Chapters 1 and 2: First Order Differential Equations

(a) [20%] An equation \( y' = f(x, y) \) is **Separable** provided functions \( F, G \) exist such that \( f(x, y) = F(x)G(y) \). Apply a test to an equation showing it fails to be separable.

(b) [30%] Problem \( \frac{dy}{dx} = f(x, y) \) is both linear and separable. It can be solved by superposition \( y = y_h + y_p \), where \( y_h \) is the homogeneous solution and \( y_p \) is an equilibrium solution. Find \( y_h \) and \( y_p \).

(c) [20%] Solve a linear homogeneous equation \( \frac{dy}{dx} + p(x)y = 0 \).

(d) [30%] Solve \( y' + p(x)y = q(x) \) by the linear integrating factor method. Show all steps.

Chapter 3: Linear Equations of Higher Order

(a) [20%] Solve for the general solution: \( y'' + ay' + by = f(x) \)

(c) [30%] Given a damped forced spring-mass system, answer questions about classification, resonance and beats.

(d) [20%] Construct the characteristic equation of a linear \( n \)th order homogeneous differential equation of least order \( n \) which has a given particular solution.

(e) [30%] An \( n \)th order non-homogeneous differential equation is specified by its characteristic equation and the forcing term \( f(x) \). Find the shortest trial solution for \( y_p \) according to the method of undetermined coefficients. **Do not evaluate** undetermined coefficients.

Chapters 4 and 5: Systems of Differential Equations

(a) [20%]

(a1) Use the Eigenanalysis Method to solve \( \frac{d}{dt} \vec{x}(t) = A\vec{x}(t) \).

(a2) Show details for computing an eigenpair.

**Expected:** Show linear algebra details for computing eigenvector \( \vec{v} \) for eigenvalue \( \lambda \). This involves row reduction plus display of the scalar solution and the vector solution.

(b) [20%] Find the scalar general solution of a \( 2 \times 2 \) system by the Cayley-Hamilton-Ziebur Method, using the textbook’s Chapter 4 shortcut.

(c) [30%] Assume a \( 3 \times 3 \) system \( \frac{d}{dt} \vec{u} = A\vec{u} \). has a given scalar general solution.
(c1) Compute a $3 \times 3$ fundamental matrix $\Phi(t)$.
(c2) Write a formula for the exponential matrix $e^{At}$.
(c3) Display the solution of the initial value problem $\frac{d}{dt} \vec{u} = A\vec{u}$, $\vec{u}(0) = \vec{c}$.

(d) [30%] Consider a given $3 \times 3$ linear homogeneous system
\[
\begin{aligned}
x' &= \cdots \\
y' &= \cdots \\
z' &= \cdots
\end{aligned}
\]
Solve the system by the most efficient method.

Chapter 6: Dynamical Systems

Consider the nonlinear dynamical system
\[
\begin{aligned}
x' &= \cdots \\
y' &= \cdots \\
z' &= \cdots
\end{aligned}
\]

(a) [20%] Find the equilibrium points for nonlinear system (1).
(b) [20%] Compute the Jacobian matrix $J(x, y)$ for nonlinear system (1). Then evaluate $J(x, y)$ at each of the equilibrium points found in part (a).
(c) [30%] Consider nonlinear system (1). Classify the linearization at each equilibrium point found in part (a) as a node, spiral, center, saddle. Do not sub-classify a node.
(d) [30%] Consider nonlinear system (1). Determine the possible classifications of node, spiral, center or saddle and corresponding stability for each equilibrium determined in part (a), according to the Pasting Theorem, which is Theorem 2 in section 6.2 (Stability of Almost Linear Systems).

Chapter 7: Laplace Theory

Symbol $\delta(t)$ is the Dirac impulse. Symbol $u(t)$ is the unit step. Assumed below is experience with the following rules. Each rule has precise hypotheses, omitted here for brevity.

