)$

(f) The characteristic equation is $(r-2)^2(r^2-4) = 0$. Then $(r-2)^3(r+2) = 0$ with root list $2, 2, 2, -2$ and Euler atoms $e^{2x}, xe^{2x}, x^2e^{2x}, e^{-2x}$. The general solution is a linear combination of these four atoms.

(g) The characteristic equation is $(r-1)^2(r^2-1)((r+2)^2+4) = 0$. The root list is $1, 1, 1, -1, -2 \pm 2i$ with Euler atoms $e^x, xe^x, x^2e^x, e^{-x}, e^{-2x}\cos(2x), e^{-2x}\sin(2x)$. The general solution is a linear combination of these six atoms.

(h) The characteristic equation roots, listed according to multiplicity, are $0, 0, 0, -1, 2, 2, 3 + 4i, 3 - 4i$. Then the Euler solution atoms are $e^{0x}, xe^{0x}, x^2e^{0x}, e^{-x}, e^{2x}, xe^{2x}, e^{3x}\cos(4x), e^{3x}\sin(4x)$. The general solution is a linear combination of these eight atoms.

Chapter 7: Laplace Theory

Sample Problem 10.

> Laplace theory implements the *method of quadrature* for higher order differential equations, linear systems of differential equations, and certain partial differential equations.
>
> Laplace's method **solves differential equations.**

The Problem. Solve by table methods or Laplace's method.

(a) Forward table. Find $\mathcal{L}(f(t))$ for $f(t) = te^{2t} + 2t\sin(3t) + 3e^{-t}\cos(4t)$.

(b) Backward table. Find $f(t)$ for

$$\mathcal{L}(f(t)) = \frac{16}{s^2+4} + \frac{s+1}{s^2-2s+10} + \frac{2}{s^2+16}.$$

(c) Solve the initial value problem $x''(t) + 256x(t) = 1$, $x(0) = 1$, $x'(0) = 0$.

Solution (a).

$$\begin{array}{rcll}
\mathcal{L}(f(t)) & = & \mathcal{L}(te^{2t} + 2t\sin(3t) + 3e^{-t}\cos(4t)) & \\
& = & \mathcal{L}(te^{2t}) + 2\mathcal{L}(t\sin(3t)) + 3\mathcal{L}(e^{-t}\cos(4t)) & \text{Linearity} \\
& = & -\frac{d}{ds}\mathcal{L}(e^{2t}) - 2\frac{d}{ds}\mathcal{L}(\sin(3t)) + 3\mathcal{L}(e^{-t}\cos(4t)) & \text{Differentiation rule} \\
& = & -\frac{d}{ds}\mathcal{L}(e^{2t}) - 2\frac{d}{ds}\mathcal{L}(\sin(3t)) + 3\,\mathcal{L}(\cos(4t))\big|_{s=s+1} & \text{Shift rule} \\
& = & -\frac{d}{ds}\frac{1}{s-2} - 2\frac{d}{ds}\frac{3}{s^2+9} + 3\left.\frac{s}{s^2+16}\right|_{s=s+1} & \text{Forward table} \\
& = & \frac{1}{(s-2)^2} + \frac{12s}{(s^2+9)^2} + 3\frac{s+1}{(s+1)^2+16} & \text{Calculus}
\end{array}$$

Solution (b).

$$\begin{array}{rcll}
\mathcal{L}(f(t)) & = & \frac{16}{s^2+4} + \frac{s+1}{s^2-2s+10} + \frac{2}{s^2+16} & \\
& = & 8\frac{2}{s^2+4} + \frac{s+1}{(s-1)^2+9} + \frac{1}{2}\frac{4}{s^2+16} & \text{Prep for backward table} \\
& = & 8\mathcal{L}(\sin 2t) + \frac{s+1}{(s-1)^2+9} + \frac{1}{2}\mathcal{L}(\sin 4t) & \text{backward table} \\
& = & 8\mathcal{L}(\sin 2t) + \left.\frac{s+2}{s^2+9}\right|_{s=s-1} + \frac{1}{2}\mathcal{L}(\sin 4t) & \text{shift rule} \\
& = & 8\mathcal{L}(\sin 2t) + \mathcal{L}(\cos 3t + \frac{2}{3}\sin 3t)\Big|_{s=s-1} + \frac{1}{2}\mathcal{L}(\sin 4t) & \text{backward table} \\
& = & 8\mathcal{L}(\sin 2t) + \mathcal{L}(e^t\cos 3t + e^t\frac{2}{3}\sin 3t) + \frac{1}{2}\mathcal{L}(\sin 4t) & \text{shift rule} \\
& = & \mathcal{L}(8\sin 2t) + e^t\cos 3t + e^t\frac{2}{3}\sin 3t + \frac{1}{2}\sin 4t) & \text{Linearity} \\
f(t) & = & 8\sin 2t + e^t\cos 3t + e^t\frac{2}{3}\sin 3t + \frac{1}{2}\sin 4t & \text{Lerch's cancel rule}
\end{array}$$

Solution (c).

