Outline for Final Exam Differential Equations 2280

Tuesday, 30 April 2019, 7:30am to 10:15am

Chapters 1 and 2: First Order Differential Equations

(a) [20%] An equation $y' = f(x, y)$ is **Separable** provided functions F, G exist such that $f(x, y) = F(x)G(y)$. Apply a test to an equation showing it fails to be separable.

(b) [30%] Problem $\dfrac{dy}{dx} = f(x, y)$ is both linear and separable. It can be solved by superposition $y = y_h + y_p$, where y_h is the homogeneous solution and y_p is an equilibrium solution. Find y_h and y_p.

(c) [20%] Solve a linear homogeneous equation $\dfrac{dy}{dx} + p(x)y = 0$.

(d) [30%] Solve $y' + p(x)y = q(x)$ by the linear integrating factor method. Show all steps.

Chapter 3: Linear Equations of Higher Order

(a) [20%] Solve for the general solution: $y'' + ay' + by = f(x)$

(c) [30%] Given a damped forced spring-mass system, answer questions about classification, resonance and beats.

(d) [20%] Construct the characteristic equation of a linear nth order homogeneous differential equation of least order n which has a given particular solution.

(e) [30%] An nth order non-homogeneous differential equation is specified by its characteristic equation and the forcing term $f(x)$. Find the shortest trial solution for y_p according to the method of undetermined coefficients. **Do not evaluate** undetermined coefficients.

Chapters 4 and 5: Systems of Differential Equations

(a) [20%]

(a1) Use the Eigenanalysis Method to solve $\dfrac{d}{dt}\vec{\mathbf{x}}(t) = A\vec{\mathbf{x}}(t)$.

(a2) Show details for computing an eigenpair.

Expected: Show linear algebra details for computing eigenvector $\vec{\mathbf{v}}$ for eigenvalue λ. This involves row reduction plus display of the scalar solution and the vector solution.

(b) [20%] Find the scalar general solution of a 2×2 system by the Cayley-Hamilton-Ziebur Method, using the textbook's Chapter 4 shortcut.

(c) [30%] Assume a 3×3 system $\dfrac{d}{dt}\vec{\mathbf{u}} = A\vec{\mathbf{u}}$. has a given scalar general solution.

(c1) Compute a 3×3 fundamental matrix $\Phi(t)$.

(c2) Write a formula for the exponential matrix e^{At}.

(c3) Display the solution of the initial value problem $\frac{d}{dt}\vec{\mathbf{u}} = A\vec{\mathbf{u}}$, $\vec{\mathbf{u}}(0) = \vec{\mathbf{c}}$.

(d) [30%] Consider a given 3×3 linear homogeneous system

$$\begin{cases} x' &= \cdots \\ y' &= \cdots \\ z' &= \cdots \end{cases}$$

Solve the system by the most efficient method.

Chapter 6: Dynamical Systems

Consider the nonlinear dynamical system

$$\begin{cases} x' &= \cdots \\ y' &= \cdots \end{cases} \tag{1}$$

(a) [20%] Find the equilibrium points for nonlinear system (1).

(b) [20%] Compute the Jacobian matrix $J(x, y)$ for nonlinear system (1). Then evaluate $J(x, y)$ at each of the equilibrium points found in part (a).

(c) [30%] Consider nonlinear system (1). Classify the linearization at each equilibrium point found in part (a) as a node, spiral, center, saddle. Do not sub-classify a node.

(d) [30%] Consider nonlinear system (1). Determine the possible classifications of node, spiral, center or saddle and corresponding stability for each equilibrium determined in part (a), according to the **Pasting Theorem**, which is Theorem 2 in section 6.2 (Stability of Almost Linear Systems).

Chapter 7: Laplace Theory

Symbol $\delta(t)$ is the Dirac impulse. Symbol $u(t)$ is the unit step. Assumed below is experience with the following rules. Each rule has precise hypotheses, omitted here for brevity.

