Systems of Second Order Differential Equations
Cayley-Hamilton-Ziebur

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Characteristic Equation

Definition 1 (Characteristic Equation)
Given a square matrix $\boldsymbol{A}$, the **characteristic equation** of $\boldsymbol{A}$ is the polynomial equation

$$\mathbf{det}(\boldsymbol{A} - \boldsymbol{\lambda} \boldsymbol{I}) = \mathbf{0}.$$

The determinant $|\boldsymbol{A} - \boldsymbol{\lambda} \boldsymbol{I}|$ is formed by subtracting $\boldsymbol{\lambda}$ from the diagonal of $\boldsymbol{A}$.
The polynomial $\boldsymbol{p}(\boldsymbol{x}) = |\boldsymbol{A} - \boldsymbol{x}\boldsymbol{I}|$ is called the **characteristic polynomial** of matrix $\boldsymbol{A}$.

- If $\boldsymbol{A}$ is $\mathbf{2} \times \mathbf{2}$, then $\boldsymbol{p}(\boldsymbol{x})$ is a quadratic.
- If $\boldsymbol{A}$ is $\mathbf{3} \times \mathbf{3}$, then $\boldsymbol{p}(\boldsymbol{x})$ is a cubic.
- The determinant is expanded by the cofactor rule, in order to preserve factorizations.

Characteristic Equation Examples

Create $|\boldsymbol{A} - \boldsymbol{x}\boldsymbol{I}|$ by subtracting $\boldsymbol{x}$ from the diagonal of $\boldsymbol{A}$.

Evaluate by the cofactor rule.

$$\boldsymbol{A} = \begin{pmatrix} \mathbf{2} & \mathbf{3} \\ \mathbf{0} & \mathbf{4} \end{pmatrix}, \quad \boldsymbol{p}(\boldsymbol{x}) = \begin{vmatrix} \mathbf{2} - \boldsymbol{x} & \mathbf{3} \\ \mathbf{0} & \mathbf{4} - \boldsymbol{x} \end{vmatrix} = (\mathbf{2} - \boldsymbol{x})(\mathbf{4} - \boldsymbol{x})$$

$$\boldsymbol{A} = \begin{pmatrix} \mathbf{2} & \mathbf{3} & \mathbf{4} \\ \mathbf{0} & \mathbf{5} & \mathbf{6} \\ \mathbf{0} & \mathbf{0} & \mathbf{7} \end{pmatrix}, \quad \boldsymbol{p}(\boldsymbol{x}) = \begin{vmatrix} \mathbf{2} - \boldsymbol{x} & \mathbf{3} & \mathbf{4} \\ \mathbf{0} & \mathbf{5} - \boldsymbol{x} & \mathbf{6} \\ \mathbf{0} & \mathbf{0} & \mathbf{7} - \boldsymbol{x} \end{vmatrix} = (\mathbf{2} - \boldsymbol{x})(\mathbf{5} - \boldsymbol{x})(\mathbf{7} - \boldsymbol{x})$$

Cayley-Hamilton

Theorem 1 (Cayley-Hamilton)
A square matrix A satisfies its own characteristic equation.

If $p(x) = (-x)^n + a_{n-1}(-x)^{n-1} + \cdots a_0$, then the result is the equation

$$(-A)^n + a_{n-1}(-A)^{n-1} + \cdots + a_1(-A) + a_0 I = 0,$$

where I is the $n \times n$ identity matrix and 0 is the $n \times n$ zero matrix.

The 2×2 Case

Then $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and for $a_1 = \mathbf{trace}(A)$, $a_0 = \det(A)$ we have $p(x) = x^2 + a_1(-x) + a_0$. The Cayley-Hamilton theorem says

$$A^2 + a_1(-A) + a_0 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Cayley-Hamilton Example

Assume

$$\boldsymbol{A} = \begin{pmatrix} \mathbf{2} & \mathbf{3} & \mathbf{4} \\ \mathbf{0} & \mathbf{5} & \mathbf{6} \\ \mathbf{0} & \mathbf{0} & \mathbf{7} \end{pmatrix}$$

Then

$$\boldsymbol{p(x)} = \begin{vmatrix} 2-x & 3 & 4 \\ 0 & 5-x & 6 \\ 0 & 0 & 7-x \end{vmatrix} = (2-x)(5-x)(7-x)$$

and the Cayley-Hamilton Theorem says that

$$(2\boldsymbol{I} - \boldsymbol{A})(5\boldsymbol{I} - \boldsymbol{A})(7\boldsymbol{I} - \boldsymbol{A}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Euler's Substitution and the Characteristic Equation

Definition. Euler's Substitution for the second order equation $\vec{\mathbf{u}}'' = A\vec{\mathbf{u}}$ is

$$\vec{\mathbf{u}} = \vec{\mathbf{v}}e^{rt}.$$

The symbol r is a real or complex constant and symbol $\vec{\mathbf{v}}$ is a constant vector.

Theorem 2 (Euler Solution Equation from Euler's Substitution)
Euler's substitution applied to $\vec{\mathbf{u}}'' = A\vec{\mathbf{u}}$ leads directly to the equation

$$|A - r^2 I| = 0.$$

This is perhaps the premier method for remembering the characteristic equation for the second order vector-matrix equation $\vec{\mathbf{u}}'' = A\vec{\mathbf{u}}$.

Proof: Substitute $\vec{\mathbf{u}} = \vec{\mathbf{v}}e^{rt}$ into $\vec{\mathbf{u}}'' = A\vec{\mathbf{u}}$ to obtain $r^2 e^{rt}\vec{\mathbf{v}} = A\vec{\mathbf{v}}e^{rt}$. Cancel the exponential, then $r^2\vec{\mathbf{v}} = A\vec{\mathbf{v}}$. Re-arrange to the homogeneous system $\left(A - r^2 I\right)\vec{\mathbf{v}} = \vec{\mathbf{0}}$. This homogeneous linear algebraic equation has a nonzero solution $\vec{\mathbf{v}}$ if and only if the determinant of coefficients vanishes: $|A - r^2 I| = 0$.