$$\begin{array}{lcll}
\mathcal{L}(x''(t) + 256x(t)) & = & \mathcal{L}(1) & \mathcal{L}\text{ acts like matrix mult} \\
s\mathcal{L}(x') - x'(0) + 256\mathcal{L}(x) & = & \mathcal{L}(1) & \text{Parts rule} \\
s(s\mathcal{L}(x) - x(0)) - x'(0) + 256\mathcal{L}(x) & = & \mathcal{L}(1) & \text{Parts rule} \\
s^2\mathcal{L}(x) - s + 256\mathcal{L}(x) & = & \mathcal{L}(1) & \text{Use } x(0) = 1, x'(0) = 0 \\
(s^2 + 256)\mathcal{L}(x) & = & s + \mathcal{L}(1) & \text{Collect } \mathcal{L}(x) \text{ left}
\end{array}$$

$$\begin{array}{rcll}
\mathcal{L}(x) & = & \frac{s+\mathcal{L}(1)}{(s^2+256)} & \text{Isolate } \mathcal{L}(x) \text{ left} \\
\mathcal{L}(x) & = & \frac{s+1/s}{(s^2+256)} & \text{Forward table} \\
\mathcal{L}(x) & = & \frac{s^2+1}{s(s^2+256)} & \text{Algebra} \\
\mathcal{L}(x) & = & \frac{A}{s} + \frac{Bs+C}{s^2+256} & \text{Partial fractions} \\
\mathcal{L}(x) & = & A\mathcal{L}(1) + B\mathcal{L}(\cos 16t) + \frac{C}{16}\mathcal{L}(\sin 16t) & \text{Backward table} \\
\mathcal{L}(x) & = & \mathcal{L}(A + B\cos 16t + \frac{C}{16}\sin 16t) & \text{Linearity} \\
x(t) & = & A + B\cos 16t + \frac{C}{16}\sin 16t & \text{Lerch's rule}
\end{array}$$

The partial fraction problem remains:

$$\frac{s^2+1}{s(s^2+256)} = \frac{A}{s} + \frac{Bs+C}{s^2+256}$$

This problem is solved by clearing the fractions, then swapping sides of the equation, to obtain

$$A(s^2+256) + (Bs+C)(s) = s^2+1.$$

Substitute three values for s to find 3 equations in 3 unknowns A, B, C:

$$\begin{array}{llcl} s=0 & 256A & = & 1 \\ s=1 & 257A+B+C & = & 2 \\ s=-1 & 257A+B-C & = & 2 \end{array}$$

Then $A = 1/256, B = 255/256, C = 0$ and finally

$$x(t) = A + B\cos 16t + \frac{C}{16}\sin 16t = \frac{1+255\cos 16t}{256}$$

Answer Checks

```
# Sample Problem 10
# answer check part (a)
f:=t*exp(2*t)+2*t*sin(3*t)+3*exp(-t)*cos(4*t);
with(inttrans): # load laplace package
laplace(f,t,s);
# The last two fractions simplify to 3(s+1)/((s+1)^2+16).
# answer check part (b)
F:=16/(s^2+4)+(s+1)/(s^2-2*s+10)+2/(s^2+16);
invlaplace(F,s,t);
# answer check part (c)
de:=diff(x(t),t,t)+256*x(t)=1;ic:=x(0)=1,D(x)(0)=0;
dsolve([de,ic],x(t));
# answer check part (c), partial fractions
convert((s^2+1)/(s*(s^2+256)),parfrac,s);
```

The output appears on the next page

```
> # Sample Problem 10
> # answer check part (a)
> f:=t*exp(2*t)+2*t*sin(3*t)+3*exp(-t)*cos(4*t);
```

$$f := t\,e^{2t} + 2\,t\sin(3\,t) + 3\,e^{-t}\cos(4\,t) \quad (1)$$

```
> with(inttrans): # load laplace package
> laplace(f,t,s);
```

$$\frac{1}{(s-2)^2} + \frac{12\,s}{(s^2+9)^2} + \frac{3}{2\,(s+1-4\,I)} + \frac{3}{2\,(s+1+4\,I)} \quad (2)$$

```
> # The last two fractions simplify to 3(s+1)/((s+1)^2+16).
> # answer check part (b)
> F:=16/(s^2+4)+(s+1)/(s^2-2*s+10)+2/(s^2+16);
```

$$F := \frac{16}{s^2+4} + \frac{s+1}{s^2-2\,s+10} + \frac{2}{s^2+16} \quad (3)$$

```
> invlaplace(F,s,t);
```

$$8\sin(2\,t) + \frac{1}{2}\sin(4\,t) + \frac{1}{3}\,e^{t}\,(3\cos(3\,t) + 2\sin(3\,t)) \quad (4)$$

```
> # answer check part (c)
> de:=diff(x(t),t,t)+256*x(t)=1;ic:=x(0)=1,D(x)(0)=0;
```

$$de := \frac{d^2}{dt^2}\,x(t) + 256\,x(t) = 1$$

$$ic := x(0) = 1,\ D(x)(0) = 0 \quad (5)$$

```
> dsolve([de,ic],x(t));
```

$$x(t) = \frac{1}{256} + \frac{255}{256}\cos(16\,t) \quad (6)$$

```
> # answer check part (c), partial fractions
> convert((s^2+1)/(s*(s^2+256)),parfrac,s);
```

$$\frac{255}{256}\,\frac{s}{s^2+256} + \frac{1}{256\,s} \quad (7)$$