Convolution Theorem. $\mathcal{L}(g_1)\mathcal{L}(g_2) = \mathcal{L}\left(\int_0^t g_1(t-x)g_2(x)dx\right)$

Periodic Function Theorem. $f(t+p) = f(t)$ implies $\mathcal{L}(f(t)) = \dfrac{\int_0^p f(t)dt}{1 - e^{-ps}}$

Second Shifting Theorem Forward. $\mathcal{L}(g(t)u(t-a)) = e^{-as}\mathcal{L}\left(g(t)|_{t->t+a}\right)$

Second Shifting Theorem Backward. $e^{-as}\mathcal{L}(f(t)) = \mathcal{L}(f(t-a)u(t-a))$

Dirac Impulse Formula. Formally $\delta(t) = du(t)$. Then $\int_0^\infty W(x)du(t-a) = W(a)$.

Resolvent Identity. $\vec{\mathbf{u}}' = A\vec{\mathbf{u}} + \vec{\mathbf{F}}(t)$ has identity $(sI - A)\mathcal{L}(\vec{\mathbf{u}}) = \vec{\mathbf{u}}(0) + \mathcal{L}(\vec{\mathbf{F}})$.

(a) [20%] Let $f(t)$ be continuous and of exponential order. Prove a Laplace Rule (no choice as to which rule).

(b) [20%] Illustrate the convolution theorem by solving for $f(t)$ in the equation $\mathcal{L}(f(t)) = \cdots$. Check the answer with partial fractions.

(c) [20%] Solve for $f(t)$ using the second shifting theorem: $\mathcal{L}(f(t)) = \cdots$

(d) [20%] Derive a formula for $\mathcal{L}(x(t))$ for an impulse problem like

$$x''(t) + p\,x'(t) + q\,x(t) = k\delta(t-a), \quad x(0) = x_0, \quad x'(0) = v_0.$$

(e) [20%] **Laplace Theory** applied to a specific forced linear dynamical system (a, b, c, d are known constants)

$$\begin{cases} x' = ax + by + f_1(t), \\ y' = cx + dy + f_2(t) \\ x(0) = 0, y(0) = 0, \end{cases}$$

produces formulas like

$$\mathcal{L}(x(t)) = \frac{1}{s^2(s+2)(s+6)}, \quad \mathcal{L}(y(t)) = \frac{s^2 - s - 1}{s^2(s+2)(s+6)}.$$

Display the **Resolvent Method** solution steps that produce these formulas.

Chapter 9: Fourier Series and Partial Differential Equations

In part (a), function $f_0(x)$ is given on the interval $-L \le x \le L$. Let $f(x)$ be the periodic extension of f_0 to the whole real line, of period $2L$.

(a) [20%] Compute the Fourier coefficients a_n and b_n of $f(x)$ on $[-L, L]$.

In part (b), function $f_0(x)$ is given on the interval $-L \le x \le L$. Let $f(x)$ be the periodic extension of f_0 to the whole real line, of period $2L$.

(b) [10%] Find all values of x for which the Fourier series of f will exhibit Gibb's over-shoot.

(c) [10%] Question about the Fourier Convergence Theorem plus integration and differentiation of Fourier series.

(d) [30%] **Heat Conduction in a Rod**.
Let $L = 2$ (rod length), $k = 1$ (conduction constant). Solve the rod problem on $0 \le x \le L$, $t \ge 0$:

$$\begin{cases} u_t &= k\,u_{xx}, \\ u(0,t) &= 0, \\ u(L,t) &= 0, \\ u(x,0) &= \text{given specific } f(x) \end{cases}$$

(e) [30%] **Vibration of a Finite String**.
Let $L = 4$ (string length), $c = 4$ (wave speed). Solve the finite string vibration problem on $0 \le x \le L$, $t > 0$:

$$\begin{cases} u_{tt}(x,t) &= c^2\, u_{xx}(x,t), \\ u(0,t) &= 0, \\ u(L,t) &= 0, \\ u(x,0) &= \text{given specific } f(x) \\ u_t(x,0) &= \text{given specific } g(x) \end{cases}$$