Chapter 1. Sample Problem 1

An answer check for the differential equation and initial condition

$$\frac{dy}{dx} = -y(x) + 23, \quad y(0) = 5 \tag{1}$$

requires substitution of the candidate solution $y(x) = 23 - 18\,e^{-x}$ into the left side (LHS) and right side (RHS), then compare the expressions for equality for all symbols. The process of testing LHS = RHS applies to both the differential equation and the initial condition, making the answer check have **two** presentation panels. Complete the following:

1. Show the two panels in an answer check for initial value problem (1).

2. Relate (1) to a Newton cooling model for warming a 5 C apple to room temperature 23 C.

References. Edwards-Penney sections 1.1, 1.4, 1.5. Newton cooling in Serway and Vuille, *College Physics 9/E*, Brooks-Cole (2011), ISBN-10: 0840062060.
Newton cooling differential equation $\frac{du}{dt} = -h(u(t) - u_1)$, slide:
`http://www.math.utah.edu/~gustafso/s2019/2280/lectureslides/2250ThreeExamples.pdf`
Slide on answer checks:
`http://www.math.utah.edu/~gustafso/s2019/2280/lectureslides/FTC-Method-of-Quadrature.pdf`

Chapter 1. Sample Problem 2

A 2-ft high institutional coffee maker serves coffee from an orifice 5 inches above the base of the cylindrical tank. The tank drains according to the Torricelli model

$$\frac{dy}{dx} = -0.02\sqrt{|y(x)|}, \quad y(0) = y_0. \tag{2}$$

Symbol $y(x) \geq 0$ is the tank coffee height in feet above the orifice at time x seconds, while $y_0 \geq 0$ is the coffee height at time $x = 0$.

Establish these facts about the physical problem.

1. If $y_0 = 0$, then $y(x)$ is not determined by the model. A physical explanation is expected, based on possible past tank levels. Numerical solutions are therefore technological nonsense.

2. If $y_0 > 0$, then the solution $y(x)$ is uniquely determined and computable by numerical software. Justify using Picard's existence-uniqueness theorem.

3. Solve equation (2) using separation of variables when y_0 is 19 inches, then numerically find the drain time (about 125 seconds).

References. Edwards-Penney, Picard's theorem 1 section 1.3 and Torricelli's Law section 1.4. Tank draining `Mathematica` demo at Wolfram Research:

Carl Schaschke, *Fluid Mechanics: Worked Examples for Engineers*, The Institution of Chemical Engineers (2005), ISBN-10: 0852954980, Chapter 6.
Slide on Picard and Peano Theorems:
`http://www.math.utah.edu/~gustafso/s2019/2280/lectureslides/Picard+DirectionFields.pdf`

Chapter 1. Sample Problem 1.

Part 1

panel 1.

$$LHS = \frac{dy}{dx}$$
$$= \frac{d}{dx}(23 - 18e^{-x})$$
$$= 0 + 18e^{-x}$$

$$RHS = -y + 23$$
$$= -(23 - 18e^{-x}) + 23$$
$$= 18e^{-x}$$

$\therefore$ LHS = RHS, DE ✓

panel 2.

$$LHS = y(0)$$
$$= (23 - 18e^{-x})\big|_{x=0}$$
$$= 23 - 18e^{0}$$
$$= 5$$
$$= RHS,\ IC\ ✓$$

Part 2

Newton cooling is $u' = -h(u - u_1)$, $u(0) = u_0$. Changing $y \mapsto u$ and $x \mapsto t$ for the given DE + IC produces

$$\begin{cases} u' = -(u - 23), \\ u(0) = 5. \end{cases}$$

Then $h = +1$ is the cooling constant, $23 = u_1 =$ ambient temperature, $5 = u_0 =$ initial temperature. Then

$$\begin{cases} y(x) = u(t) = \text{apple temperature}, \\ 23 = u_1 = \text{wall thermometer temp}, \\ 5 = u_0 = \text{apple initial temp}, \\ -1 = h = \text{Newton cooling constant}, \\ x = t = \text{time}. \end{cases}$$

Chapter 1. Sample Problem 2.

Part 1

The tank could drain any time $t_0 < 0$ in the past, meaning there is a solution $y(x)$ such that $y(x) > 0$ for $x < t_0$ and $y(x) = 0$ for $x \geq t_0$. In short, ∞-many solutions. The model fails to determine a unique solution.

Part 2

If $y_0 > 0$, then $f(x,y) = -0.02\sqrt{|y|}$ and $\frac{\partial f}{\partial y} = -0.01\,|y|^{-1/2}$ on box $B = \{(x,y) : |x| \leq 10,\ \frac{1}{2}y_0 \leq y \leq 10\}$. Picard's Theorem says there is a smaller box $B_1 = \{(x,y) : |x| \leq H,\ \frac{1}{2}y_0 \leq y \leq 10\}$ on which a unique edge-to-edge solution $y(x)$ exists, $y(0) = y_0$.

