Systems of Differential Equations
Matrix Methods

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Definition 1 (Characteristic Equation)

Given a square matrix $A$, the **characteristic equation** of $A$ is the polynomial equation

$$\det(A - rI) = 0.$$ 

The determinant $|A - rI|$ is formed by subtracting $r$ from the diagonal of $A$. The polynomial $p(r) = |A - rI|$ is called the **characteristic polynomial**.

- If $A$ is $2 \times 2$, then $p(r)$ is a quadratic.
- If $A$ is $3 \times 3$, then $p(r)$ is a cubic.
- The determinant is generally expanded by the cofactor rule, in order to preserve factorizations.
- If $A$ is triangular, then $|A - rI|$ is the product of diagonal entries.
Characteristic Equation Examples

Create the determinant $|A - rI|$ by subtracting $r$ from the diagonal of $A$. Evaluate by the cofactor rule or the triangular rule.

$$A = \begin{pmatrix} 2 & 3 & 4 \\ 0 & 5 & 6 \\ 0 & 0 & 7 \end{pmatrix}, \quad p(r) = \begin{vmatrix} 2 - r & 3 & 4 \\ 0 & 5 - r & 6 \\ 0 & 0 & 7 - r \end{vmatrix} = (2-r)(5-r)(7-r)$$

$$A = \begin{pmatrix} 2 & 0 & 3 \\ 0 & 4 & 4 \\ 0 & 0 & 7 \end{pmatrix}, \quad p(r) = \begin{vmatrix} 2 - r & 3 & 4 \\ 0 & 4 - r & 6 \\ 0 & 0 & 7 - r \end{vmatrix} = (2-r)(4-r)(7-r)$$
Theorem 1 (Cayley-Hamilton)
A square matrix $A$ satisfies its own characteristic equation.

If $p(r) = (-r)^n + a_{n-1}(-r)^{n-1} + \cdots + a_0$, then the result is the equation

$$(-A)^n + a_{n-1}(-A)^{n-1} + \cdots + a_1(-A) + a_0 I = 0,$$

where $I$ is the $n \times n$ identity matrix and $0$ is the $n \times n$ zero matrix.

The 2 × 2 Case

Then $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and for $a_1 = \text{trace}(A)$, $a_0 = \text{det}(A)$ we have $p(r) = r^2 + a_1(-r) + a_0$. The Cayley-Hamilton theorem says

$$A^2 + a_1(-A) + a_0 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$
Cayley-Hamilton Example

Assume

\[ A = \begin{pmatrix} 2 & 3 & 4 \\ 0 & 5 & 6 \\ 0 & 0 & 7 \end{pmatrix} \]

Then

\[ p(r) = \begin{vmatrix} 2 - r & 3 & 4 \\ 0 & 5 - r & 6 \\ 0 & 0 & 7 - r \end{vmatrix} = (2 - r)(5 - r)(7 - r) \]

and the Cayley-Hamilton Theorem says that

\[ (2I - A)(5I - A)(7I - A) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \]
Cayley-Hamilton-Ziebur Theorem

Theorem 2 (Cayley-Hamilton-Ziebur Structure Theorem for $\vec{u}' = A\vec{u}$)
Each of the components $u_1(t), \ldots, u_n(t)$ of the vector solution $\vec{u}(t)$ of system
$\vec{u}'(t) = A\vec{u}(t)$ is a solution of the $n$th order scalar linear homogeneous constant-coefficient differential equation whose characteristic equation is $|A - rI| = 0$.

Meaning: The vector solution $\vec{u}(t)$ of

$$\vec{u}' = A\vec{u}$$

is a vector linear combination of the Euler solution atoms constructed from the roots of the characteristic equation $|A - rI| = 0$. 
Proof of the Cayley-Hamilton-Ziebur Theorem

Consider the case $n = 2$, because the proof details are similar in higher dimensions.

