Systems of Second Order Differential Equations
Cayley-Hamilton-Ziebur

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Characteristic Equation

Definition 1 (Characteristic Equation)
Given a square matrix $A$, the characteristic equation of $A$ is the polynomial equation
\[ \det(A - \lambda I) = 0. \]
The determinant $|A - \lambda I|$ is formed by subtracting $\lambda$ from the diagonal of $A$. The polynomial $p(x) = |A - xI|$ is called the characteristic polynomial of matrix $A$.

- If $A$ is $2 \times 2$, then $p(x)$ is a quadratic.
- If $A$ is $3 \times 3$, then $p(x)$ is a cubic.
- The determinant is expanded by the cofactor rule, in order to preserve factorizations.
Characteristic Equation Examples

Create $|A - xI|$ by subtracting $x$ from the diagonal of $A$.

Evaluate by the cofactor rule.

$$A = \begin{pmatrix} 2 & 3 & 4 \\ 0 & 5 & 6 \\ 0 & 0 & 7 \end{pmatrix}, \quad p(x) = \begin{vmatrix} 2 - x & 3 \\ 0 & 4 - x \end{vmatrix} = (2 - x)(4 - x)$$

$$A = \begin{pmatrix} 2 & 3 & 4 \\ 0 & 5 & 6 \\ 0 & 0 & 7 \end{pmatrix}, \quad p(x) = \begin{vmatrix} 2 - x & 3 & 4 \\ 0 & 5 - x & 6 \\ 0 & 0 & 7 - x \end{vmatrix} = (2 - x)(5 - x)(7 - x)$$
Cayley-Hamilton

Theorem 1 (Cayley-Hamilton)
A square matrix $A$ satisfies its own characteristic equation.

If $p(x) = (-x)^n + a_{n-1}(-x)^{n-1} + \cdots + a_0$, then the result is the equation

$(-A)^n + a_{n-1}(-A)^{n-1} + \cdots + a_1(-A) + a_0I = 0,$

where $I$ is the $n \times n$ identity matrix and $0$ is the $n \times n$ zero matrix.

The $2 \times 2$ Case

Then $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and for $a_1 = \text{trace}(A)$, $a_0 = \det(A)$ we have $p(x) = x^2 + a_1(-x) + a_0$. The Cayley-Hamilton theorem says

$A^2 + a_1(-A) + a_0 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$
Cayley-Hamilton Example

Assume

\[ A = \begin{pmatrix} 2 & 3 & 4 \\ 0 & 5 & 6 \\ 0 & 0 & 7 \end{pmatrix} \]

Then

\[ p(x) = \begin{vmatrix} 2 - x & 3 & 4 \\ 0 & 5 - x & 6 \\ 0 & 0 & 7 - x \end{vmatrix} = (2 - x)(5 - x)(7 - x) \]

and the Cayley-Hamilton Theorem says that

\[ (2I - A)(5I - A)(7I - A) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \]
Euler’s Substitution and the Characteristic Equation

**Definition.** Euler’s Substitution for the second order equation \( \vec{u}'' = A\vec{u} \) is

\[ \vec{u} = \vec{v}e^{rt}. \]

The symbol \( r \) is a real or complex constant and symbol \( \vec{v} \) is a constant vector.

**Theorem 2 (Euler Solution Equation from Euler’s Substitution)**

Euler’s substitution applied to \( \vec{u}'' = A\vec{u} \) leads directly to the equation

\[ |A - r^2 I| = 0. \]

This is perhaps the premier method for remembering the characteristic equation for the second order vector-matrix equation \( \vec{u}'' = A\vec{u} \).

**Proof:** Substitute \( \vec{u} = \vec{v}e^{rt} \) into \( \vec{u}'' = A\vec{u} \) to obtain \( r^2e^{rt}\vec{v} = A\vec{v}e^{rt} \). Cancel the exponential, then \( r^2\vec{v} = A\vec{v} \). Re-arrange to the homogeneous system \((A - r^2I)\vec{v} = \vec{0}\). This homogeneous linear algebraic equation has a nonzero solution \( \vec{v} \) if and only if the determinant of coefficients vanishes: \( |A - r^2 I| = 0 \).
Theorem 3 (Cayley-Hamilton-Ziebur Structure Theorem for $\ddot{\mathbf{u}} = A\mathbf{u}$)

The solution $\mathbf{u}(t)$ of second order equation $\ddot{\mathbf{u}}(t) = A\mathbf{u}(t)$ is a vector linear combination of Euler solution atoms corresponding to roots of the equation $\det(A - r^2\mathbf{I}) = 0$.

The equation $|A - r^2\mathbf{I}| = 0$ is formed by substitution of $\lambda = r^2$ into the eigenanalysis characteristic equation of $A$.