Part 3

The IC is $y(0) = 19/12$ feet. Because $y > 0$, then $f(x,y) = F(x)\,G(y)$ with $F = -0.02$ and $G = \sqrt{y}$. Separation gives:

$$\frac{y'}{y^{1/2}} = -0.02$$

$$\int \frac{du}{u^{1/2}} = -0.02\int dx, \quad u = y(x)$$

method of quadrature ↓

$$\frac{u^{1/2}}{1/2} = -0.02x + C_1$$

$$y^{1/2} = -0.01x + C$$

$$y = (-0.01x + c)^2$$

square both sides,

$$\sqrt{\frac{19}{12}} = (0 + c)$$

$c = \sqrt{19/12}$ from 2 lines up ↑

$$y = \left(-0.01x + \sqrt{19/12}\right)^2$$

Drain time is x when $y = 0$, or $x = \dfrac{\sqrt{19/12}}{0.01} = 125.83$

Answer checked in Wolfram Alpha and Waterloo Maple.

Chapters 1,2. Sample Problem 3.

Suppose a cup of hot chocolate has an initial temperature of $185°F$ when freshly poured and the desired drinking temperature is $160°F$. After 50 seconds in a room at $68°F$, the temperature has cooled to $181°F$. **Newton's Law of Cooling** applies to model the temperature $u(t)$ of the chocolate by the initial value problem

$$\frac{du}{dt} = -h\left(u(t) - 68\right), \quad u(0) = 185,$$

where $h > 0$ is the cooling constant, to be determined from supplied information.

1. Find an equation for the temperature $u(t)$ at any time t.

2. Find the Newton cooling constant h.

3. Determine the time required for the chocolate to cool to $160°F$.

References. Edwards-Penney section 1.5. Serway and Vuille, *College Physics 9/E*, Brooks-Cole (2011), ISBN-10: 0840062060. An answer check might use *The Coffee Cooling Problem*, a Wolfram Demonstration Project contributed by S.M. Binder, which can be found at http://demonstrations.wolfram.com/TheCoffeeCoolingProblem/.

Credits. Created by Rebecca Terry, January 2014.

Chapters 1,2. Sample Problem 4. Logistic growth $F(x) = rx(1 - x/K)$ can be used to describe the annual natural growth of a fish stock. Symbol $x(t)$ is the stock biomass in number of fish at the start of month t. The intrinsic growth rate is symbol r. The environmental carrying capacity in stock biomass terms is symbol K.

1. Assume a fish pond has carrying capacity $K = 780500$ and that 80% of the the fish survive to maturity. We'll assume 6 months to maturity and $r = 0.8$. Write in detail the no-harvesting model $x'(t) = F(x(t))$ and find the equilibrium values.

2. Assume constant harvesting H to give the model $x'(t) = F(x(t)) - H$. Use the quadratic formula from algebra to find the equilibrium points as a function of symbol $H \geq 0$. Then verify the following results.

> If $H = 156100$, then there are two states: extinction for $x(0) < 390250$ and limiting population 390250 otherwise.
>
> If $H > 156100$, then the extinction state is the only possibility.
>
> If $H < 156100$, then there are two equilibria. The larger equilibrium population size is stable and the smaller is unstable. These numbers imply sustainable harvest for certain population sizes, but not all.

3. Assume a constant harvest rate H. Create two graphics of the population $x(t)$ over 36 months. The first uses a harvesting size H to show sustainable harvest. The second uses a different size H to show non-sustainable harvest. Handwritten plots are expected, or a computer plot, if you know how.

References. Edwards-Penney sections 2.1, 2.2. Course documents:
`http://www.math.utah.edu/~gustafso/s2019/2280/lectureslides/2250logistic.pdf` **Logistic Equation**,
`http://www.math.utah.edu/~gustafso/s2019/2280/lectureslides/2250phaseline.pdf` **Stability**,
`http://www.math.utah.edu/~gustafso/s2019/2280/lectureslides/fishFarming2014.pdf` **Fish Farming**
and a logistic investigation in Malaysia by **M.F. Laham 2012**:
`http://www.ukm.my/jsm/pdf_files/SM-PDF-41-2-2012/04%20Mohamed%20Faris.pdf`

Chapters 1,2. Sample solutions plus maple code

Sample Problem 3

Answers: (1) $u = 68 + 117e^{-ht}$

(2) $h = (-1/50)\ln\left(\frac{113}{117}\right) \cong 0.0006957$

(3) $t \cong 345.52$ seconds, about 6 min.

Details (1). Because u = degrees F and t = seconds, then the model is
$$\begin{cases} u' = -h(u-68), \\ u(0) = 185, \quad u(50) = 181. \end{cases}$$

The DE is solved by superposition $u = u_h + u_p$. The equilibrium solution is $u_p = 68$. Then $u_h = \frac{c}{\text{integ. factor}} = ce^{-ht}$, using standard linear form $u' + hu = 68h$. Condition $u(0) = 185$ is used on $u = u_h + u_p = ce^{-ht} + 68$ to evaluate $185 = ce^0 + 68$, then $c = 185 - 68 = 117$.