\[ r^2 + a_1 r + a_0 = 0 \]  
Expanded characteristic equation

\[ A^2 + a_1 A + a_0 I = 0 \]  
Cayley-Hamilton matrix equation

\[ A^2 \vec{u} + a_1 A \vec{u} + a_0 \vec{u} = \vec{0} \]  
Right-multiply by $\vec{u} = \vec{u}(t)$

\[ \vec{u}'' = A \vec{u}' = A^2 \vec{u} \]  
Differentiate $\vec{u}' = A \vec{u}$

\[ \vec{u}'' + a_1 \vec{u}' + a_0 \vec{u} = \vec{0} \]  
Replace $A^2 \vec{u} \rightarrow \vec{u}''$, $A \vec{u} \rightarrow \vec{u}'$

Then the components $x(t), y(t)$ of $\vec{u}(t)$ satisfy the two differential equations

\[
\begin{align*}
  x''(t) + a_1 x'(t) + a_0 x(t) &= 0, \\
  y''(t) + a_1 y'(t) + a_0 y(t) &= 0.
\end{align*}
\]

This system implies that the components of $\vec{u}(t)$ are solutions of the second order DE with characteristic equation $|A - rI| = 0$. 
The Cayley-Hamilton-Ziebur Method for $\vec{u}' = A\vec{u}$

Let $\text{atom}_1, \ldots, \text{atom}_n$ denote the Euler solution atoms constructed from the $n$th order characteristic equation $\det(A - rI) = 0$ by Euler’s Theorem. The solution of

$$\vec{u}' = A\vec{u}$$

is given for some constant vectors $\vec{d}_1, \ldots, \vec{d}_n$ by the equation

$$\vec{u}(t) = (\text{atom}_1)\vec{d}_1 + \cdots + (\text{atom}_n)\vec{d}_n$$

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**Warning**: The vectors $\vec{d}_1, \ldots, \vec{d}_n$ are not arbitrary; they depend on the $n$ initial conditions $u_k(0) = c_k$, $k = 1, \ldots, n$. The number of constants in these $n$ vectors is $n^2$. Picard’s Existence Theorem implies that exactly $n$ of these constant are arbitrary, while the remaining $n^2 - n$ (or $n(n - 1)$) constants are completely determined in terms of the $n$ arbitrary constants. In the $2 \times 2$ case there are 2 arbitrary constants $a, b$. The remaining two constants $c, d$ are completely determined in terms of $a, b$. 
Cayley-Hamilton-Ziebur Method Conclusions

- Solving $\vec{u}' = A\vec{u}$ is reduced to finding the $n$ constant vectors $\vec{d}_1, \ldots, \vec{d}_n$.
- The vectors $\vec{d}_j$ are not arbitrary. They are uniquely determined by $A$ and $\vec{u}(0)$! In particular, the $n$ constant vectors have a total of $n^2$ components, whereas Picard’s theorem says that the general solution of $\vec{u}' = A\vec{u}$ has exactly $n$ arbitrary constants (not $n^2$).

A general method to find them is to differentiate the equation

$$\vec{u}(t) = (\text{atom}_1)\vec{d}_1 + \cdots + (\text{atom}_n)\vec{d}_n$$

$n - 1$ times, then set $t = 0$ and replace $\vec{u}^{(k)}(0)$ by $A^k\vec{u}(0)$ [because $\vec{u}' = A\vec{u}$, $\vec{u}'' = A\vec{u}' = AA\vec{u}$, etc]. The resulting $n$ equations in vector unknowns $\vec{d}_1, \ldots, \vec{d}_n$ can be solved by elimination.

- If all atoms constructed are base atoms constructed from real roots, then each $\vec{d}_j$ is a constant multiple of a real eigenvector of $A$. Atom $e^{rt}$ corresponds to the eigenpair equation $A\vec{v} = r\vec{v}$. 
Cayley-Hamilton-Ziebur Shortcut for $2 \times 2$ systems

Example

Let’s solve $\vec{u}' = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \vec{u}$, $\vec{u}(0) = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$, which is the $2 \times 2$ system

$$\begin{cases} x'(t) = x(t) + 2y(t), & x(0) = -1, \\ y'(t) = 2x(t) + y(t), & y(0) = 2, \end{cases} \quad \vec{u}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}.$$

The characteristic polynomial of the non-triangular matrix $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ is

$$\begin{vmatrix} 1 - r & 2 \\ 2 & 1 - r \end{vmatrix} = (1 - r)^2 - 4 = (r + 1)(r - 3).$$

Because the roots are $r = -1, r = 3$, then the Euler solution atoms are $e^{-t}, e^{3t}$.

