Semester Project Part 3: Due by May 7, 2019

Instructions. Please prepare your own report on 8x11 paper, handwritten. Work alone or in groups. Help is available by telephone, office visit or email. All problems in Part 3 of the semester project reference Chapters 3, 4, 5, 7 in Edwards-Penney. Use the sample problems with solutions to fully understand the required details:
`http://www.math.utah.edu/~gustafso/s2019/2280/quiz/sampleQuizzes/project-part3.pdf`
Visit the Math Center in building LCB for assistance on problem statements, references and technical details.

Problem 1. Piecewise Continuous Inputs

Consider a passenger SUV on a short trip from Salt Lake City to Evanston, on the Wyoming border. The route is I-80 E, 75 miles through Utah. Google maps estimates 1 hour and 11 minutes driving time. The table below shows the distances, time, road segment and average speed with total trip time 1 hour and 38 minutes. Cities enroute reduce the freeway speed by 10 mph, the trip time effect not shown in the table.

Miles	Minutes	Speed mph	Road Segment	Posted limit mph
18.1	20	54.3	Parley's Walmart to Kimball	65
11.3	12	56.5	Kimball to Wanship	65 – 55
9.1	11	49.6	Wanship to Coalville	70
5.7	6	57	Coalville to Echo Dam	70
16.5	16	61.9	Echo Dam to 75 mph sign	70
39	33	70.9	75 mph sign to Evanston	75

The **velocity function** for the SUV is approximated by

$$V_{\text{pc}}(t) = \begin{cases} \text{Speed mph} & \text{Time interval minutes} & \text{Road segment} \\ \hline \\ 54.3 & 0 < t < 20 & \text{Parley's Walmart to Kimball} \\ 56.5 & 20 < t < 32 & \text{Kimball to Wanship} \\ 49.6 & 32 < t < 43 & \text{Wanship to Coalville} \\ 57.0 & 43 < t < 49 & \text{Coalville to Echo Dam} \\ 61.9 & 49 < t < 65 & \text{Echo Dam to 75 mph sign} \\ 70.1 & 65 < t < 98 & \text{75 mph sign to Evanston} \end{cases}$$

The velocity function $V_{\text{pc}}(t)$ is piecewise continuous, because it has the general form

$$f(t) = \begin{cases} f_1(t) & t_1 < t < t_2 \\ f_2(t) & t_2 < t < t_3 \\ \vdots & \vdots \\ f_n(t) & t_n < t < t_{n+1} \end{cases}$$

where functions $f_1, f_2, \ldots, f_n$ are **continuous on the whole real line** $-\infty < t < \infty$. We don't define $f(t)$ at division points, because of many possible ways to make the definition. As long as these values are not used, then it will make no difference. Both right and left hand limits exist at a division point. For Laplace theory, we like the definition $f(0) = \lim_{h \to 0+} f(h)$, which allows the parts rule $\mathcal{L}(f'(t)) = s\mathcal{L}(f(t)) - f(0)$.

The Problem. The SUV travels from $t = 0$ to $t = \frac{98}{60} = 1.6$ hours. The odometer trip meter reading $x(t)$ is in miles (assume $x(0) = 0$). The function $V_{\text{pc}}(t)$ is an approximation to the speedometer reading. Laplace's method can solve the approximation model

$$\frac{dx}{dt} = V_{\text{pc}}(60t), \quad x(0) = 0, \quad x \text{ in miles, } t \text{ in hours,}$$

obtaining $x(t) = \int_0^t V_{\text{pc}}(60w)dw$, the same result as the method of quadrature. Show the details. Then display the piecewise linear continuous trip meter reading $x(t)$.

Laplace's method solves differential equations. It is the premier method for solving equations containing switches or impulses.

Unit Step Define $u(t-a) = \begin{cases} 1 & t \geq a, \\ 0 & t < a. \end{cases}$. It is a **switch**, turned on at $t = a$.

Ramp Define **ramp**$(t-a) = (t-a)u(t-a) = \begin{cases} t-a & t \geq a, \\ 0 & t < a. \end{cases}$, whose graph shape is a continuous **ramp** at 45-degree incline starting at $t = a$.