Details (2). Start with answer (1) and use $u(50) = 181$. Then

$$181 = 68 + 117e^{-ht} \quad \text{when } t = 50$$
$$113 = 117e^{-ht}$$
$$e^{-ht} = \frac{113}{117} \Rightarrow -ht = \ln\left(\frac{113}{117}\right)$$
$$\Rightarrow h = \frac{-1}{50}\ln\left(\frac{113}{117}\right)$$

Details (3). As in details (2),

$$160 = 68 + 117e^{-ht}$$
$$e^{-ht} = \frac{160-68}{117}$$
$$-ht = \ln\left(\frac{92}{117}\right) \quad \text{take log across eqtn.}$$
$$t = \left(\frac{-1}{h}\right)\ln\left(\frac{92}{117}\right)$$
$$t = 50\,\frac{\ln(92/117)}{\ln(113/117)} \cong 345.52 \text{ seconds}$$

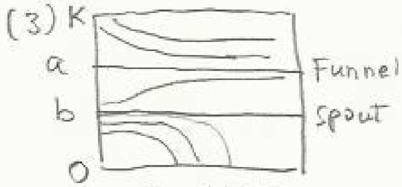
Chapters 1,2. Sample solutions plus maple code

Sample Problem 4

Answers (1) $x' = 0.8x\left(1 - \frac{x}{780500}\right)$, $x = 0, 780500$

(2) $x^2 - Kx + \frac{HK}{r} = 0$, roots $= \frac{K}{2} \pm \frac{1}{2}\sqrt{K^2 - \frac{4HK}{r}}$

(3)

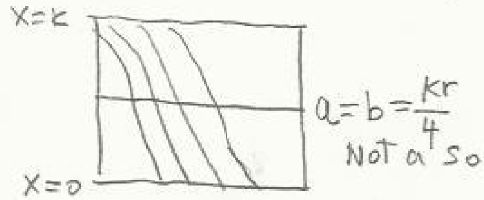


$a = \frac{K}{2} + \frac{1}{2}\sqrt{D}$

$b = \frac{K}{2} - \frac{1}{2}\sqrt{D}$

$D = K^2 - \frac{4HK}{r}$

Choose $H = \frac{Kr}{4} - 5000$



Choose $H = \frac{Kr}{4} + 5000$
All sols decrease to zero (and beyond), meaning extinction

Details (1). $x' = rx\left(1 - \frac{x}{K}\right)$, Substitute $r = 0.8$, $K = 780500$

Details (2). $x' = rx\left(1 - \frac{x}{K}\right) - H = rx - \frac{r}{K}x^2 - H$

$$x' = -\frac{r}{K}\left(x^2 - Kx + \frac{HK}{r}\right)$$

Apply Quadratic formula to $x^2 - Kx + \frac{HK}{r} = 0$ to find roots, reported in both (2), (3) above.

Details (3). A double real root is when the discriminant $D = K^2 - \frac{4HK}{r} = 0$, requiring $H = \frac{Kr}{4}$. For $H < \frac{Kr}{4}$, there are 2 real roots a, b as given in the answer. For $H > \frac{Kr}{4}$ there are no real roots, therefore $x' < 0$ and x decreases to zero (extinction) and beyond.

The larger root $a = \frac{K}{2} + \frac{1}{2}\sqrt{D}$ is a stable funnel.
The smaller root $b = \frac{K}{2} - \frac{1}{2}\sqrt{D}$ is an unstable spout.
When $D = 0$, then $a = b = K/2$ is a node. When a, b are real, extinction for $x < b$, carrying capacity $= a$, sustainable harvest for $x > b$.

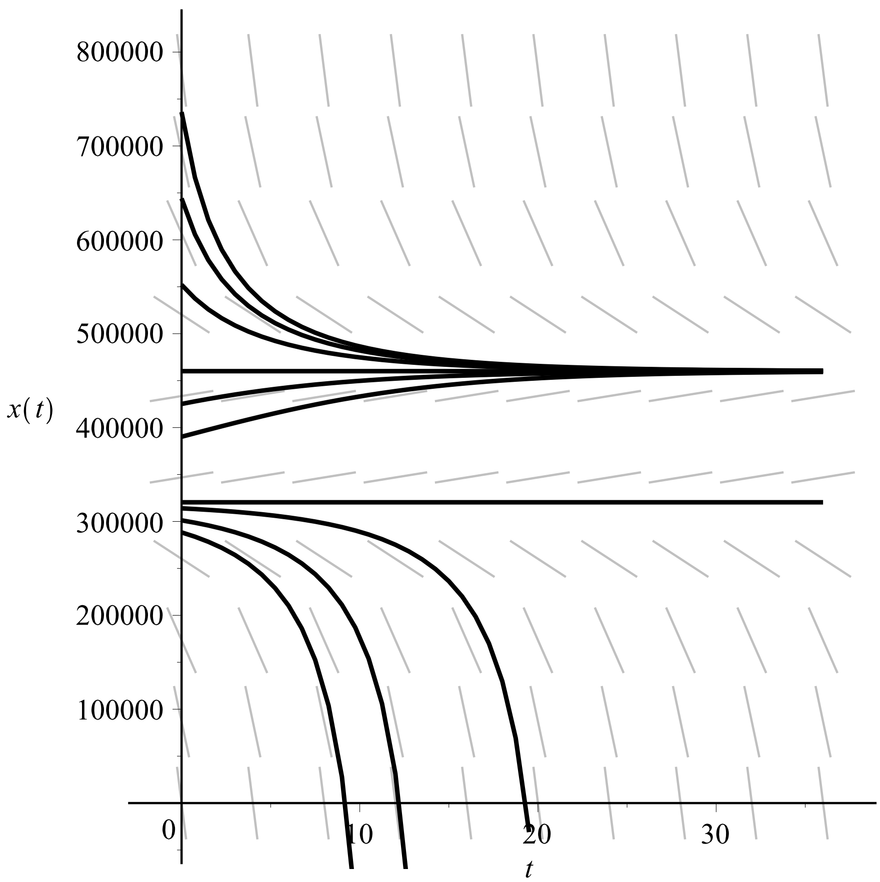
```
> F:=x->r*x*(1-x/K):G:=x->r*x*(1-x/K)-H:
> solve(G(x)=0,x);
```

