Name ____________________________ Class Time ______

Math 2280 Maple Project 3: Laplace Applications
S2019

Maple lab 3 has five problems L3-1, L3-2, L3-3, L3-4, L3-5.

References: Code in `maple` appears in file `2280mapleL3-laplace-S2019.txt` located at URL `http://www.math.utah.edu/~gustafso/`. This document: `2280mapleL3-laplace-S2019.pdf`. Other related and required documents are available at the web site.

Problem L3-1. (Periodic Wave Plots)

In the table are examples of standard periodic waves used in engineering applications. The last table column has piecewise expressions h defined on the base interval $[0, T]$.

Given an expression h on the base $[0, T]$, then its T-periodic extension H to $(-\infty, \infty)$ is always $H(x) = h(g(x))$ where $g(x) = x - T\,\mathbf{floor}(x/T)$.

In terms of the *triangular wave* $\mathbf{twave}(x) = x - \mathbf{floor}(x)$, we may write $g(x) = T\,\mathbf{twave}(x/T)$. The triangular wave is remembered as a **T-periodic train of ramps**. The *staircase function* $\mathbf{floor}(x)$ used in the construction of the ramp train is **not periodic**; it is a standard math library function in major programming languages.

(a) Plot all the periodic examples `f1 to f7`. Please choose an appropriate graph window for each.

(b) Justify every maple expression in the table. Use the example below as a guide.

Example. To justify the first table entry, which contains math definition `f1(x)` and piecewise definition `h1(x)` for the square wave, define `H1(x)` to be the T-periodic extension of `h1(x)` to the whole real line, then plot `H1(x)-f1(x)` over three periods. The plot should be the zero function, i.e., the x-axis.

```
# Maple details for the example
opt1:=ytickmarks=3,color=red,labels=[x,'f(x)'],title="square wave";
opt2:=numpoints=100,thickness=2,discont=true;
opts:=opt1,opt2;
f1:=x -> (-1)^(floor(x));
T:=2; # T = period = 2
g:=x -> x-T*floor(x/T);
h1:=x -> piecewise(x<1,1,x<2,-1,0);
H1:=x -> h1(g(x)); # T-periodic extension
plot(H1(x)-f1(x),x=0..3*T,opts); # Should plot as y=0 (the x-axis)
```

Maple Expression	Name	T	Piecewise Definition on $[0, T]$
`f1:=x ->(-1)^floor(x);`	square wave	2	$h_1(x) = \begin{cases} 1 & 0 \le x < 1, \\ -1 & 1 \le x < 2 \end{cases}$
`f2:=x -> x-floor(x);`	triangular wave	1	$h_2(x) = \begin{cases} x & 0 \le x < 1, \\ 0 & x = 1 \end{cases}$
`f3:=x -> 1/2+(f2(x)-1/2)*f1(x);`	sawtooth wave	2	$h_3(x) = \begin{cases} x & 0 \le x < 1, \\ 2 - x & 1 \le x < 2 \end{cases}$
`f4:=x->abs(sin(x));`	rectified sine	2π	$h_4(x) = \begin{cases} \sin(x) & 0 \le x < \pi, \\ -\sin(x) & \pi \le x < 2\pi \end{cases}$
`f5:=x->(sin(x)+abs(sin(x)))/2;`	half-wave rectified sine	2π	$h_5(x) = \begin{cases} \sin(x) & 0 \le x < \pi, \\ 0 & \pi \le x < 2\pi \end{cases}$
`p:= x -> (2-x)*x:` `f6:=t->p(2*f2(x/2))*f1(x/2);`	parabolic wave	4	$h_6(x) = \begin{cases} p(x) & 0 \le x < 2, \\ -p(x-2) & 2 \le x < 4 \end{cases}$
`q:=x->` `piecewise(x<Pi,sin(x),x<2*Pi,-1):` `f7:=x->q(2*Pi*f2(x/2/Pi));`	piecewise sine pulse	2π	$h_7(x) = \begin{cases} \sin(x) & 0 \le x < \pi, \\ -1 & \pi \le x < 2\pi \end{cases}$

Problem L3-2. (Hammer Hit Oscillation)

An attached mass in an undamped spring-mass system is released from rest 1 meter below the equilibrium position. After 3 seconds of oscillation, the mass is struck by a hammer with force of 5 Newtons in a downward direction.