Unit Pulse Define **pulse**$(t,a,b) = \begin{cases} 1 & a \leq t < b, \\ 0 & \text{otherwise} \end{cases} = u(t-a) - u(t-b)$. The switch is **ON** at time $t = a$ and then **OFF** at time $t = b$.

Impulse of a Force

Define the **impulse** of an applied force $F(t)$ on time interval $a \leq t \leq b$ by the equation

$$\text{Impulse of } F = \int_a^b F(t)dt = \left(\frac{\int_a^b F(t)dt}{b-a} \right) (b-a) = \text{Average Force} \times \text{Duration Time}.$$

Dirac Unit Impulse

A Dirac impulse acts like a hammer hit, a brief injection of energy into a system. It is a special idealization of a real hammer hit, in which only the **impulse** of the force is deemed important, and not its magnitude nor duration.

Define the **Dirac Unit Impulse** by the equation $\delta(t-a) = \dfrac{du}{dt}(t-a)$, where $u(t-a)$ is the unit step. Symbol δ makes sense only under an integral sign, and the integral in question must be a generalized Riemann-Steiltjes integral (definition pending), with new evaluation rules. Symbol δ is an abbreviation like **etc** or **e.g.**, because it abbreviates a paragraph of descriptive text.

- Symbol $M\delta(t-a)$ represents an ideal impulse of magnitude M at time $t = a$. Value M is the change in momentum, but $M\delta(t-a)$ contains no detail about the applied force or the duraction. A common force approximation for a hammer hit of very small duration $2h$ and impulse M is Dirac's approximation
$$F_h(t) = \frac{M}{2h}\,\mathbf{pulse}(t, a-h, a+h).$$
- The fundamental equation is $\int_{-\infty}^{\infty} F(x)\delta(x-a)dx = F(a)$. Symbol $\delta(t-a)$ is not manipulated as an ordinary function, but regarded as $du(t-a)/dt$ in a Riemann-Stieltjes integral.

THEOREM (Second Shifting Theorem). Let $f(t)$ and $g(t)$ be piecewise continuous and of exponential order. Then for $a \geq 0$,

Forward table	**Backward table**
$\mathcal{L}\left(f(t-a)u(t-a)\right) = e^{-as}\,\mathcal{L}(f(t))$	$e^{-as}\,\mathcal{L}(f(t)) = \mathcal{L}\left(f(t-a)u(t-a)\right)$
$\mathcal{L}(g(t)u(t-a)) = e^{-as}\,\mathcal{L}\left(g(t)\big\vert_{t:=t+a}\right)$	$e^{-as}\,\mathcal{L}(f(t)) = \mathcal{L}\left(f(t)u(t)\big\vert_{t:=t-a}\right)$.

Problem 2. Laplace's method for piecewise functions and impulses.

(a) Forward table. Unit step, ramp and pulse. Evaluate the expressions as functions of s.

$$(1)\ \mathcal{L}((t-1)u(t-1)) \quad (2)\ \mathcal{L}(e^t\,\textbf{ramp}(t-2)), \quad (3)\ \mathcal{L}(5\,\textbf{pulse}(t,2,4)).$$

(b) Backward table. Find $f(t)$ in the following special cases.

$$(1)\ \mathcal{L}(f) = \frac{e^{-2s}}{s} \quad (2)\ \mathcal{L}(f) = \frac{e^{-s}}{(s+1)^2} \quad (3)\ \mathcal{L}(f) = e^{-s}\frac{3}{s} - e^{-2s}\frac{3}{s}.$$

Problem 3. Evaluate the expressions as functions of s.

(c) Forward table. Dirac Impulse and the Second Shifting theorem.

$$(1)\ \mathcal{L}(2\delta(t-5)), \quad (2)\ \mathcal{L}(2\delta(t-1)+5\delta(t-3)), \quad (3)\ \mathcal{L}(e^t\delta(t-2)).$$

The sum of Dirac impulses in (2) is called an **impulse train**. The numbers 2 and 5 represent the applied **impulse** at times 1 and 3, respectively.