$$\frac{1}{2}\,\frac{K\,r+\sqrt{K^2\,r^2-4\,H\,K\,r}}{r},\ \frac{1}{2}\,\frac{K\,r-\sqrt{K^2\,r^2-4\,H\,K\,r}}{r} \quad (1)$$

```
> de:=diff(x(t),t)=G(x(t)):r:=0.8:K:=780500:H0:=K*r/4:H:=H0-5000:
  a:=K/2+(1/2)*sqrt(K^2-4*H*K/r);b:=K/2-(1/2)*sqrt(K^2-4*H*K/r);
  ic:=[[0,0.9*b],[0,0.94*b],[0,0.98*b],[0,b],[0,(a+b)/2],[0,(3*a+b)
  /4],
  [0,a],[0,1.2*a],[0,1.4*a],[0,1.6*a]]:
  opts:=dirfield=[10,10],arrows=line,color=gray,linecolor=black,
  thickness=2:
> DEtools[DEplot](de,x(t),t=0..36,x=0..K,ic,opts);
```

$$a := 4.600935752\ 10^5$$

$$b := 3.204064248\ 10^5$$



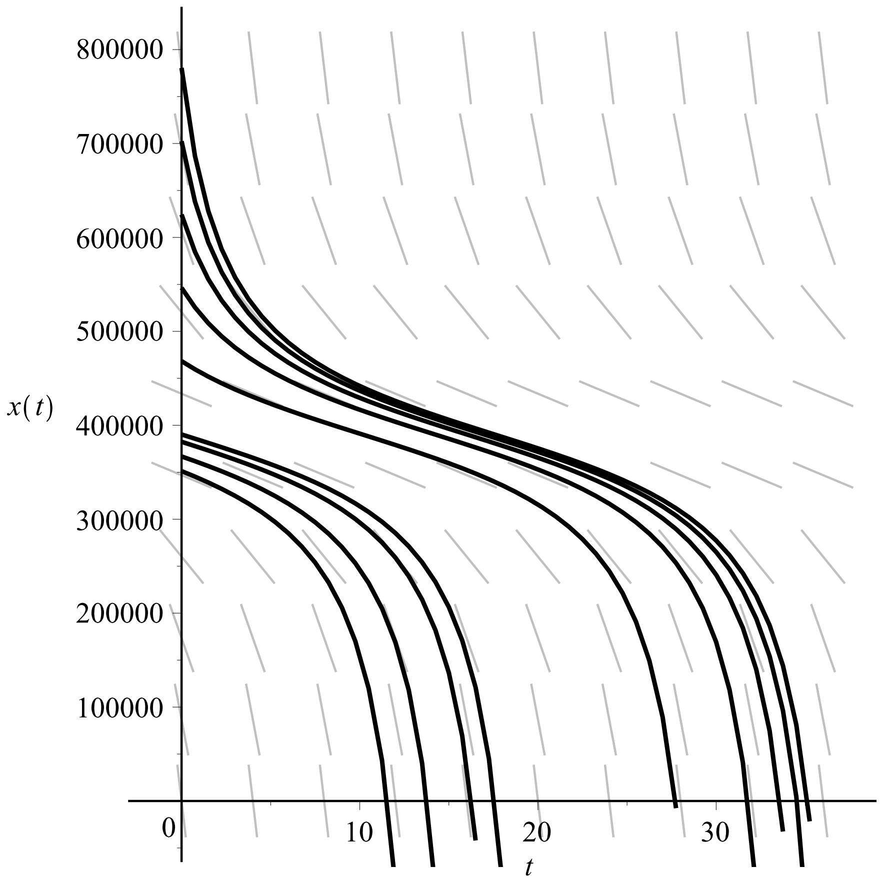

```
> de:=diff(x(t),t)=G(x(t)):
  r:=0.8:K:=780500:H0:=K*r/4:
  H:=H0+6000;a:=K/2;b:=K/2;
  ic:=[[0,0.9*b],[0,0.94*b],[0,0.98*b],[0,b],[0,a],[0,1.2*a],[0,
  1.4*a],
  [0,1.6*a],[0,1.8*a],[0,2*a]]:
```

```
opts:=dirfield=[10,10],arrows=line,color=gray,linecolor=black,
thickness=2:
DEtools[DEplot](de,x(t),t=0..36,x=0..K,ic,opts);
```

$$H := 1.621000000\ 10^5$$

$$a := 390250$$

$$b := 390250$$



All solutions decrease.