(a) Assume the model

$$\frac{d^2x}{dt^2} + 9x = 5\delta(t-3); x(0) = 1, \frac{dx}{dt}(0) = 0,$$

where $x(t)$ denotes the displacement from equilibrium at time t and $\delta(t-3)$ denotes the Dirac impulse. Determine, using the `dsolve` example below, a piecewise-defined formula for $x(t)$. Plot $x(t)$ for $0 \le t \le 7$.

(b) Solve the following hammer-hit models `DE1` to `DE4`, given as maple expressions, using the `dsolve` example for `DE`, `IC` as a template for the solution.

(c) Express the symbolic answer for each of `DE1` to `DE4` as a piecewise-defined function. Interpret each answer physically from the initial conditions and the applied impulse.

```
DE:=diff(x(t),t,t)+9*x(t)=3*Dirac(t-3); IC:=x(0)=1,D(x)(0)=0;
dsolve({DE,IC},x(t),method=laplace);
# x(t) = cos(3*t)+Heaviside(t-3)*sin(-9+3*t)
convert(%,piecewise);combine(%,trig);
# x(t) = cos(3*t) for t < 3,cos(3*t)+sin(-9+3*t) for t>3, undef t=3.

DE1:=diff(x(t),t,t)+9*x(t)=5*Dirac(t-3); IC1:=x(0)=-1,D(x)(0)=1;
DE2:=diff(x(t),t,t)+9*x(t)=6*Dirac(t-3); IC2:=x(0)=1,D(x)(0)=-1;
DE3:=diff(x(t),t,t)+9*x(t)=8*Dirac(t-3); IC3:=x(0)=0,D(x)(0)=-1;
DE4:=diff(x(t),t,t)+9*x(t)=9*Dirac(t-3); IC4:=x(0)=1,D(x)(0)=0;
# The input 3*Dirac(t-3) is an impulse or hammer hit of impulse=3 at time t=3
# The mass is pulled down 1 unit below equilibrium (x(0)=1) and then
# released from rest (x'(0)=0).
```

Problem L3-3. (Maple Solution of Initial Value Problems)

(a) Solve the IVP $y''(t) - y'(t) - 2y(t) = 5\sin(t)$, $y(0) = 1$, $y'(0) = -1$. Please use the `inttrans` package. Show the steps in Laplace's method, entirely in maple, with explicit use of maple functions `laplace(f(t),t,s)` and `invlaplace(F(s),s,t)`. The solution should duplicate the major steps that would be done by hand, table details omitted. See the example below.

(b) Solve the pulse-input IVP

$$3y'' + 3y' + 2y = \begin{cases} 0 & \text{for} \quad t < 0, \\ 3 & \text{for} \quad 0 \le t < 4, \\ 0 & \text{for} \quad t \ge 4, \end{cases}$$

with initial data $y(0) = 0$, $y'(0) = 0$. Use any maple method. Express your answer as a piecewise-defined function.

(c) Solve the IVP $y'' + y = 1 + \delta(t - 2\pi)$, $y(0) = 1$, $y'(0) = 0$. Use maple `dsolve`. Express the answer as a piecewise-defined function.

```
# Example (a)
# 5y ''(t) + 4y(t) = cos(t), y (0) = 1, y'(0)=0
 DE:=5*diff(y(t),t,t)+4*y(t)=cos(t);IC:=y(0)=1,D(y)(0)=0;
with(inttrans):
# PHASE 1: Laplace's method, then isolate L(y(t)) left
 phase1a:=laplace(DE,t,s);
 phase1b:=subs(IC,phase1a);
phase1c:=solve(phase1b,laplace(y(t),t,s));
# PHASE 2: Backward table and Lerch's theorem
 phase2:=invlaplace(phase1c,s,t);
# Answer check with dsolve
 dsolve([DE,IC],y(t));
# Answer: y(t)=-cos(t)+2*cos(2*t/sqrt(5))
```

Problem L3-4. (Expressions for Periodic Waves)

Let h be the T-periodic extension to $-\infty < x < \infty$ of $f(x)$, which is only defined on $0 \le x \le T$. Define $T = 2$ and $f(x) = 2/10 + (7/10)\sin x + (1/10)\cos 5x$ on $[0, T]$.