Reference: The Riemann-Stieltjes Integral

Definition

The Riemann-Stieltjes integral of a real-valued function f of a real variable with respect to a real monotone non-decreasing function g is denoted by

$$\int_a^b f(x)\,dg(x)$$

and defined to be the limit, as the mesh of the partition

$$P = \{a = x_0 < x_1 < \cdots < x_n = b\}$$

of the interval $[a, b]$ approaches zero, of the approximating RiemannStieltjes sum

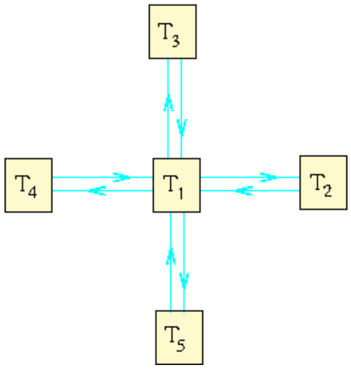
$$S(P, f, g) = \sum_{i=0}^{n-1} f(c_i)(g(x_{i+1}) - g(x_i))$$

where c_i is in the i-th subinterval $[x_i, x_{i+1}]$. The two functions f and g are respectively called the **integrand** and the **integrator**.

The **limit** is a number A, the value of the Riemann-Stieltjes integral. The meaning of the limit: Given $\varepsilon > 0$, then there exists $\delta > 0$ such that for every partition $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ with $\textbf{mesh}(P) = \max_{0\le i<n}\ (x_{i+1} - x_i) < \delta$, and for every choice of points c_i in $[x_i, x_{i+1}]$,

$$|S(P, f, g) - A| < \varepsilon.$$

Problem 4. Brine Tanks.



Flow through each pipe is f gallons per unit time.

Each tank has constant volume V.

Symbols $x_1(t)$ to $x_5(t)$ are the salt amounts in tanks T_1 to T_5, respectively.

The differential equations are obtained by the classical balance law, which says that the rate of change in salt amount is the rate in minus the rate out. Individual rates in/out are of the form (flow rate)(salt concentration), where flow rate f has units volume per unit time and $x_i(t)/V$ is the concentration = amount/volume.

$$\begin{aligned}
x_1'(t) &= \frac{f}{V}\left(x_2(t)+x_3(t)+x_4(t)+x_5(t)-4x_1(t)\right)\\
x_2'(t) &= \frac{f}{V}\left(x_1(t)-x_2(t)\right),\\
x_3'(t) &= \frac{f}{V}\left(x_1(t)-x_3(t)\right),\\
x_4'(t) &= \frac{f}{V}\left(x_1(t)-x_4(t)\right),\\
x_5'(t) &= \frac{f}{V}\left(x_1(t)-x_5(t)\right).
\end{aligned}$$

Solve Parts (a) to (e) below.

Part (a). Change variables $t = Vr/f$ to obtain the new system

$$\begin{aligned}
\frac{dx_1}{dr} &= x_2+x_3+x_4+x_5-4x_1\\
\frac{dx_2}{dr} &= x_1-x_2,\\
\frac{dx_3}{dr} &= x_1-x_3,\\
\frac{dx_4}{dr} &= x_1-x_4,\\
\frac{dx_5}{dr} &= x_1-x_5.
\end{aligned}$$

Part (b). Formulate the equations in Part (a) in the system form $\frac{d}{dr}\vec{u} = A\vec{u}$.

Answer:

$$A = \begin{pmatrix} -4 & 1 & 1 & 1 & 1\\ 1 & -1 & 0 & 0 & 0\\ 1 & 0 & -1 & 0 & 0\\ 1 & 0 & 0 & -1 & 0\\ 1 & 0 & 0 & 0 & -1 \end{pmatrix}, \quad \vec{u} = \begin{pmatrix} x_1\\ x_2\\ x_3\\ x_4\\ x_5\end{pmatrix}$$

Part (c). Show details for finding the eigenvalues of A: $\lambda = 0, -1, -1, -1, -5$.