Chapter 2. Sample Problem 5. A graphic called a **phase diagram** displays the behavior of all solutions of $u' = F(u)$. A **phase line diagram** is an abbreviation for a direction field on the vertical axis (u-axis). It consists of equilibrium points and signs of $F(u)$ between equilibria. A phase diagram can be created solely from a phase line diagram, using just three drawing rules:

1. Solutions don't cross.
2. Equilibrium solutions are horizontal lines $u = c$. All other solutions are increasing or decreasing.
3. A solution curve can be moved rigidly left or right to create another solution curve.

Use these tools on the equation $u' = u(u^2 - 4)$ to make a phase line diagram, and then make a phase diagram with at least 8 threaded solutions. Label the equilibria as stable, unstable, funnel, spout, node.

References. Edwards-Penney section 2.2. Course document on **Stability**:
`http://www.math.utah.edu/~gustafso/s2019/2280/lectureslides/2250phaseline.pdf`

Chapter 2. Sample Problem 6. An autonomous differential equation $\frac{dy}{dx} = F(x)$ with initial condition $y(0) = y_0$ has a formal solution

$$y(x) = y_0 + \int_0^x F(u)du.$$

The integral may not be solvable by calculus methods. In this case, the integral is evaluated numerically to compute $y(x)$ or to plot a graphic. There are three basic numerical methods that apply, the rectangular rule (RECT), the trapezoidal rule (TRAP)and Simpson's rule (SIMP).

Apply the three methods for $F(x) = \sin(x^2)$ and $y_0 = 0$ using step size $h = 0.2$ from $x = 0$ to $x = 1$. Then fill in the blanks in the following table. Use technology if it saves time. Lastly, compare the four data sets in a plot, using technology.

x – values	0.0	0.2	0.4	0.6	0.8	1.0
y – to 10 digits	0.0	0.0026663619	0.02129435557	0.07133622797	0.1657380596	0.3102683017
y – RECT values	0.0	0.0	0.007997866838			0.2297554431
y – TRAP values	0.0		0.02392968750	0.07508893150		0.3139025416
y – SIMP values	0.0	0.002666288917	0.02129368017		0.1657330636	

References. Edwards-Penney Sections 2.4, 2.5, 2.6, because methods Euler, Modified Euler and RK4 reduce to RECT, TRAP, SIMP methods when $f(x, y)$ is independent of y, i.e., an equation $y' = F(x)$. Course document on numerical solution of $y' = F(x)$, RECT, TRAP, SIMP methods:
`http://www.math.utah.edu/~gustafso/s2019/2280/lectureslides/solve-quadrature-numerically.pdf`

Wolfram Alpha at `http://www.wolframalpha.com/` can do the RECT rule and graphics with input string

```
integrate sin(x^2) using left endpoint method with interval width 0.2 from
x=0 to x=1
```

Chapter 2. Sample Problem 5.

