Introduction to Fourier Series

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Fourier Series: The Square Wave Example

A model problem in Fourier Series is to compute the coefficients in the classical Fourier series

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx).$$

The definition of function f is:

$$f(x) = \begin{cases} 1 & 0 < x < \pi, \\ -1 & -\pi < x < 0. \end{cases} \tag{1}$$

This example is called the **square wave**.

Classical Method: Fourier Coefficients of the Square Wave

Classical Method. Historically the coefficients in a Fourier series are computed by formulas derived from a simple calculus idea:

> Multiply the series expression on both sides by a function $g(x)$, then integrate across the equation from $x = -\pi$ to $x = \pi$.

The plan is to choose $g(x)$ as one of the functions from the **Trig System**:

$$1, \quad \cos(nx), \quad \sin(nx), \quad n = 1, \ldots, \infty.$$

The trig system is a sequence $\{\phi_k\}_{k=1}^{\infty}$ of **orthogonal functions** with respect to the inner product $\langle f, g\rangle = \int_{-\pi}^{\pi} f(x)g(x)dx$, which means

$$\langle \phi_i, \phi_j\rangle = 0 \quad i \neq j, \quad \langle \phi_i, \phi_i\rangle > 0 \quad \text{for all } i.$$

Square Wave Fourier Coefficients. Displayed here without details are the coefficients in the relation

$$f(x) = \begin{cases} 1 & 0 < x < \pi, \\ -1 & -\pi < x < 0. \end{cases} = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx).$$

Square Wave Coefficients: $a_0 = 0, \quad a_n = 0, \quad b_n = \frac{2}{n\pi}\left(1 - (-1)^n\right)$

Inner Product and the Trig System

Definition. A set of functions $\{\phi_k\}_{k=1}^{\infty}$ is said to be **orthogonal** with respect to inner product $\langle f, g\rangle$ if

$$\langle \phi_i, \phi_j\rangle = 0 \quad i \neq j, \quad \langle f, g\rangle > 0 \quad \text{for all } i.$$

The **orthogonality relations** for the sequence $\{\phi_k\}_{k=1}^{\infty}$ are the equations displayed above.

Trig System. Define inner product $\langle f, g\rangle = \int_{-\pi}^{\pi} f(x)g(x)dx$. The trig system is orthogonal with respect to this inner product. The enumeration of elements in set $\{1, \cos(nx), \sin(nx)\}$ can be defined by $\phi_1(x) = 1$, $\phi_2(x) = \sin(x)$, $\phi_3(x) = \cos(x)$, $\phi_4(x) = \sin(2x)$ and so on. The trig system **orthogonality relations** are in three equations, in order (1) i, j odd, (2) i, j even, (3) i odd and j even:

$$(1) \quad \langle \phi_i, \phi_j\rangle = \int_{-\pi}^{\pi} \cos(nx)\cos(mx)dx = 0, \quad n \neq m,$$

$i \neq j$ both odd, $i = 2n+1$, $j = 2m+1$,

$$(2) \quad \langle \phi_i, \phi_j\rangle = \int_{-\pi}^{\pi} \sin(nx)\sin(mx)dx = 0, \quad n \neq m,$$

$i \neq j$ both even, $i = 2n$, $j = 2m$, $n \neq m$,

$$(3) \quad \langle \phi_i, \phi_j\rangle = \int_{-\pi}^{\pi} \cos(nx)\sin(mx)dx = 0, \quad n \geq 0, m \geq 1,$$

odd $i = 2n+1$, even $j = 2m$.

Inner Product Space Guide to Fourier Coefficients

The intervals chosen for Fourier series expansions vary with the application, making it necessary to use different orthogonal systems for the series representation. Always, the expansion is an orthogonal series of the form

$$f(x) = \sum_{n=1}^{\infty} c_n \phi_n(x), \quad \{\phi_n\}_{n=1}^{\infty} \quad \text{orthogonal.}$$

Symmetric Interval Expansions

Interval	Inner product	Orthogonal set $\{\phi_n\}_{n=1}^{\infty}$
$[-\pi, \pi]$	$\langle f, g\rangle = \int_{-\pi}^{\pi} fg\,dx$	$\{1, \cos u, \sin u : u = nx\}$
$[-L, L]$	$\langle f, g\rangle = \int_{-L}^{L} fg\,dx$	$\{1, \cos u, \sin u : u = n\pi x/L\}$

Half Range Expansions

Even-odd f	Interval	Inner product	Orthogonal set $\{\phi_n\}_{n=1}^{\infty}$
EVEN	$[0, M]$	$\langle f, g\rangle = \int_{0}^{M} fg\,dx$	$\{1, \cos u : u = n\pi x/M\}$
ODD	$[0, M]$	$\langle f, g\rangle = \int_{0}^{M} fg\,dx$	$\{\sin u : u = n\pi x/M\}$