In symbols, the structure theorem says

$$\mathbf{u} = \mathbf{d}_1 A_1 + \cdots + \mathbf{d}_k A_k,$$

where $A_1, \ldots, A_k$ are Euler solution atoms corresponding to the roots $r$ of the determining equation $|A - r^2\mathbf{I}| = 0$. Therefore, all vectors in the relation have real entries. However, only $2n$ entries of vectors $\mathbf{d}_1, \ldots, \mathbf{d}_k$ are arbitrary constants, the remaining entries being dependent on them.
Proof of the Cayley-Hamilton-Ziebur Theorem

Consider the case when $A$ is $2 \times 2$ ($n = 2$), because the proof details are similar in higher dimensions. Expand $|A - xI| = 0$ to find the characteristic equation $x^2 + cx + d = 0$, for some constants $c, d$. The Cayley-Hamilton theorem says that $A^2 + cA + d \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Let $\vec{u}$ be a solution of $\vec{u}''(t) = A\vec{u}(t)$. Multiply the Cayley-Hamilton identity by vector $\vec{u}$ and simplify to obtain

$$A^2\vec{u} + cA\vec{u} + d\vec{u} = \vec{0}.$$ 

Using equation $\vec{u}''(t) = A\vec{u}(t)$ backwards, we compute $A^2\vec{u} = A\vec{u}'' = \vec{u}'''$. Replace the terms of the displayed equation to obtain the relation

$$\vec{u}''' + c\vec{u}'' + d\vec{u} = \vec{0}.$$ 

Each component $y$ of vector $\vec{u}$ then satisfies the 4th order linear homogeneous equation $y^{(4)} + cy^{(2)} + dy = 0$, which has characteristic equation $r^4 + cr^2 + d = 0$. This equation is the expansion of determinant equation $|A - r^2I| = 0$. Therefore $y$ is a linear combination of the Euler solution atoms found from the roots of this equation. It follows then that $\vec{u}$ is a vector linear combination of the Euler solution atoms so identified. This completes the proof.
A 2 × 2 Illustration

Solve the system \( \ddot{\vec{u}} = A\vec{u} \), \( A = \begin{pmatrix} -75 & 25 \\ 50 & -50 \end{pmatrix} \), which is a spring-mass system with \( k_1 = 100, k_2 = 50, m_1 = 2, m_2 = 1 \).

Solution: The eigenvalues of \( A \) are \( \lambda = -25 \) and \( -100 \). Then the determining equation \( |A - r^2 I| = 0 \) has complex roots \( r = \pm 5i \) and \( \pm 10i \) with corresponding Euler solution atoms \( \cos(4t), \sin(5t), \cos(10t), \sin(10t) \). The eigenpairs of \( A \) are

\[
\begin{pmatrix} -25, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \end{pmatrix}, \begin{pmatrix} -100, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \end{pmatrix}.
\]

Then \( \vec{u} \) is a vector linear combination of the Euler solution atoms

\[
\vec{u}(t) = \vec{d}_1 \cos(5t) + \vec{d}_2 \sin(5t) + \vec{d}_3 \cos(10t) + \vec{d}_4 \sin(10t).
\]
A 2 × 2 Illustration continued

How to Find $\vec{d}_1$ to $\vec{d}_4$

Substitute the formula

$$\vec{u}(t) = \vec{d}_1 \cos(5t) + \vec{d}_2 \sin(5t) + \vec{d}_3 \cos(10t) + \vec{d}_4 \sin(10t)$$

into $\vec{u}'' = A\vec{u}$, then solve for the unknown vectors $\vec{d}_j$, $j = 1, 2, 3, 4$, by equating coefficients of Euler solution atoms matching left and right:

$$A\vec{d}_1 = -25\vec{d}_1, \quad A\vec{d}_2 = -25\vec{d}_2, \quad A\vec{d}_3 = -100\vec{d}_3, \quad A\vec{d}_4 = -100\vec{d}_4.$$ 

These eigenpair relationships imply formulas involving the eigenvectors of $A$. We get, for some constants $a_1, a_2, b_1, b_2$,

$$\vec{d}_1 = a_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad \vec{d}_2 = b_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad \vec{d}_3 = a_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \vec{d}_4 = b_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$
Summary for the $2 \times 2$ Illustration

$$\vec{u}(t) = \vec{d}_1 \cos(5t) + \vec{d}_2 \sin(5t) + \vec{d}_3 \cos(10t) + \vec{d}_4 \sin(10t)$$

$$\vec{u}(t) = (a_1 \cos(5t) + b_1 \sin(5t)) \begin{pmatrix} 1 \\ 2 \end{pmatrix} + (a_2 \cos(10t) + b_2 \sin(10t)) \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$