$u' = u(u^2-4)$

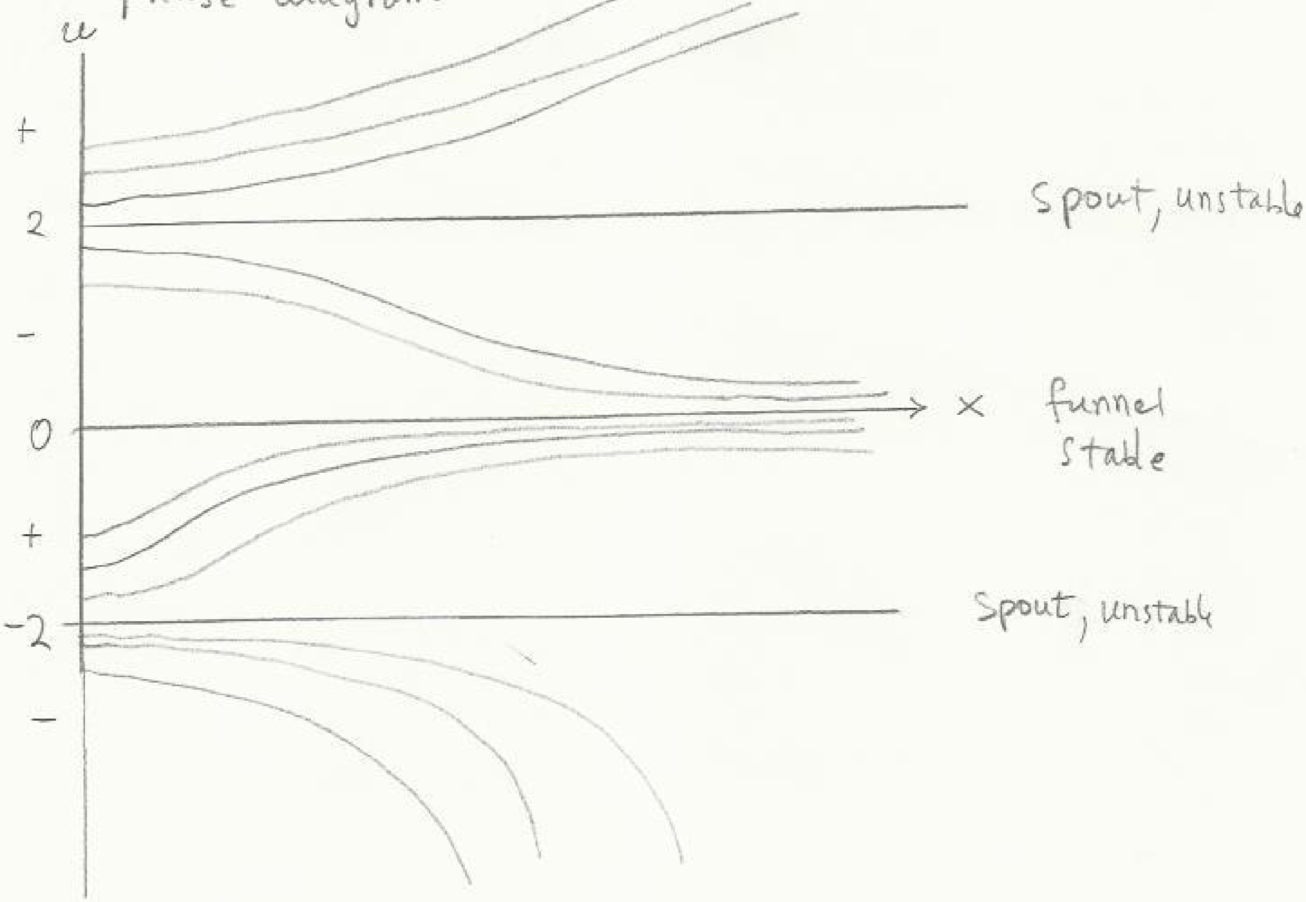
phase line diagram, $F(u) = u(u^2-4) = u(u-2)(u+2)$

Equilibria are roots of $F(u)=0$. Then $u = 0, 2, -2$

$F(-3) =$ Neg, $F(1) =$ Neg
$F(-1) =$ pos, $F(3) =$ pos
Example: Sign $-$ on $0 < u < 2$ because $F(1) = -3$ (F is one-signed for $0 < u < 2$)

The phase line diagram was constructed from F. We move it to the vertical u-axis, below. Sign + means increasing, − means decreasing.

phase diagram



Theorem Between 2 adjacent equilibrium points, $F(u)$ is one-signed. Alternate: if $F(a)=F(b)=0$ and F has no roots in $a<u<b$, Then F is one-signed on $a<u<b$.

Chapter 2. Sample Problem 6.

$$\begin{cases} y' = \sin(x^2) \\ y(0) = 0 \end{cases}$$

Solve numerically using Rect, Trap, Simp

RECT The approx formula is $y(x_0+h) = y(x_0) + h\,F(x_0)$, because $\int_a^b F\,dx \cong F(a)(b-a)$ for $b-a$ small. Then the supplied table implies

$$y(0.6) = y(0.4) + \int_{0.4}^{0.6} F\,dx \cong 0.007997866838 + (0.2)\,F(0.4)$$
$$= 0.007997866838 + (0.2)\sin(0.4^2)$$
$$= 0.03986150816$$
$$y(0.8) = 0.03986150816 + 0.2\sin(0.6^2)$$
$$= 0.1103163548$$

TRAP The approx formula $\int_a^b F\,dx \cong (b-a)\frac{F(a)+F(b)}{2}$ implies that $y(x_0+h) = y(x_0) + \frac{h}{2}\left(F(x_0) + F(x_0+h)\right)$. The Table implies

$$y(0.2) = y(0.0) + \frac{0.2}{2}\left(F(0) + F(0.2)\right)$$
$$= 0 + 0.1\left(\sin(0) + \sin(0.2^2)\right)$$
$$= 0.003998933419$$
$$y(0.8) = y(0.6) + 0.1\left(F(0.6) + F(0.8)\right)$$
$$= 0.07508893150 + 0.1\left(\sin(0.6^2) + \sin(0.8^2)\right)$$
$$= 0.1700358989$$

RK4 Use $\int_a^b F\,dx \cong \frac{(b-a)}{6}\left(F(a) + 4F\left(\frac{a+b}{2}\right) + F(b)\right)$ to get $y(x_0+h) = y(x_0) + \frac{h}{6}\left(F(x_0) + 4F(x_0+h/2) + F(x_0+h)\right)$. The Table implies