Then $\vec{u}$ is a vector linear combination of the solution atoms, $\vec{u} = e^{-t}\vec{d}_1 + e^{3t}\vec{d}_2$, or equivalently,

$$x(t) = ae^{-t} + be^{3t} \quad \text{and} \quad y(t) = ce^{-t} + de^{3t}.$$
How to Find $a, b, c, d$: C-H-Z Shortcut for $2 \times 2$

Known:
\[
\begin{align*}
    x'(t) &= x(t) + 2y(t), \\
    y'(t) &= 2x(t) + y(t), \\
    \mathbf{u}(t) &= \begin{pmatrix} x(t) \\ y(t) \end{pmatrix},
\end{align*}
\]

and
\[
\begin{align*}
    x(t) &= ae^{-t} + be^{3t}, \\
    y(t) &= ce^{-t} + de^{3t}.
\end{align*}
\]

The symbols $a, b$ will be arbitrary constants in the solution, expected because 2 initial conditions are required by Picard’s Theorem. We will find $c, d$ in terms of $a, b$, by this Cayley-Hamilton-Ziebur shortcut:

Given $x(t) = ae^{-t} + be^{3t}$, then $y(t)$ if found by solving for $y(t)$ in one of the differential equations, namely the first one: $x'(t) = x(t) + 2y(t)$. Therefore, $y(t) = \frac{1}{2}(x' - x)$. Substitute the known expression for $x(t)$ to find $y(t)$:
\[
\begin{align*}
    y(t) &= \frac{1}{2}x' - \frac{1}{2}x \\
    &= \frac{1}{2}(ae^{-t} + be^{3t})' - \frac{1}{2}(ae^{-t} + be^{3t}) \\
    &= \frac{1}{2}(-ae^{-t} + 3be^{3t} - ae^{-t} - be^{3t}) \\
    &= -ae^{-t} + be^{3t}.
\end{align*}
\]

The answer: $c = -a, d = b$. Then the general solution of the system has two arbitrary constants $a, b$:
\[
\begin{align*}
    x(t) &= ae^{-t} + be^{3t}, \\
    y(t) &= -ae^{-t} + be^{3t}.
\end{align*}
\]
How to Find $a, b, c, d$ from $x(0) = -1, y(0) = 2$.

Known:

\[
\begin{align*}
  x(t) &= ae^{-t} + be^{3t}, \\
  y(t) &= -ae^{-t} + be^{3t}.
\end{align*}
\]

Set $t = 0$ in these two equations to obtain the linear system of algebraic equations for $a, b$:

\[
\begin{align*}
  -1 &= ae^0 + be^0, \\
  2 &= -ae^0 + be^0.
\end{align*}
\]

Because $e^0 = 1$, then these equations become

\[
\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix},
\]

which is a system studied in Linear Algebra. The solution is $a = -\frac{3}{2}, b = \frac{1}{2}$. Then the solution of the system is

\[
\begin{align*}
  x(t) &= -\frac{3}{2}e^{-t} + \frac{1}{2}e^{3t}, \\
  y(t) &= \frac{3}{2}e^{-t} + \frac{1}{2}e^{3t}.
\end{align*}
\]
How to Directly Find Vectors $\vec{d}_1$ and $\vec{d}_2$: C-H-Z Elimination Method for $2 \times 2$

We solve for vectors $\vec{d}_1$, $\vec{d}_2$ in the equation

$$\vec{u} = e^{-t}\vec{d}_1 + e^{3t}\vec{d}_2.$$ 

Advice: Define $\vec{d}_0 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$. Differentiate the above relation. Replace $\vec{u}'$ via the differential equation $\vec{u}' = A\vec{u}$. Set $t = 0$ in the equations and replace $\vec{u}(0)$ by $\vec{d}_0$ in the two formulas to obtain relations

$$\vec{d}_0 = e^0\vec{d}_1 + e^0\vec{d}_2$$
$$A\vec{d}_0 = -e^0\vec{d}_1 + 3e^0\vec{d}_2$$