Part (d). Find the eigenvectors of A.

Part (e). Solve the differential equation $\frac{d\vec{u}}{dr} = A\vec{u}$ by the eigenanalysis method.

Three Methods for Solving $\frac{d}{dt}\vec{u}(t) = A\vec{u}(t)$

- **Eigenanalysis Method**. The eigenpairs of matrix A are required. The matrix A must be diagonalizable, meaning there are n eigenpairs $(\lambda_1, \vec{v}_1), (\lambda_2, \vec{v}_2), \ldots, (\lambda_n, \vec{v}_n)$. The main theorem says that the general solution of $\vec{u}' = A\vec{u}$ is

$$\vec{u}(t) = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2 + \cdots + c_n e^{\lambda_n t}\vec{v}_n.$$

- **Laplace's Method**. Solve the scalar equations by the Laplace transform method. The **resolvent method** automates this process: $\vec{u}(t) = \mathcal{L}^{-1}\left((sI - A)^{-1}\right)\vec{u}(0)$.

- **Cayley-Hamilton-Ziebur Method**. The solution $\vec{u}(t)$ is a vector linear combination of the Euler solution atoms $f_1, \ldots, f_n$ found from the roots of the characteristic equation $|A - \lambda I| = 0$. The vectors $\vec{d}_1, \ldots, \vec{d}_n$ in the linear combination

$$\vec{u}(t) = f_1(t)\vec{d}_1 + f_2(t)\vec{u}_2 + \cdots + f_n(t)\vec{d}_n$$

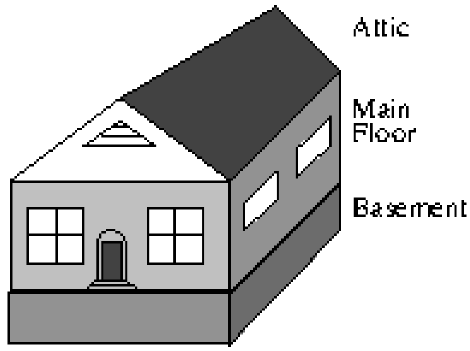
are determined by the explicit formula

$$< \vec{d}_1 \,|\, \vec{d}_2 \,|\, \cdots \,|\, \vec{d}_n > = < \vec{u}_0 \,|\, A\vec{u}_0 \,|\, \cdots \,|\, A^{n-1}\vec{u}_0 > \left(W(0)^T\right)^{-1},$$

where $W(t)$ is the Wronskian matrix of atoms $f_1, \ldots, f_n$ and $\vec{u}_0$ is the initial data.

Problem 5. Home Heating

Consider a typical home with attic, basement and insulated main floor.



Heating Assumptions and Variables

- It is usual to surround the main living area with insulation, but the attic area has walls and ceiling without insulation.
- The walls and floor in the basement are insulated by earth.
- The basement ceiling is insulated by air space in the joists, a layer of flooring on the main floor and a layer of drywall in the basement.

The changing temperatures in the three levels is modeled by Newton's cooling law and the variables

$$\begin{aligned} z(t) &= \text{Temperature in the attic,} \\ y(t) &= \text{Temperature in the main living area,} \\ x(t) &= \text{Temperature in the basement,} \\ t &= \text{Time in hours.} \end{aligned}$$

A typical mathematical model is the set of equations

$$\begin{aligned} x' &= \frac{3}{4}(45 - x) + \frac{1}{4}(y - x), \\ y' &= \frac{1}{4}(x - y) + \frac{1}{4}(40 - y) + \frac{1}{2}(z - y) + 20, \\ z' &= \frac{1}{2}(y - z) + \frac{1}{2}(35 - z). \end{aligned}$$

Solve Parts (a) to (c) below.

Part (a). Formulate the system of differential equations as a matrix system $\frac{d}{dt}\vec{u}(t) = A\vec{u}(t) + \vec{b}$. Show details.