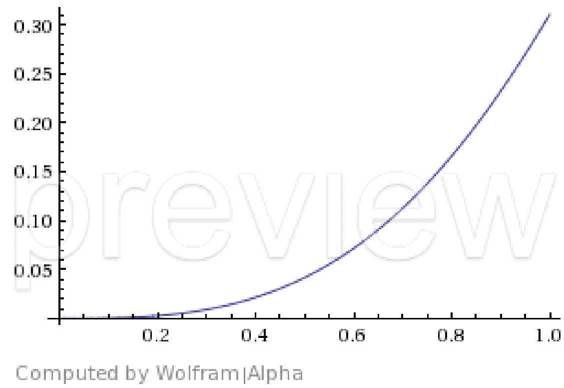
$$y(0.6) = y(0.4) + \frac{0.2}{6}\left(F(0.4) + 4F(0.5) + F(0.6)\right)$$
$$= 0.02129368017 + \frac{0.1}{3}\left(\sin(0.4^2) + 4\sin(0.5^2) + \sin(0.6^2)\right)$$
$$= 0.07133395608$$
$$y(1.0) = y(0.8) + \frac{0.1}{3}\left(\sin(0.8^2) + 4\sin(0.9^2) + \sin(1.0^2)\right)$$
$$= 0.3102602343$$

$\boxed{\text{Solution key}}$

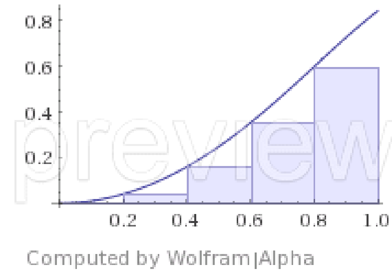
$\boxed{0.03986150816}$ $\boxed{0.1103163548}$

$\boxed{0.003998933419}$ $\boxed{0.1700358989}$

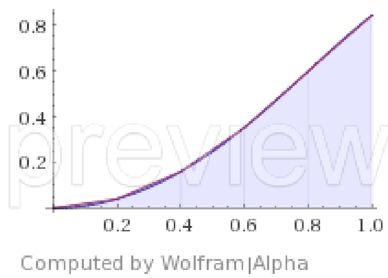
$\boxed{0.07133395608}$ $\boxed{0.3102602343}$



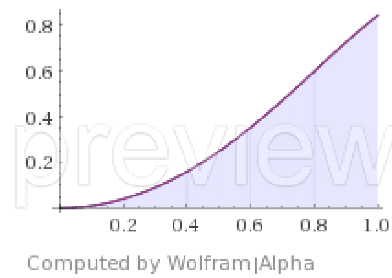
10-digit integral of sin(x^2)



RECT rule plot of y'=sin(x^2), h=0.2



TRAP rule plot of y'=sin(x^2), h=0.2



SIMP rule plot of y'=sin(x^2), h=0.2

Chapters 1,2: Sample Problem 7

The velocity of a crossbow arrow fired upward from the ground is given at different times in the following table.

Time t in seconds	Velocity $v(t)$ in ft/sec	Location
0.000	50	Ground
1.413	0	Maximum
2.980	-45	Near Ground Impact

(a) The velocity $v(t)$ can be approximated by a quadratic polynomial

$$z(t) = at^2 + bt + c$$

which reproduces the table data. Find three equations for the coefficients a, b, c. Then solve for them to obtain $a \approx 2.238$, $b \approx -38.55$, $c = 50$.

(b) Assume a linear drag model $v' = -32 - \rho v$. Substitute the polynomial answer $v = z(t)$ of (a) into this differential equation, then substitute $t = 0$ and solve for $\rho \approx 0.131$.

(c) Solve the model $w' = -32 - \rho w$, $w(0) = 50$ to get $w(t) = -\frac{32}{\rho} + \left(50 + \frac{32}{\rho}\right) e^{-\rho t}$. Substitute $\rho = 0.131$. Then $w(t) = -244.2748092 + 294.2748092\, e^{-0.131\, t}$ is an exponential model for linear drag which might reproduce the crossbow data.

(d) Compare $w(t)$ and $z(t)$ in a plot. Comment on the plot and what it means. Bear in mind that $w(t)$ is an exponential model while $z(t)$ is a quadratic model. Neither of them are the true velocty $v(t)$ which produced the crossbow data.

References. Edwards-Penney sections 2.3, 3.1, 3.2. Course document on **Linear algebraic equations**:

http://www.math.utah.edu/~gustafso/s2019/2280/lectureslides/linearequDRAFT.pdf

Course document on **Newton kinematics**:

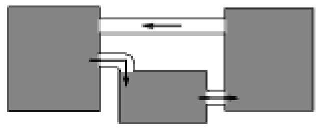
http://www.math.utah.edu/~gustafso/s2019/2280/lectureslides/newtonModelsDE2008.pdf

Chapters 1,2. Sample Problem 8

Consider the system of differential equations

$$\begin{aligned} x_1' &= -\tfrac{1}{6}x_1 & & + \tfrac{1}{6}x_3, \\ x_2' &= \tfrac{1}{6}x_1 - \tfrac{1}{3}x_2, & & \\ x_3' &= & \tfrac{1}{3}x_2 & - \tfrac{1}{6}x_3, \end{aligned}$$

for the amounts x_1, x_2, x_3 of salt in recirculating brine tanks, as in the figure:



Recirculating Brine Tanks A, B, C

The volumes are $60, 30, 60$ for A, B, C, respectively.