(a) Plot $h(t)$ on the interval $[-10, 10]$. Use the composition formula $h(t) = f(g(t))$, where $g(t) = t - T\,\mathbf{floor}(t/T)$.

(b) Compute the Laplace of $h(t)$ directly from the periodic function theorem, using the sample maple code

```
int(f(g(t))*exp(-s*t),t=0..T)/(1-exp(-s*T));
```

Replacing $f(x)$ by $(1/10)\cos(5x)$ should give the answer below. The answer for $2/10 + (7/10)\sin x + (1/10)\cos 5x$ has many more terms.

$$\frac{1}{10}\,\frac{se^{2\,s} - s\cos(10) + 5\,\sin(10)}{(s^2 + 25)\,(-1 + e^{2\,s})}$$

(c) Maple directly finds the laplace of $g(t) = t - T\,\mathbf{floor}(t/T)$, but not the laplace of $h(t) = f(g(t))$. Truncating $f(x) = \frac{2}{10} + \frac{7}{10}\sin(x) + \frac{1}{10}\cos(5x)$ to the constant term $2/10$ allows maple to compute the Laplace of $f(g(t))$. But the sine and cosine terms do not evaluate.

To get help from maple, the function $h(t)$ is expressed as a series of pulses. The laplace of the series $h(t)$ can be computed, provided $\frac{1}{10}\cos(5x)$ is removed from $f(x)$. This example shows that the periodic function theorem is a basic tool in Laplace theory. Here's the success story for this example:

```
pulse:=(t,a,b)->Heaviside(t-a)-Heaviside(t-b);
f := x -> 2/10+7/10*sin(x):  h:= t->sum(f(t-n*T)*pulse(t,n*T,n*T+T),n=0..infinity);
inttrans[laplace](h(t),t,s);
eval(%) assuming n::positive;
```

Type this code into maple and report the answer. Check the answer by comparing terms in the solution to part (b) above.

REMARK. Here's what does not work. Beware of testing the code below: it uses about 800mb memory and may finish with no answer. If you find a way to resolve the difficulty for all versions of maple, then please send email, detailing how to do it.

```
pulse:=(t,a,b)->Heaviside(t-a)-Heaviside(t-b);
f := x -> (1/10)*cos(5*x):
h:= t->sum(f(t-n*T)*pulse(t,n*T,n*T+T),n=0..infinity);
inttrans[laplace](h(t),t,s);
eval(%) assuming n::positive;
```

Problem L3-5. (Resolvent Method)

The Laplace resolvent formula for the problem $\mathbf{u}' = A\mathbf{u}$, $\mathbf{u}(0) = \mathbf{u}_0$ is

$$\mathcal{L}(\mathbf{u}(t)) = (sI - A)^{-1}\mathbf{u}_0.$$

For example, $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ gives

$$\mathcal{L}(\mathbf{u}(t))) = \begin{pmatrix} s-1 & 0 \\ 0 & s-2 \end{pmatrix}^{-1} \mathbf{u}_0 = \begin{pmatrix} \frac{1}{s-1} & 0 \\ 0 & \frac{1}{s-2} \end{pmatrix} \mathbf{u}_0 = \begin{pmatrix} \mathcal{L}(e^t) & 0 \\ 0 & \mathcal{L}(e^{2t}) \end{pmatrix} \mathbf{u}_0,$$

which implies $\mathbf{u}(t) = \begin{pmatrix} e^t & 0 \\ 0 & e^{2t} \end{pmatrix} \mathbf{u}_0$.

The answers for the components of $\mathbf{u}$ are αe^t, βe^{2t}, according to the following maple code:

```
with(LinearAlgebra):with(inttrans):
A:=Matrix([[1,0],[0,2]]);
u0:=Vector([alpha,beta]);C:=s*IdentityMatrix(2)-A;
B:=(1/C).u0; # 1/C is the inverse of C
u:=Map(invlaplace,B,s,t);
```

Compute the solution $\mathbf{u}(t)$ using the resolvent formula for the following cases.

(a) $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, $\mathbf{u}(0) = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$

(b) $A = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix}$, $\mathbf{u}(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

(c) $A = \begin{pmatrix} 2 & -5 \\ 4 & -2 \end{pmatrix}$, $\mathbf{u}(0) = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$