- **Convolution Theorem.** $\mathcal{L}(g_1)\mathcal{L}(g_2) = \mathcal{L}\left(\int_0^t g_1(t-x)g_2(x)dx\right)$
- **Periodic Function Theorem.** $f(t+p) = f(t)$ implies $\mathcal{L}(f(t)) = \frac{\int_0^p f(t)dt}{1 - e^{-ps}}$
- **Second Shifting Theorem Forward.** $\mathcal{L}(g(t)u(t-a)) = e^{-as}\mathcal{L}\left(g(t)1_{t>a}\right)$
- **Second Shifting Theorem Backward.** $e^{-as}\mathcal{L}(f(t)) = \mathcal{L}(f(t-a)u(t-a))$
- **Dirac Impulse Formula.** Formally $\delta(t) = du(t)$. Then $\int_0^\infty W(x)du(t-a) = W(a)$.
- **Resolvent Identity.** $\vec{u}' = A\vec{u} + \vec{F}(t)$ has identity $(sI - A)\mathcal{L}(\vec{u}) = \vec{u}(0) + \mathcal{L}(\vec{F})$. 
(a) [20%] Let \( f(t) \) be continuous and of exponential order. Prove a Laplace Rule (no choice as to which rule).

(b) [20%] Illustrate the convolution theorem by solving for \( f(t) \) in the equation \( \mathcal{L}(f(t)) = \cdots \). Check the answer with partial fractions.

(c) [20%] Solve for \( f(t) \) using the second shifting theorem: \( \mathcal{L}(f(t)) = \cdots \)

(d) [20%] Derive a formula for \( \mathcal{L}(x'(t)) \) for an impulse problem like

\[
x''(t) + px'(t) + q x(t) = k\delta(t-a), \quad x(0) = x_0, \quad x'(0) = v_0.
\]

(e) [20%] Laplace Theory applied to a specific forced linear dynamical system \((a, b, c, d) \) are known constants

\[
\begin{cases}
x' = ax + by + f_1(t), \\
y' = cx + dy + f_2(t) \\
x(0) = 0, y(0) = 0,
\end{cases}
\]

produces formulas like

\[
\mathcal{L}(x(t)) = \frac{1}{s^2(s+2)(s+6)}, \quad \mathcal{L}(y(t)) = \frac{s^2 - s - 1}{s^2(s+2)(s+6)}.
\]

Display the Resolvent Method solution steps that produce these formulas.

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**Chapter 9: Fourier Series and Partial Differential Equations**

In part (a), function \( f_0(x) \) is given on the interval \(-L \leq x \leq L\). Let \( f(x) \) be the periodic extension of \( f_0 \) to the whole real line, of period \( 2L \).

(a) [20%] Compute the Fourier coefficients \( a_n \) and \( b_n \) of \( f(x) \) on \([-L, L] \).

In part (b), function \( f_0(x) \) is given on the interval \(-L \leq x \leq L\). Let \( f(x) \) be the periodic extension of \( f_0 \) to the whole real line, of period \( 2L \).

(b) [10%] Find all values of \( x \) for which the Fourier series of \( f \) will exhibit Gibb’s over-shoot.

(c) [10%] Question about the Fourier Convergence Theorem plus integration and differentiation of Fourier series.

(d) [30%] **Heat Conduction in a Rod.**

Let \( L = 2 \) (rod length), \( k = 1 \) (conduction constant). Solve the rod problem on \( 0 \leq x \leq L, t \geq 0 \):

\[
\begin{cases}
\frac{du}{dt} = ku_{xx}, \\
u(0,t) = 0, \\
u(L,t) = 0, \\
u(x,0) = \text{given specific } f(x)
\end{cases}
\]
(e) [30%] Vibration of a Finite String. Let $L = 4$ (string length), $c = 4$ (wave speed). Solve the finite string vibration problem on $0 \leq x \leq L$, $t > 0$:

\[
\begin{aligned}
&u_{tt}(x,t) = c^2 u_{xx}(x,t), \\
&u(0,t) = 0, \\
&u(L,t) = 0, \\
&u(x,0) = \text{given specific } f(x) \\
&u_t(x,0) = \text{given specific } g(x)
\end{aligned}
\]