The steady-state salt amounts in the three tanks are found by formally setting $x_1' = x_2' = x_3' = 0$ and then solving for the symbols x_1, x_2, x_3. Solve the corresponding linear system of algebraic equations to obtain the answer $x_1 = x_3 = 2c$, $x_2 = c$, which means the total amount of salt is uniformly distributed in the tanks in ratio $2 : 1 : 2$.

References. Edwards-Penney sections 3.1, 3.2, 7.3 Figure 5. Course document on **Linear algebraic equations**:

`http://www.math.utah.edu/~gustafso/s2019/2280/lectureslides/linearequDRAFT.pdf`

Course document on **Systems and Brine Tanks**:

`http://www.math.utah.edu/~gustafso/s2019/2280/lectureslides/systemsBrineTank.pdf`

Chapters 1,2. Sample Problem 7.

(a) Let $t_1 = 1.413$, $t_2 = 2.98$. Use $at^2+bt+c = v(t)$ for $t=0, t_1, t_2$ to obtain the system

$$\begin{cases} a\cdot 0^2 + b\cdot 0 + c = 50 \\ a\cdot t_1^2 + b\cdot t_1 + c = 0 \\ a\cdot t_2^2 + b t_2 + C = -45 \end{cases}$$

Then $\boxed{c=50}$. The 3X3 system reduces to a 2X2 system

$$\begin{cases} a t_1^2 + b t_1 = -50 \\ a t_2^2 + b t_2 = -95 \end{cases}$$

$$\begin{cases} a + b/t_1 = -50/t_1^2 \\ a t_2^2 + b t_2 = -95 \end{cases} \qquad \text{mult}(1, 1/t_1^2)$$

$$\begin{cases} a + b/t_1 = -50 t_1^2 \\ 0 + b\cdot t_3 = -95 + \dfrac{50 t_2^2}{t_1^2} \end{cases} \qquad \text{combo}(1,2,-t_2^2) \text{ where } t_3 = t_2 - \frac{t_2^2}{t_1}$$

Then $\boxed{b} = \dfrac{1}{t_2-t_1}\left(\dfrac{-50 t_2}{t_1} + 95\dfrac{t_1}{t_2}\right) = \boxed{-38.54760463}$

$\boxed{a} = \dfrac{1}{t_2-t_1}\left(\dfrac{50}{t_1} - \dfrac{95}{t_2}\right) = \boxed{2.23772148}$

The example gives evidence for why technology is used on systems of equations. For 2-digit accuracy, it is less hand work, and quite fast with a calculator.

(b) Substitution gives $2at + b = -32 - \rho(at^2+bt+c)$, Then $t=0$ implies $b = -32 - \rho c$. Calculator gives $\rho = (-32-b)/c = 0.1309520926 \cong 0.131$

(c) $w = \text{equil sol} + \dfrac{c_1}{\text{integ factor}} = \dfrac{-32}{\rho} + \dfrac{c_1}{e^{\rho t}}$. Then $w(0)=50$ implies $c_1 = 50 + 32/\rho$.

(d) A good plot is $|v(t) - w(t)|$ on $0 \le t \le 3$. It shows max error of 0.3. typo: v(t) above should be z(t)

Chapters 1,2. Sample Problem 8.

System $\begin{cases} -\frac{1}{6}x_1 + \frac{1}{6}x_3 = 0 \\ \frac{1}{6}x_1 - \frac{1}{3}x_2 = 0 \\ \frac{1}{3}x_2 - \frac{1}{6}x_3 = 0 \end{cases}$ has augmented matrix equal to

$$\left(\begin{array}{ccc|c} -\frac{1}{6} & 0 & \frac{1}{6} & 0 \\ \frac{1}{6} & -\frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & -\frac{1}{6} & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} -1 & 0 & 1 & 0 \\ 1 & -2 & 0 & 0 \\ 0 & 2 & -1 & 0 \end{array}\right) \quad \text{mult}(1,6), \text{mult}(2,6), \text{mult}(3,6)$$

$$\left(\begin{array}{ccc|c} -1 & 0 & 1 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 2 & -1 & 0 \end{array}\right) \quad \text{combo}(1,2,1)$$

$$\left(\begin{array}{ccc|c} -1 & 0 & 1 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad \text{combo}(2,3,1)$$

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad \text{mult}(1,-1), \text{mult}(2,-1)$$

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad \text{mult}(2, \tfrac{1}{2}) \qquad \text{Last frame}$$

$$\begin{cases} x_1 - x_3 = 0 \\ x_2 - \frac{1}{2}x_3 = 0 \\ 0 = 0 \end{cases}$$

reduced echelon system
last frame algorithm applies

Answer

$$\begin{cases} x_1 = t_1 \\ x_2 = \frac{1}{2}t_1 \\ x_3 = t_1 \end{cases} \qquad -\infty < t_1 < \infty$$

Let $t_1 = 2c$
Then

$$\begin{cases} x_1 = 2c \\ x_2 = c \\ x_3 = 2c \end{cases}$$