Laplace's method solves differential equations. It is the premier method for solving equations containing switches or impulses.

Unit Step Define $u(t-a) = \begin{cases} 1 & t \ge a, \\ 0 & t < a. \end{cases}$ It is a **switch**, turned on at $t = a$.

Ramp Define **ramp**$(t-a) = (t-a)u(t-a) = \begin{cases} t-a & t \ge a, \\ 0 & t < a. \end{cases}$, whose graph shape is a continuous **ramp** at 45-degree incline starting at $t = a$.

Unit Pulse Define **pulse**$(t, a, b) = \begin{cases} 1 & a \le t < b, \\ 0 & \text{otherwise} \end{cases} = u(t-a) - u(t-b)$. The switch is **ON** at time $t = a$ and then **OFF** at time $t = b$.

Impulse of a Force

Define the **impulse** of an applied force $F(t)$ on time interval $a \le t \le b$ by the equation

$$\text{Impulse of } F = \int_a^b F(t)dt = \left(\frac{\int_a^b F(t)dt}{b-a} \right)(b-a) = \text{Average Force} \times \text{Duration Time}.$$

Dirac Unit Impulse

A Dirac impulse acts like a hammer hit, a brief injection of energy into a system. It is a special idealization of a real hammer hit, in which only the **impulse** of the force is deemed important, and not its magnitude nor duration.

Define the **Dirac Unit Impulse** by the equation $\delta(t-a) = \dfrac{du}{dt}(t-a)$, where $u(t-a)$ is the unit step. Symbol δ makes sense only under an integral sign, and the integral in question must be a generalized Riemann-Steiltjes integral (definition pending), with new evaluation rules. Symbol δ is an abbreviation like **etc** or **e.g.**, because it abbreviates a paragraph of descriptive text.

- Symbol $M\delta(t-a)$ represents an ideal impulse of magnitude M at time $t = a$. Value M is the change in momentum, but $M\delta(t-a)$ contains no detail about the applied force or the duraction. A common force approximation for a hammer hit of very small duration $2h$ and impulse M is Dirac's approximation
$$F_h(t) = \frac{M}{2h}\,\mathbf{pulse}(t, a-h, a+h).$$
- The fundamental equation is $\int_{-\infty}^{\infty} F(x)\delta(x-a)dx = F(a)$. Symbol $\delta(t-a)$ is not manipulated as an ordinary function, but regarded as $du(t-a)/dt$ in a Riemann-Stieltjes integral.

THEOREM (Second Shifting Theorem). Let $f(t)$ and $g(t)$ be piecewise continuous and of exponential order. Then for $a \ge 0$,

Forward table	**Backward table**
$\mathcal{L}(f(t-a)u(t-a)) = e^{-as}\,\mathcal{L}(f(t))$	$e^{-as}\,\mathcal{L}(f(t)) = \mathcal{L}(f(t-a)u(t-a))$
$\mathcal{L}(g(t)u(t-a)) = e^{-as}\,\mathcal{L}\left(g(t)\vert_{t:=t+a}\right)$	$e^{-as}\,\mathcal{L}(f(t)) = \mathcal{L}\left(f(t)u(t)\vert_{t:=t-a}\right).$

Problem 2. Laplace's method for piecewise functions and impulses.

(a) Forward table. Unit step, ramp and pulse. Evaluate the expressions as functions of s.

$$(1)\ \mathcal{L}((t-1)u(t-1)) \quad (2)\ \mathcal{L}(e^t\,\mathbf{ramp}(t-2)), \quad (3)\ \mathcal{L}(5\,\mathbf{pulse}(t,2,4)).$$

(b) Backward table. Find $f(t)$ in the following special cases.

$$(1)\ \mathcal{L}(f) = \frac{e^{-2s}}{s} \quad (2)\ \mathcal{L}(f) = \frac{e^{-s}}{(s+1)^2} \quad (3)\ \mathcal{L}(f) = e^{-s}\frac{3}{s} - e^{-2s}\frac{3}{s}.$$

Problem 3. Evaluate the expressions as functions of s.

(c) Forward table. Dirac Impulse and the Second Shifting theorem.

$$(1)\ \mathcal{L}(2\delta(t-5)), \quad (2)\ \mathcal{L}(2\delta(t-1)+5\delta(t-3)), \quad (3)\ \mathcal{L}(e^t\delta(t-2)).$$

The sum of Dirac impulses in (2) is called an **impulse train**. The numbers 2 and 5 represent the applied **impulse** at times 1 and 3, respectively.

Reference: The Riemann-Stieltjes Integral

Definition

The Riemann-Stieltjes integral of a real-valued function f of a real variable with respect to a real monotone non-decreasing function g is denoted by

$$\int_a^b f(x)\,dg(x)$$

and defined to be the limit, as the mesh of the partition

$$P = \{a = x_0 < x_1 < \cdots < x_n = b\}$$

of the interval $[a,b]$ approaches zero, of the approximating RiemannStieltjes sum

$$S(P,f,g) = \sum_{i=0}^{n-1} f(c_i)(g(x_{i+1}) - g(x_i))$$

where c_i is in the i-th subinterval $[x_i, x_{i+1}]$. The two functions f and g are respectively called the **integrand** and the **integrator**.

The **limit** is a number A, the value of the Riemann-Stieltjes integral. The meaning of the limit: Given $\varepsilon > 0$, then there exists $\delta > 0$ such that for every partition $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ with $\mathbf{mesh}(P) = \max_{0\le i<n}\ (x_{i+1} - x_i) < \delta$, and for every choice of points c_i in $[x_i, x_{i+1}]$,

$$|S(P,f,g) - A| < \varepsilon.$$