Answer. $\vec{u} = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} \frac{3}{4}(45) \\ 20 + \frac{40}{4} \\ \frac{35}{2} \end{pmatrix}, \quad A = \begin{pmatrix} -1 & \frac{1}{4} & 0 \\ \frac{1}{4} & -1 & \frac{1}{2} \\ 0 & \frac{1}{2} & -1 \end{pmatrix}$

Part (b). The heating problem has an **equilibrium solution** $\vec{u}_p(t)$ which is a constant vector of temperatures for the three floors. It is formally found by setting $\frac{d}{dt}\vec{u}(t) = 0$, and then $\vec{u}_p = -A^{-1}\vec{b}$. Justify the algebra and explicitly find $\vec{u}_p(t)$.

Answer. $\vec{u}_p(t) = -A^{-1}\vec{b} = \begin{pmatrix} \frac{560}{11} \\ \frac{755}{11} \\ \frac{570}{11} \end{pmatrix} = \begin{pmatrix} 50.91 \\ 68.64 \\ 51.82 \end{pmatrix}.$

Part (c). The homogeneous problem is $\frac{d}{dt}\vec{u}(t) = A\vec{u}(t)$. It can be solved by a variety of methods, three major methods enumerated below. Choose a method and solve for $\vec{x}(t)$.

Answer: The homogenous scalar general solution is

$$\begin{aligned} x_1(t) &= -2c_1e^{-t} + \frac{1}{2}c_2e^{-at} + \frac{1}{2}c_3e^{-bt}, \\ x_2(t) &= -\frac{1}{2}\sqrt{5}c_2e^{-at} - \frac{1}{2}\sqrt{5}c_3e^{-bt}, \\ x_3(t) &= c_1e^{-t} + c_2e^{-at} + c_3e^{-bt}. \end{aligned}$$

Four Methods for Solving $\vec{u}' = A\vec{u}$

- **Eigenanalysis Method**. Three eigenpairs of matrix A are required. The matrix A must be diagonalizable, meaning there are 3 eigenpairs $(\lambda_1, \vec{v}_1), (\lambda_2, \vec{v}_2), (\lambda_3, \vec{v}_3)$. The main theorem says that the general solution of $\vec{u}' = A\vec{u}$ is

$$\vec{u}(t) = c_1e^{\lambda_1 t}\vec{v}_1 + c_2e^{\lambda_2 t}\vec{v}_2 + c_3e^{\lambda_3 t}\vec{v}_3.$$

- **Laplace's Method**. Solve the scalar equations by the Laplace transform method. The **resolvent method** automates this process: $\vec{u}(t) = \mathcal{L}^{-1}\left((sI - A)^{-1}\right)\vec{u}(0)$.

- **Cayley-Hamilton-Ziebur Method**. The solution $\vec{u}(t)$ is a vector linear combination

$$\vec{u}(t) = \vec{d}_1 f_1(t) + \vec{d}_2 f_2(t) + \vec{d}_3 f_3(t)$$

  of the Euler solution atoms f_1, f_2, f_3 found from the roots of the characteristic equation $|A - \lambda I| = 0$.

  The vectors $\vec{d}_1, \vec{d}_2, \vec{d}_3$ are determined by the explicit formula

$$< \vec{d}_1 \,|\, \vec{d}_2 \,|\, \vec{d}_3 > = < \vec{u}_0 \,|\, A\vec{u}_0 \,|\, A^2\vec{u}_0 > \left(W(0)^T\right)^{-1},$$

  where $W(t)$ is the Wronskian matrix of atoms f_1, f_2, f_3 and $\vec{u}_0$ is the initial data.

- **Exponential Matrix Method**. The method uses e^{At}, which is a fundamental matrix $\Phi(t)$ for the system $\frac{d}{dt}\vec{u} = A\vec{u}$, which satisfies the extra condition $\Phi(0) = I$=$n \times n$ identity matrix. Then the solution of the system can be written $\vec{u}(t) = e^{At}\vec{u}(0)$. Putzer's method applies to find e^{At} in any dimension n. More practical is a computer algebra system. For instance, `maple` finds the exponential matrix by this sample code:

```
A:=<1,2|3,4>; LinearAlgebra[MatrixExponential](A,t);
```