We solve for $\vec{d}_1$, $\vec{d}_2$ by elimination. Adding the equations gives $\vec{d}_0 + A\vec{d}_0 = 4\vec{d}_2$ and then $\vec{d}_0 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ implies

$$\vec{d}_1 = \frac{3}{4}\vec{d}_0 - \frac{1}{4}A\vec{d}_0 = \begin{pmatrix} -3/2 \\ 3/2 \end{pmatrix},$$
$$\vec{d}_2 = \frac{1}{4}\vec{d}_0 + \frac{1}{4}A\vec{d}_0 = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}.$$
Summary of the $2 \times 2$ Illustration

The solution $\vec{u}(t)$ of the dynamical system

$$\vec{u}' = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \vec{u}, \quad \vec{u}(0) = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

is a vector linear combination of solution atoms $e^{-t}$, $e^{3t}$ given by the equation

$$\vec{u} = e^{-t} \begin{pmatrix} -3/2 \\ 3/2 \end{pmatrix} + e^{3t} \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}.$$  

Eigenpairs for Free

Each vector appearing in the formula is a scalar multiple of an eigenvector, because eigenvalues $-1$, $3$ are real and distinct. Simplified eigenpairs are

$$\left( -1, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right), \quad \left( 3, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right).$$
A Matrix Method for Finding $\vec{d}_1$ and $\vec{d}_2$

The Cayley-Hamilton-Ziebur Method produces a unique solution for $\vec{d}_1$, $\vec{d}_2$ because the coefficient matrix

$$
\begin{pmatrix}
e^0 & e^0 \\
-e^0 & 3e^0
\end{pmatrix}
$$

is exactly the Wronskian $W$ of the basis of atoms $e^{-t}$, $e^{3t}$ evaluated at $t = 0$. This same fact applies no matter the number of coefficients $\vec{d}_1$, $\vec{d}_2$, ... to be determined.

Let $\vec{d}_0 = \vec{u}(0)$, the initial condition. The answer for $\vec{d}_1$ and $\vec{d}_2$ can be written in matrix form in terms of the transpose $W^T$ of the Wronskian matrix as

$$
\langle \vec{d}_1 | \vec{d}_2 \rangle = \langle \vec{d}_0 | A\vec{d}_0 \rangle (W^T)^{-1}.
$$

Symbol $\langle \vec{A} | \vec{B} \rangle$ is the augmented matrix of column vectors $\vec{A}$, $\vec{B}$. 
Solving a $2 \times 2$ Initial Value Problem by the Matrix Method

$$\vec{u}' = A\vec{u}, \quad \vec{u}(0) = \begin{pmatrix} -\frac{1}{2} \\ 1 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}.$$ 

Then $\vec{d}_0 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$, $A\vec{d}_0 = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$ and

$$\langle \vec{d}_1 | \vec{d}_2 \rangle = \begin{pmatrix} -1 & 3 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}^T \begin{pmatrix} -1 & 3 \\ 2 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} -\frac{3}{2} & 1 \\ 3 & \frac{1}{2} \end{pmatrix}.$$ 

Extract $\vec{d}_1 = \begin{pmatrix} -\frac{3}{2} \\ 3 \\ 2 \end{pmatrix}$, $\vec{d}_2 = \begin{pmatrix} 1 \\ \frac{1}{2} \end{pmatrix}$. Then the solution of the initial value problem is

$$\vec{u}(t) = e^{-t} \begin{pmatrix} -\frac{3}{2} \\ 3 \\ 2 \end{pmatrix} + e^{3t} \begin{pmatrix} 1 \\ \frac{1}{2} \end{pmatrix} = \begin{pmatrix} -\frac{3}{2}e^{-t} + \frac{1}{2}e^{3t} \\ 3 \frac{1}{2}e^{-t} + \frac{1}{2}e^{3t} \end{pmatrix}.$$
Other Representations of the Solution $\vec{u}$

Let $y_1(t), \ldots, y_n(t)$ be a solution basis for the $n$th order linear homogeneous constant-coefficient differential equation whose characteristic equation is $\det(A - rI) = 0$.