Cayley-Hamilton-Ziebur Method for Second Order Systems

Theorem 3 (Cayley-Hamilton-Ziebur Structure Theorem for $\vec{u}'' = A\vec{u}$)
The solution $\vec{u}(t)$ of second order equation $\vec{u}''(t) = A\vec{u}(t)$ is a vector linear combination of Euler solution atoms corresponding to roots of the equation $\det(A - r^2 I) = 0$.

The equation $|A - r^2 I| = 0$ is formed by substitution of $\lambda = r^2$ into the eigenanalysis characteristic equation of A.

In symbols, the structure theorem says

$$\vec{u} = \vec{d}_1 A_1 + \cdots + \vec{d}_k A_k,$$

where $A_1, \ldots, A_k$ are Euler solution atoms corresponding to the roots r of the determining equation $|A - r^2 I| = 0$. Therefore, all vectors in the relation have real entries. However, only $2n$ entries of vectors $\vec{d}_1, \ldots, \vec{d}_k$ are arbitrary constants, the remaining entries being dependent on them.

Proof of the Cayley-Hamilton-Ziebur Theorem

Consider the case when A is 2×2 ($n = 2$), because the proof details are similar in higher dimensions. Expand $|A - xI| = 0$ to find the characteristic equation $x^2 + cx + d = 0$, for some constants c, d. The Cayley-Hamilton theorem says that $A^2 + cA + d\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Let $\vec{\mathbf{u}}$ be a solution of $\vec{\mathbf{u}}''(t) = A\vec{\mathbf{u}}(t)$. Multiply the Cayley-Hamilton identity by vector $\vec{\mathbf{u}}$ and simplify to obtain

$$A^2\vec{\mathbf{u}} + cA\vec{\mathbf{u}} + d\vec{\mathbf{u}} = \vec{0}.$$

Using equation $\vec{\mathbf{u}}''(t) = A\vec{\mathbf{u}}(t)$ backwards, we compute $A^2\vec{\mathbf{u}} = A\vec{\mathbf{u}}'' = \vec{\mathbf{u}}''''$. Replace the terms of the displayed equation to obtain the relation

$$\vec{\mathbf{u}}'''' + c\vec{\mathbf{u}}'' + d\vec{\mathbf{u}} = \vec{0}.$$

Each component y of vector $\vec{\mathbf{u}}$ then satisfies the 4th order linear homogeneous equation $y^{(4)} + cy^{(2)} + dy = 0$, which has characteristic equation $r^4 + cr^2 + d = 0$. This equation is the expansion of determinant equation $|A - r^2 I| = 0$. Therefore y is a linear combination of the Euler solution atoms found from the roots of this equation. It follows then that $\vec{\mathbf{u}}$ is a vector linear combination of the Euler solution atoms so identified. This completes the proof.

A 2 × 2 Illustration

Solve the system $\vec{u}'' = A\vec{u}$, $A = \begin{pmatrix} -75 & 25 \\ 50 & -50 \end{pmatrix}$, which is a spring-mass system with $k_1 = 100$, $k_2 = 50$, $m_1 = 2$, $m_2 = 1$.

Solution: The eigenvalues of A are $\lambda = -25$ and -100. Then the determining equation $|A - r^2 I| = 0$ has complex roots $r = \pm 5i$ and $\pm 10i$ with corresponding Euler solution atoms $\cos(4t), \sin(5t), \cos(10t), \sin(10t)$. The eigenpairs of A are

$$\left(-25, \begin{pmatrix} 1 \\ 2 \end{pmatrix}\right), \quad \left(-100, \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right).$$

Then $\vec{u}$ is a vector linear combination of the Euler solution atoms

$$\vec{u}(t) = \vec{d}_1 \cos(5t) + \vec{d}_2 \sin(5t) + \vec{d}_3 \cos(10t) + \vec{d}_4 \sin(10t).$$

A 2×2 Illustration continued

How to Find $\vec{\mathrm{d}}_1$ to $\vec{\mathrm{d}}_4$

Substitute the formula

$$\vec{u}(t) = \vec{d}_1 \cos(5t) + \vec{d}_2 \sin(5t) + \vec{d}_3 \cos(10t) + \vec{d}_4 \sin(10t)$$

into $\vec{u}'' = A\vec{u}$, then solve for the unknown vectors $\vec{d}_j$, $j = 1, 2, 3, 4$, by equating coefficients of Euler solution atoms matching left and right:

$$A\vec{d}_1 = -25\vec{d}_1, \quad A\vec{d}_2 = -25\vec{d}_2, \quad A\vec{d}_3 = -100\vec{d}_3, \quad A\vec{d}_4 = -100\vec{d}_4.$$

These eigenpair relationships imply formulas involving the eigenvectors of A. We get, for some constants a_1, a_2, b_1, b_2,

$$\vec{d}_1 = a_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad \vec{d}_2 = b_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad \vec{d}_3 = a_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \vec{d}_4 = b_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Summary for the 2×2 Illustration

$$\vec{u}(t) = \vec{d}_1 \cos(5t) + \vec{d}_2 \sin(5t) + \vec{d}_3 \cos(10t) + \vec{d}_4 \sin(10t)$$

$$\vec{u}(t) = (a_1 \cos(5t) + b_1 \sin(5t)) \begin{pmatrix} 1 \\ 2 \end{pmatrix} + (a_2 \cos(10t) + b_2 \sin(10t)) \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$