Consider the solution basis $\text{atom}_1, \text{atom}_2, \ldots, \text{atom}_n$. Each atom is a linear combination of $\vec{y}_1, \ldots, \vec{y}_n$. Replacing the atoms in the formula

$$\vec{u}(t) = (\text{atom}_1)\vec{d}_1 + \cdots + (\text{atom}_n)\vec{d}_n$$

by these linear combinations implies there are constant vectors $\vec{D}_1, \ldots, \vec{D}_n$ such that

$$\vec{u}(t) = y_1(t)\vec{D}_1 + \cdots + y_n(t)\vec{D}_n$$
Another General Solution of $\vec{u}' = \vec{A} \vec{u}$

**Theorem 3 (General Solution)**
The unique solution of $\vec{u}' = \vec{A} \vec{u}$, $\vec{u}(0) = \vec{d}_0$ is

$$\vec{u}(t) = \phi_1(t)\vec{u}_0 + \phi_2(t)\vec{A}\vec{u}_0 + \cdots + \phi_n(t)\vec{A}^{n-1}\vec{u}_0$$

where $\phi_1, \ldots, \phi_n$ are linear combinations of atoms constructed from roots of the characteristic equation $\det(\vec{A} - r\vec{I}) = 0$, such that

$$\text{Wronskian}(\phi_1(t), \ldots, \phi_n(t))|_{t=0} = \vec{I}.$$
Proof of the theorem

Proof: Details will be given for \( n = 3 \). The details for arbitrary matrix dimension \( n \) is a routine modification of this proof. The Wronskian condition implies \( \phi_1, \phi_2, \phi_3 \) are independent. Then each atom constructed from the characteristic equation is a linear combination of \( \phi_1, \phi_2, \phi_3 \). It follows that the unique solution \( \vec{u} \) can be written for some vectors \( \vec{d}_1, \vec{d}_2, \vec{d}_3 \) as

\[
\vec{u}(t) = \phi_1(t)\vec{d}_1 + \phi_2(t)\vec{d}_2 + \phi_3(t)\vec{d}_3.
\]

Differentiate this equation twice and then set \( t = 0 \) in all 3 equations. The relations \( \vec{u}' = A\vec{u} \) and \( \vec{u}'' = A\vec{u}' = AA\vec{u} \) imply the 3 equations

\[
\begin{align*}
\vec{d}_0 &= \phi_1(0)\vec{d}_1 + \phi_2(0)\vec{d}_2 + \phi_3(0)\vec{d}_3 \\
A\vec{d}_0 &= \phi'_1(0)\vec{d}_1 + \phi'_2(0)\vec{d}_2 + \phi'_3(0)\vec{d}_3 \\
A^2\vec{d}_0 &= \phi''_1(0)\vec{d}_1 + \phi''_2(0)\vec{d}_2 + \phi''_3(0)\vec{d}_3
\end{align*}
\]

Because the Wronskian is the identity matrix \( I \), then these equations reduce to

\[
\begin{align*}
\vec{d}_0 &= 1\vec{d}_1 + 0\vec{d}_2 + 0\vec{d}_3 \\
A\vec{d}_0 &= 0\vec{d}_1 + 1\vec{d}_2 + 0\vec{d}_3 \\
A^2\vec{d}_0 &= 0\vec{d}_1 + 0\vec{d}_2 + 1\vec{d}_3
\end{align*}
\]

which implies \( \vec{d}_1 = \vec{d}_0, \vec{d}_2 = A\vec{d}_0, \vec{d}_3 = A^2\vec{d}_0 \).

The claimed formula for \( \vec{u}(t) \) is established and the proof is complete.
Illustrated here is the change of basis formula for $n = 3$. The formula for general $n$ is similar.

Let $\phi_1(t), \phi_2(t), \phi_3(t)$ denote the linear combinations of atoms obtained from the vector formula

$$(\phi_1(t), \phi_2(t), \phi_3(t)) = (\text{atom}_1(t), \text{atom}_2(t), \text{atom}_3(t)) C^{-1}$$

where

$$C = \text{Wronskian}(\text{atom}_1, \text{atom}_2, \text{atom}_3)(0).$$

The solutions $\phi_1(t), \phi_2(t), \phi_3(t)$ are called the principal solutions of the linear homogeneous constant-coefficient differential equation constructed from the characteristic equation $\det(A - rI) = 0$. They satisfy the initial conditions

$$\text{Wronskian}(\phi_1, \phi_2, \phi_3)(0) = I.$$